McGill University
Math 325A: Differential Equations

LECTURE 5: FIRST ORDER DIFFERENTIAL EQUATIONS (IV)

(Text: Sections 2.5,2.6)

1 Change of Variables.

Sometimes it is possible by means of a change of variable to transform a DE into one of the known types. For example, homogeneous equations can be transformed into separable equations and Bernoulli equations can be transformed into linear equations. The same idea can be applied to some other types of equations, as described as follows.

1.1 $y' = f(ax + by),\ b \neq 0$

Here, if we make the substitution $u = ax + by$ the differential equation becomes

$$\frac{du}{dx} = bf(u) + a$$

which is separable.

Example 1. The DE $y' = 1 + \sqrt{y - x}$ becomes $u' = \sqrt{u}$ after the change of variable $u = y - x$.

1.2 $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

Here, we assume that $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ are distinct lines meeting in the point (x_0, y_0). The above DE can be written in the form

$$\frac{dy}{dx} = \frac{a_1(x - x_0) + b_1(y - y_0)}{a_2(x - x_0) + b_2(y - y_0)}$$

which yields the DE

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

after the change of variables $X = x - x_0$, $Y = y - y_0$.

1.3 Riccati equation: $y' = p(x)y + q(x)y^2 + r(x)$

Suppose that $u = u(x)$ is a solution of this DE and make the change of variables $y = u + 1/v$. Then $y' = u' - v'/v^2$ and the DE becomes

$$\begin{aligned} u' - v'/v^2 &= p(x)(u + 1/v) + q(x)(u^2 + 2u/v + 1/v^2) + r(x) \\ &= p(x)u + q(x)u^2 + r(x) + (p(x) + 2uq(x))/v + q(x)/v^2 \end{aligned}$$

from which we get $v' + (p(x) + 2uq(x))v = -q(x)$, a linear equation.

Example 2. $y' = 1 + x^2 - y^2$ has the solution $y = x$ and the change of variable $y = x + 1/v$ transforms the equation into $v' + 2xv = 1$.

2 Orthogonal Trajectories.

An important application of first order DE's is to the computation of the orthogonal trajectories of a family of curves $f(x, y, C) = 0$. An orthogonal trajectory of this family is a curve that, at each point of intersection with a member of the given family, intersects that member orthogonally. To find the orthogonal trajectories, we may derive the ODE, whose solutions are described by these trajectories. For this purpose, we are going first to derive the ODE, whose solutions have the implicit form, $f(x, y, C) = 0$. In doing so, we differentiate $f(x, y, C) = 0$ implicitly with respect to x we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' = 0$$

from which we get

$$y' = -\frac{f_x(x, y, C)}{f_y(x, y, C)}.$$

Now we solve for $C = C(x, y)$ from the equation $f(x, y, C) = 0$, and substitute $C(x, y)$ in the above formula for y'. This gives the equation:

$$y' = g(x, y) = -\frac{f_x\Big[x, y, C(x, y)\Big]}{f_y\Big[x, y, C(x, y)\Big]}.$$

Note that $y'(x)$ yields the slope of the tangent line at the point (x, y) of a curve of the given family passing through (x, y). The slope of the orthogonal trajectory at the passing point (x, y) must be

$$y'(x) = -\frac{1}{g(x, y)}.$$

Therefore, the ODE governing the orthogonal trajectories is derived as

$$y' = \frac{f_y\Big[x, y, C(x, y)\Big]}{f_x\Big[x, y, C(x, y)\Big]}.$$

Example 3. Let us find the orthogonal trajectories of the family $x^2 + y^2 = Cx$, the family of circles with center on the x-axis and passing through the origin. Here

$$2x + 2yy' = C = \frac{x^2 + y^2}{x}$$

from which, we derive the ODE: $y' = g(x, y) = (y^2 - x^2)/2xy$. Then the ODE governing the orthogonal trajectories can be written as

$$y' = -\frac{1}{g(x, y)},$$

or,

$$y' = 2xy/(x^2 - y^2).$$

The above can be re-written in the form:

$$2xy + (y^2 - x^2)y' = 0.$$

If we let $M = 2xy$, $N = y^2 - x^2$ we have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{4x}{2xy} = \frac{2}{y}$$

so that we have an integrating factor μ which is a function of y. We have $\mu' = -2\mu/y$ from which $\mu = 1/y^2$. Multiplying the DE for the orthogonal trajectories by $1/y^2$ we get

$$\frac{2x}{y} + \left(1 - \frac{x^2}{y^2}\right) y' = 0.$$

Solving $\frac{\partial F}{\partial x} = 2x/y$, $\frac{\partial F}{\partial y} = 1 - x^2/y^2$ for F yields $F(x,y) = x^2/y + y$ from which the orthogonal trajectories are $x^2/y + y = C$, i.e., $x^2 + y^2 = Cy$. This is the family of circles with center on the y-axis and passing through the origin. Note that the line $y = 0$ is also an orthogonal trajectory that was not found by the above procedure. This is due to the fact that the integrating factor was $1/y^2$ which is not defined if $y = 0$ so we had to work in a region which does not cut the x-axis, e.g., $y > 0$ or $y < 0$.