McGill University
Math 325A: Differential Equations

LECTURE 7: FIRST ORDER DIFFERENTIAL EQUATIONS (VI)

(Text: pp. 38-40, Ch. 13)

GEOMETRICAL APPROACHES

1 Definitions and Basic Concepts

1.1 Directional Field

A plot of short line segments drawn at various points in the (x, y) plane showing the slope of the solution curve there is called *direction field* for the DE.

1.2 Integral Curves

The family of curves in the (x, y) plane, that represent all solutions of DE is called the *integral curves*.

1.3 Autonomous Systems

The first order DE

$$dy/dx = f(y)$$

is called autonomous, since the independent variable does not appear explicitly. The isoclines are made up of horizonal lines $y = m$, along which the slope of directional fields is the constant, $y' = f(m)$.

1.4 Equilibrium Points

The DE has the constant solution $y = y_0$, if and only if $f(y_0) = 0$. These values of y_0 are the **equilibrium points** or **stationary points** of the DE. $y = y_0$ is called a **source** if $f(y)$ changes sign from - to + as y increases from just below $y = y_0$ to just above $y = y_0$ and is called a **sink** if $f(y)$ changes sign from + to - as y increases from just below $y = y_0$ to just above $y = y_0$; it is called a **node** if there is no change in sign. Solutions $y(t)$ of the DE appear to be attracted by the line $y = y_0$, if y_0 is a sink and move away from the line $y = y_0$, if y_0 is a source.

2 Phase Line Analysis

The y-axis on which is plotted the equilibrium points of the DE with arrows between these points to indicate when the solution y is increasing or decreasing is called the phase line of the DE. The autonomous DE

$$dy/dx = 2y - y^2$$

has 0 and 1 as equilibrium points. The point $y = 0$ is a source and $y = 2$ is a sink (see Fig.1). This DE is a logistic model for a population having 2 as the size of a stable population. The equation

$$dy/dx = -y(2 - y)(3 - y)$$

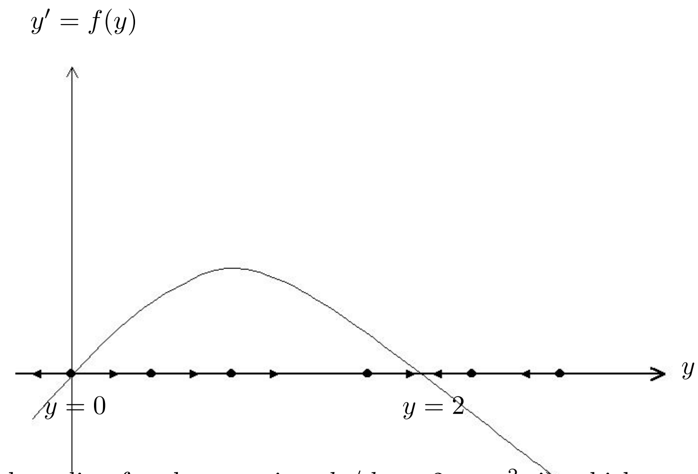

Figure 1: Sketch of the phase line for the equation $dy/dx = 2y - y^2$, in which $y = 0$ is a source, $y = 2$ is a sink.

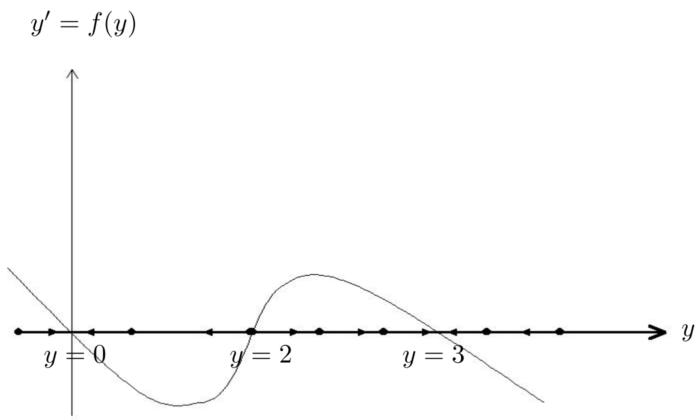

Figure 2: Sketch of the phase line for the equation $dy/dx = -y(2-y)(3-y)$, in which $y = 0, 3$ is a sink, $y = 2$ is a source.

has three equilibrium states: $y = 0, 2, 3$. Among them, $y = 0, 3$ are the sink, while $y = 2$ is the source (see Fig.2). The equation

$$dy/dx = -y(2-y)^2$$

has two equilibrium states: $y = 0, 2$. The point $y = 0$ is a sink, while $y = 2$ is a node (see Fig.3). The sink is stable, source is unstable, whereas the node is semi-stable. The node point of the equation $y = f(y)$ can either disappear, or split into one sink and one source, when the equation is perturbed with a small amount ε and becomes: $y = f(y) + \varepsilon$.

3 Bifurcation Diagram

Some dynamical system contains a parameter Λ, such as

$$y' = f(y, \Lambda).$$

Then the characteristics of its equilibrium states, such as their number and nature, depends on the value of Λ. Some times, through a special value of $\Lambda = \Lambda_*$, these characteristics of equilibrium states

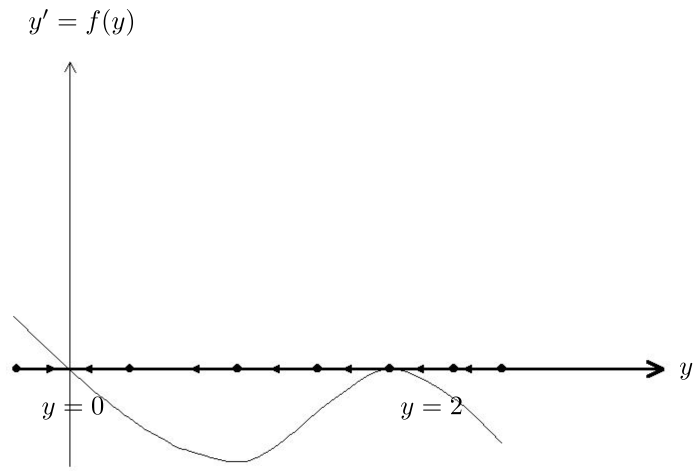

Figure 3: Sketch of the phase line for the equation $dy/dx = -y(2-y)^2$, in which the point $y = 0$ is a sink, while $y = 2$ is a node.

may change. This $\Lambda = \Lambda_*$ is called the **bifurcation point**.

Example 1. For the logistic population growth model, if the population is reduced at a constant rate $s > 0$, the DE becomes

$$dy/dx = 2y - y^2 - s$$

which has a source at the larger of the two roots of the equation

$$y^2 - 2y + s = 0$$

for $s < 2$. If $s > 2$ there is no equilibrium point and the population dies out as y is always decreasing. The point s=2 is called a bifurcation point of the DE.

Example 2. Chemical Reaction Model. One has the DE

$$dy/dx = -ay\left[y^2 - \frac{R - R_c}{a}\right],$$

where

- y is the concentration of species A;
- R is the concentration of some chemical element,

and (a, R_c) are constants (fixed). It is derived that

- If $R < R_c$, the system has one equilibrium state $y = 0$, which is stable;
- If $R > R_c$, the system has three equilibrium states: $y = 0$, which is now unstable, and $y = \pm\sqrt{\frac{R-R_c}{a}}$, which are stable.

For this system, $R = R_c$ is the bifurcation point.

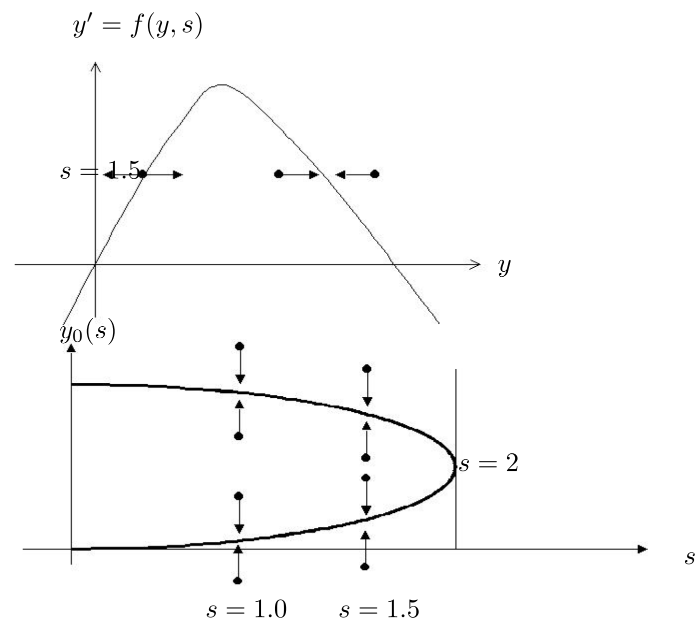

Figure 4: Sketch of the bifurcation diagram of the equation $dy/dx = y(2-y) - s$, in which the point $s = 2$ is the bifurcation point.

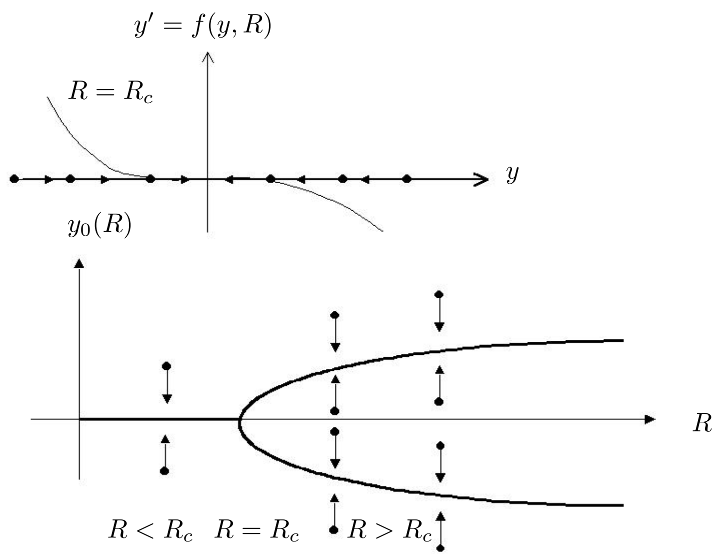

Figure 5: Sketch of the bifurcation diagram of the equation $dy/dx = -ay\left[y^2 - \frac{R-R_c}{a}\right]$, in which the point $R = R_c$ is the bifurcation point.