McGill University
Math 325A: Differential Equations

LECTURE 4: FIRST ORDER DIFFERENTIAL EQUATIONS (III)

(Text: Sections 2.4,2.5)

1 Exact Equations.

By a region of the xy-plane we mean a connected open subset of the plane. The differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is said to be exact on a region (R) if there is a function $F(x,y)$ defined on (R) such that

$$\frac{\partial F}{\partial x} = M(x,y); \quad \frac{\partial F}{\partial y} = N(x,y)$$

In this case, if M, N are continuously differentiable on (R) we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{1}$$

Conversely, it can be shown that condition (1) is also sufficient for the exactness of the given DE on (R) providing that (R) is simply connected, .i.e., has no "holes".

The exact equations are solvable. In fact, suppose $y(x)$ is its solution. Then one can write:

$$M[x, y(x)] + N[x, y(x)]\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = \frac{d}{dx}F[x, y(x)] = 0.$$

It follows that

$$F[x, y(x)] = C,$$

where C is an arbitrary constant. This is an implicit form of the solution $y(x)$. Hence, the function $F(x,y)$, if it is found, will give a family of the solutions of the given DE. The curves $F(x,y) = C$ are called integral curves of the given DE.

Example 1. $2x^2y\frac{dy}{dx} + 2xy^2 + 1 = 0$. Here $M = 2xy^2 + 1$, $N = 2x^2y$ and $R = \mathbb{R}^2$, the whole xy-plane. The equation is exact on $\mathbb{R}^2$ since $\mathbb{R}^2$ is simply connected and

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x}.$$

To find F we have to solve the partial differential equations

$$\frac{\partial F}{\partial x} = 2xy^2 + 1, \quad \frac{\partial F}{\partial y} = 2x^2y.$$

If we integrate the first equation with respect to x holding y fixed, we get

$$F(x,y) = x^2y^2 + x + \phi(y).$$

Differentiating this equation with respect to y gives

$$\frac{\partial F}{\partial y} = 2x^2 y + \phi'(y) = 2x^2 y$$

using the second equation. Hence $\phi'(y) = 0$ and $\phi(y)$ is a constant function. The solutions of our DE in implicit form is $x^2y^2 + x = C$.

Example 2. We have already solved the homogeneous DE

$$\frac{dy}{dx} = \frac{x-y}{x+y}.$$

This equation can be written in the form

$$y - x + (x+y)\frac{dy}{dx} = 0$$

which is an exact equation. In this case, the solution in implicit form is $x(y-x) + y(x+y) = C$, i.e., $y^2 + 2xy - x^2 = C$.

2 Theorem.

If $F(x, y)$ is homogeneous of degree n then

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF(x, y).$$

Proof. The function F is homogeneous of degree n if $F(tx, ty) = t^n F(x, y)$. Differentiating this with respect to t and setting $t = 1$ yields the result. **QED**

3 Integrating Factors.

If the differential equation $M + Ny' = 0$ is not exact it can sometimes be made exact by multiplying it by a continuously differentiable function $\mu(x, y)$. Such a function is called an *integrating factor*. An integrating factor μ satisfies the PDE $\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}$ which can be written in the form

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}.$$

This equation can be simplified in special cases, two of which we treat next.

- $\underline{\mu \text{ is a function of } x \text{ only.}}$ This happens if and only if

  $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = p(x)$$

  is a function of x only in which case $\mu' = p(x)\mu$.

- $\underline{\mu \text{ is a function of } y \text{ only.}}$ This happens if and only if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = q(y)$$

is a function of y only in which case $\mu' = -q(y)\mu$.

Example 1. $2x^2 + y + (x^2y - x)y' = 0$. Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 - 2xy}{x^2y - x} = \frac{-2}{x}$$

so that there is an integrating factor μ which is a function of x only which satisfies $\mu' = -2\mu/x$. Hence $\mu = 1/x^2$ is an integrating factor and $2 + y/x^2 + (y - 1/x)y' = 0$ is an exact equation whose general solution is $2x - y/x + y^2/2 = C$ or $2x^2 - y + xy^2/2 = Cx$.

Example 2. $y + (2x - ye^y)y' = 0$. Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-1}{y}$$

so that there is an integrating factor which is a function of y only which satisfies $\mu' = 1/y$. Hence y is an integrating factor and $y^2 + (2xy - y^2e^y)y' = 0$ is an exact equation with general solution $xy^2 + (-y^2 + 2y - 2)e^y = C$.

A word of caution is in order here. The solutions of the exact DE obtained by multiplying by the integrating factor may have solutions which are not solutions of the original DE. This is due to the fact that μ may be zero and one will have to possibly exclude those solutions where μ vanishes. However, this is not the case for the above Example 2.