McGill University
Math 325A: Differential Equations

LECTURE 3: FIRST ORDER DIFFERENTIAL EQUATIONS (II)

(Text: Sections 1.1,2.2,2.6)

We now give some more examples of separable equations.

1 Logistic Equation

$$y' = ay(b-y),$$

where $a, b > 0$ are fixed constants. This equation arises in the study of the growth of certain populations. Since the right-hand side of the equation is zero for $y = 0$ and $y = b$, the given DE has $y = 0$ and $y = b$ as solutions. More generally, if $y' = f(t, y)$ and $f(t, c) = 0$ for all t in some interval (I), the constant function $y = c$ on (I) is a solution of $y' = f(t, y)$ since $y' = 0$ for a constant function y.

To solve the logistic equation, we write it in the form

$$\frac{y'}{y(b-y)} = a.$$

Integrating both sides with respect to t we get

$$\int \frac{y'\mathrm{d}t}{y(b-y)} = at + C$$

which can, since $y'\mathrm{d}t = \mathrm{d}y$, be written as

$$\int \frac{\mathrm{d}y}{y(b-y)} = at + C.$$

Since, by partial fractions,

$$\frac{1}{y(b-y)} = \frac{1}{b}\left(\frac{1}{y} + \frac{1}{b-y}\right)$$

we obtain

$$\frac{1}{b}(\ln|y| - \ln|b-y|) = at + C.$$

Multiplying both sides by b and exponentiating both sides to the base e, we get

$$\frac{|y|}{|b-y|} = \mathrm{e}^{bC}\mathrm{e}^{abt} = C_1\mathrm{e}^{abt},$$

where the arbitrary constant $C_1 = \pm e^{bC}$ can be determined by the initial condition (IC): $y(0) = y_0$ as

$$C_1 = \frac{|y_0|}{|b-y_0|}.$$

Two cases need to be discussed separately.

Case (I), $y_0 < b$: one has $C_1 = |\frac{y_0}{b-y_0}| = \frac{y_0}{b-y_0} > 0$. So that,

$$\frac{|y|}{|b-y|} = \left(\frac{y_0}{b-y_0}\right) \mathrm{e}^{abt} > 0, \quad (t \in (I)).$$

From the above we derive $y/(b-y) = C_1\mathrm{e}^{abt}$, and $y = (b-y)C_1\mathrm{e}^{abt}$. This gives

$$y = \frac{bC_1\mathrm{e}^{abt}}{1 + C_1\mathrm{e}^{abt}} = \frac{b\left(\frac{y_0}{b-y_0}\right)\mathrm{e}^{abt}}{1 + \left(\frac{y_0}{b-y_0}\right)\mathrm{e}^{abt}}.$$

It shows that if $y_0 = 0$, one has the solution $y(t) = 0$. However, if $0 < y_0 < b$, one has the solution $0 < y(t) < b$, and as $t \to \infty$, $y(t) \to b$.

Case (II), $y_0 > b$: one has $C_1 = |\frac{y_0}{b-y_0}| = -\frac{y_0}{b-y_0} > 0$. So that,

$$\left|\frac{y}{b-y}\right| = \left(\frac{y_0}{y_0-b}\right) \mathrm{e}^{abt} > 0, \quad (t \in (I)).$$

From the above we derive $y/(y-b) = \left(\frac{y_0}{y_0-b}\right)\mathrm{e}^{abt}$, and $y = (y-b)\left(\frac{y_0}{y_0-b}\right)\mathrm{e}^{abt}$. This gives

$$y = \frac{b\left(\frac{y_0}{y_0-b}\right)\mathrm{e}^{abt}}{\left(\frac{y_0}{y_0-b}\right)\mathrm{e}^{abt} - 1}.$$

It shows that if $y_0 > b$, one has the solution $y(t) > b$, and as $t \to \infty$, $y(t) \to b$.
It is derived that

- $y(t) = 0$ is an unstable equilibrium state of the system;
- $y(t) = b$ is a stable equilibrium state of the system.

2 Fundamental Existence and Uniqueness Theorem

If the function $f(x, y)$ together with its partial derivative with respect to y are continuous on the rectangle

$$R : |x - x_0| \leq a, \ |y - y_0| \leq b$$

there is a unique solution to the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

defined on the interval $|x - x_0| < h$ where

$$h = \min(a, b/M), \quad M = \max|f(x, y)|, \ (x, y) \in R.$$

Note that this theorem indicates that a solution may not be defined for all x in the interval $|x - x_0| \leq a$. For example, the function

$$y = \frac{bCe^{abx}}{1 + Ce^{abx}}$$

is solution to $y' = ay(b-y)$ but not defined when $1 + Ce^{abx} = 0$ even though $f(x,y) = ay(b-y$ satisfies the conditions of the theorem for all x, y.

The next example show why the condition on the partial derivative in the above theorem is necessary.

Consider the differential equation $y' = y^{1/3}$. Again $y = 0$ is a solution. Separating variables and integrating, we get

$$\int \frac{dy}{y^{1/3}} = x + C_1$$

which yields $y^{2/3} = 2x/3 + C$ and hence $y = \pm(2x/3 + C)^{3/2}$. Taking $C = 0$, we get the solution $y = (2x/3)^{3/2}, (x \geq 0)$ which along with the solution $y = 0$ satisfies $y(0) = 0$. So the initial value problem $y' = y^{1/3},\ y(0) = 0$ does not have a unique solution. The reason this is so is due to the fact that $\frac{\partial f}{\partial y}(x,y) = 1/3y^{2/3}$ is not continuous when $y = 0$.

Many differential equations become linear or separable after a change of variable. We now give two examples of this.

3 Bernoulli Equation:

$$y' = p(x)y + q(x)y^n \quad (n \neq 1).$$

Note that $y = 0$ is a solution. To solve this equation, we set $u = y^\alpha$, where α is to be determined. Then, we have $u' = \alpha y^{\alpha-1}y'$, hence, our differential equation becomes

$$u'/\alpha = p(x)u + q(x)y^{\alpha+n-1}. \tag{1}$$

Now set $\alpha = 1 - n$. Thus, (1) is reduced to

$$u'/\alpha = p(x)u + q(x), \tag{2}$$

which is linear. We know how to solve this for u from which we get solve $u = y^{1-n}$ to get y.

4 Homogeneous Equation:

$$y' = F(y/x).$$

To solve this we let $u = y/x$ so that $y = xu$ and $y' = u + xu'$. Substituting for y, y' in our DE gives $u + xu' = F(u)$ which is a separable equation. Solving this for u gives y via $y = xu$. Note that $u = a$ is a solution of $xu' = F(u) - u$ whenever $F(a) = a$ and that this gives $y = ax$ as a solution of $y' = f(y/x)$.

Example. $y' = (x-y)/x + y$. This is a homogeneous equation since

$$\frac{x-y}{x+y} = \frac{1-y/x}{1+y/x}.$$

Setting $u = y/x$, our DE becomes

$$xu' + u = \frac{1-u}{1+u}$$

so that

$$xu' = \frac{1-u}{1+u} - u = \frac{1-2u-u^2}{1+u}.$$

Note that the right-hand side is zero if $u = -1 \pm \sqrt{2}$. Separating variables and integrating with respect to x, we get

$$\int \frac{(1+u)du}{1-2u-u^2} = \ln|x| + C_1$$

which in turn gives

$$(-1/2)\ln|1-2u-u^2| = \ln|x| + C_1.$$

Exponentiating, we get

$$\frac{1}{\sqrt{|1-2u-u^2|}} = e^{C_1}|x|.$$

Squaring both sides and taking reciprocals, we get

$$u^2 + 2u - 1 = C/x^2$$

with $C = \pm 1/e^{2C_1}$. This equation can be solved for u using the quadratic formula. If x_0, y_0 are given with $x_0 \neq 0$ and $u_0 = y_0/x_0 \neq -1$ there is, by the fundamental, existence and uniqueness theorem,a unique solution with $u(x_0) = y_0$. For example, if $x_0 = 1, y_0 = 2$, we have $C = 7$ and hence

$$u^2 + 2u - 1 = 7/x^2$$

Solving for u, we get

$$u = -1 + \sqrt{2 + 7/x^2}$$

where the positive sign in the quadratic formula was chosen to make $u = 2, x = 1$ a solution. Hence

$$y = -x + x\sqrt{2 + 7/x^2} = -x + \sqrt{2x^2 + 7}$$

is the solution to the initial value problem

$$y' = \frac{x-y}{x+y}, \quad y(1) = 2$$

for $x > 0$ and one can easily check that it is a solution for all x. Moreover, using the fundamental uniqueness, it can be shown that it is the only solution defined for all x.