McGill University
Math 261A: Differential Equations
Midterm

Problem 1. Find the general solution of the equation

$$(2xy^2 + 3x^2/y)dx + (3x^2y + 1/y)dy = 0 \tag{1}$$

Solution: Let $M(x,y) = 2xy^2 + 3x^2/y$, $N(x,y) = 3x^2y + 1/y$. Then

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy - \frac{3x^2}{y^2} - 6xy = -\left(2xy + \frac{3x^2}{y^2}\right) \neq 0,$$

so the equation (1) is not exact. However,

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{2xy + 3x^2/y^2}{2xy^2 + 3x^2/y} = \frac{1}{y}$$

is a function depending on y only, so the equation (1) has an integrating factor μ that is a function of y. That function is given by the formula

$$\mu(y) \;=\; \exp\left(\int \frac{dy}{y}\right) = y.$$

After a multiplication by $\mu(y) = y$, the equation becomes

$$(2xy^3 + 3x^2)dx + (3x^2y^2 + 1)dy = 0 \tag{2}$$

To solve the exact equation (2), we have to find a function $F = F(x,y)$ satisfying

$$\begin{cases} \partial F/\partial x = 2xy^3 + 3x^2, \\ \partial F/\partial y = 3x^2y^2 + 1. \end{cases} \tag{3}$$

Integrating the second equation in (3), we find that

$$F(x,y) = \int (3x^2y^2 + 1)dy = x^2y^3 + y + f(x).$$

Substituting into the first equation in (3), we get

$$\partial F/\partial x = 2xy^3 + f'(x) = 2xy^3 + 3x^2.$$

It follows that $f(x) = \int 3x^2 dx = x^3 + c$. We conclude that soltuions of (1) are given implicitly by

$$F(x,y) = x^2y^3 + y + x^3 = C.$$

Problem 2. Find the general solution of the differential equation

$$xy' + y = x^4y^2$$

Solution: First write the equation in normal form:

$$y' + y/x = x^3y^2. \tag{4}$$

This is a Bernoulli equation. Accordingly, we make a change of variable $v = y^{1-2} = 1/y$. Differentiating, we get $y' = -u'/u^2$, and the equation (4) becomes $-u'/u^2 + 1/(ux) = x^3/u^2$. Multiplying by $-u^2$, we get a linear equation

$$u' - u/x = -x^3. \tag{5}$$

The integrating factor is $\mu(x) = \exp(-\int dx/x) = 1/x$. The solution is

$$u \;=\; x\left(-\int \frac{x^3}{x}\,dx + C\right) = -x^4/3 + Cx.$$

Substituting for u, we find that a general solution of (4) is

$$\frac{1}{y} = -x^4/3 + Cx.$$

In addition, $y = 0$ is also a solution.

Problem 3. Find the general solution of the differential equation

$$y'' + 4y = xe^x \tag{6}$$

Solution: The solution of the homogeneous equation $y'' + 4y = 0$ is given by $y = c_1\cos(2x) + c_2\sin(2x)$. We want to find a particular solution y_p of the nonhomogeneous equation (6) by method of undetermined coefficients. The trial solution has the form

$$y_p(x) = axe^x + be^x.$$

We substitute y_p into (6) to find a and b. We get $y_p'' + 4y_p = 5axe^x + (2a+5b)e^x = xe^x$. Accordingly, to find a and b, we have to solve the system

$$\begin{cases} 5a = 1, \\ 2a + 5b = 0. \end{cases}$$

We find that $a = 1/5, b = -2/25$, and $y_p = xe^x/5 - 2e^x/25$. Therefore, the general solution of (6) is given by

$$y_{gen}(x) = \frac{xe^x}{5} - \frac{2e^x}{25} + c_1\cos(2x) + c_2\sin(2x).$$

Problem 4. Find the solution of the differential equation

$$y'' + 3y' + 2y = \sin x \tag{7}$$

satisfying $y(0) = 3, y'(0) = 0$.

Solution: The quadratic equation associated to the homogeneous differential equation $y'' + 3y' + 2y = 0$ is $\lambda^2 + 3\lambda + 2 = 0$. It has roots $\lambda = -2$ and $\lambda = -1$, so a general solution of the homogeneous equation is given by $y = ce^{-x} + de^{-2x}$. We want to find a particular solution y_p of the nonhomogeneous equation (7) by method of undetermined coefficients. The trial solution has the form

$$y_p(x) = a\sin x + b\cos x.$$

We substitute y_p into (7) to find a and b. We get $y_p''+3y_p'+2y_p = (a-3b)\sin x+(b+3a)\cos x = \sin x$. Accordingly, to find a and b, we have to solve the system

$$\begin{cases} a - 3b = 1, \\ b + 3a = 0. \end{cases}$$

We find that $a = 1/10, b = -3/10$, and $y_p = \sin x/10 - 3\cos x/10$. Therefore, the general solution of (7) is given by

$$y_{gen}(x) = \frac{\sin x}{10} - \frac{3\cos x}{10} + ce^{-x} + de^{-2x}.$$

To find c and d, we substitute for initial conditions: $y_{gen}(0) = -3/10 + c + d = 3$, and $y'_{gen}(0) = 1/10 - c - 2d = 0$. Accordingly, to find c and d, we have to solve the system

$$\begin{cases} c + d = 33/10, \\ c + 2d = 1/10. \end{cases}$$

We find that $d = -16/5, c = 13/2$, and the required solution of (7) is given by

$$y(x) = \frac{\sin x}{10} - \frac{3\cos x}{10} + \frac{13e^{-x}}{2} - \frac{16e^{-2x}}{5}.$$