Written Assignment #5 (Due on April 8, 2008)

This the assignment is on PDE and Fourier series. Please copy the question; write out a full solution, and hand in the section of class that you registered.

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(1) For the following problems determine the equilibrium temperature distribution (if it exists). For what values of β, are there solutions.

(a) $u_t = u_{xx} + 1, \quad u(x,0) = f(x), \quad u_x(0,t) = 1, \quad u_x(L,t) = \beta$

(b) $u_t = u_{xx} + x - \beta, \quad u(x,0) = f(x), \quad u_x(0,t) = 0, \quad u_x(L,t) = 0.$

(2) (a) show that

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = 0, \quad n \in \mathbb{N}$$

(b) Plot the function $f(x) = x$ and $g(x) = 1$ on the interval $[-L, L]$. Are these functions orthogonal? Show by computation

$$\int_{-L}^{L} f(x) g(x) \mathrm{d}x = 0.$$

(c) Plot the function $u(x) = \frac{1}{2}(3x^2 - 1)$ and $v(x) = \frac{1}{2}(5x^3 - 3x)$ on the interval $[-1, 1]$. Are these functions orthogonal? Show by computation

$$\int_{-1}^{1} u(x) v(x) \mathrm{d}x = 0.$$

Moreover yield the norms $||u(x)||$ and $||v(x)||$.

(3) (a) Find Fourier series for the function

$$f(x) = \mid \sin x \mid, \quad \mid x \mid < \pi.$$

(b) Find Fourier series for the function

$$g(x) = \begin{cases} 1, & -1 \le x < 0; \\ 2, & 0 \le x \le 1. \end{cases}$$

(c) Find Fourier cosine and Fourier sine series for the following function and sketch the Fourier series and compare function to its Fourier series:

$$f(x) = x, \quad 0 < x < \pi.$$

(d) (***Optional**) Making use of the result of last question, compute the Fourier cosine series for the function $g(x) = x - x^2/2$ define on $0 < x < 1$, and show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2.$$

(4) We consider the IBVP of heat conduction of a rod with finite length:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2},$$

with the periodic B.C's:

$$u(-\pi, t) = u(\pi, t), \quad u'(-\pi, t) = u'(\pi, t), \quad (t > 0)$$

and I.C.

$$u(x, 0) = 2x + |\sin x|, \quad (-\pi < x < \pi).$$

(5) We consider the IBVP of vibration of a string with finite length:

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2},$$

with B.C's:

$$u(-1, t) = u(1, t) = 0, \quad (t > 0)$$

and I.C.

$$\begin{cases} u(x, 0) = (1 - x^2) \\ \dfrac{\partial u}{\partial t}(x, 0) = 0. \end{cases} \quad (-1 < x < 1)$$

(6) (***Optional**) Find solution $u(x, y)$ if

$$\nabla^2 u(x, y) = 0, \quad (0 < x, y < \pi), \ t > 0),$$

and

$$u(0, y) = u(\pi, y) = 0, \quad (0 < y < \pi),$$

$$\begin{aligned} &u_y(x, 0) = x(x - \pi); \\ &u_y(x, \pi) = (\cos^2 x + 3\sin x), \\ &\qquad (0 < x < \pi). \end{aligned}$$

(7) We consider the IBVP of vibration of a string with finite length:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x\cos t,$$

with BC's:

$$u_x(0, t) = u(2\pi, t) = 0, \quad (t > 0)$$

and IC.

$$u(x, 0) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi < x \le 2\pi \end{cases}$$