McGill University
Math 325A: Differential Equations

LECTURE 8: FIRST ORDER DIFFERENTIAL EQUATIONS (VII)

Text: Section 3.6

1 Euler's Method

In this section we discuss methods for obtaining a numerical solution of the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0$$

at equally spaced points $x_0, x_1, x_2, \ldots, x_N = p, \ldots$ where $x_{n+1} - x_n = h > 0$ is called the step size. In general, the smaller the value of the better the approximations will be but the number of steps required will be larger. We begin by integrating $y' = f(x,y)$ between x_n and x_{n+1}. If $y(x) = \phi(x)$, this gives

$$\phi(x_{n+1}) = \phi(x_n) + \int_{x_n}^{x_{n+1}} f(t, \phi(t))dt.$$

As a first estimate of the integrand we use the value of $f(t, \phi(t))$ at the lower limit x_n, namely $f(x_n, \phi(x_n))$. Now, assuming that we have already found an estimate y_n for $\phi(x_n)$, we get the estimate

$$y_{n+1} = y_n + hf(x_n, y_n)$$

for $\phi(x_{n+1})$. It can be shown that

$$|y_n - \phi(x_n)| \leq Ch,$$

where C is a constant which depends on p.

2 Improved Euler's Method

The Euler method can be improved if we use the trapezoidal rule for estimating the above integral. Namely,

$$\int_a^b F(x)dx = \frac{1}{2}(F(a) + F(b))(b - a).$$

This leads to the estimate

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_{n+1}, y_{n+1})).$$

If we now use the Euler approximation y_{n+1} to compute $f(x_{n+1}, y_{n+1})$, we get

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n)).$$

This is known as the improved Euler method. It can be shown that

$$|y_n - \phi(x_n)| \leq Ch^2.$$

In general, if y_n is an approximation for $\phi(x_n)$ such that

$$|y_n - \phi(x_n)| \leq Ch^p,$$

we say that the approximation is of order p. Thus the Euler method is first order and the improved Euler is second order.

3 Higher Order Methods

On can obtain higher order approximations by using better approximations for $F(t) = f(t, \phi(t))$ on the interval $[x_n, x_{n+1}]$. For example, the Taylor series approximation

$$F(t) = F(x_n) + F'(x_n)(t - x_n) + F''(x_n)(t - x_n)^2/2 + \cdots + F^{(p-1)}(x_n)(t - x_n)^{p-1}/(p-1)!$$

yields the approximation

$$y_{n+1} = y_n + hf_1(x_x, y_n) + \frac{h^2}{2} f_2(x_n, y_n) + \cdots + \frac{h^p}{p!} f_{p-1}(x_n, y_n),$$

where

$$f_k(x_n, y_n) = F^{(k-1)}(x_n) = \left[\frac{\partial}{\partial x} + f(x, y)\frac{\partial}{\partial y}\right]^{(k-1)} f(x_n, y_n).$$

It can be show that this approximation is of order p. However it is computationally intensive as one has to compute higher derivatives.

In the case $p = 2$ this formula was simplified by Runge and Kutta to give the second order midpoint approximation

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\right].$$

In the case $p = 4$ they obtained the 4-th order approximation

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{aligned}
k_1 &= hf(x_n, y_n), \\
k_2 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}), \\
k_3 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}), \\
k_4 &= hf(x_n + h, y_n + k_3).
\end{aligned}$$

Computationally, it is much simpler than the 4-th order Taylor series approximation from which it is derived.