McGill University
Math 325A: Differential Equations

LECTURE 2: FIRST ORDER DIFFERENTIAL EQUATIONS (I)

(Text: Sections 2.1, 2.2)

In this lecture we will treat linear and separable first order ODE's.

1 Linear Equation

The general first order ODE has the form $F(x, y, y') = 0$ where $y = y(x)$. If it is linear it can be written in the form

$$a_0(x)y' + a_1(x)y = b(x)$$

where $a_0(x),\ a_(x),\ b(x)$ are continuous functions of x on some interval (I). To bring it to normal form $y' = f(x, y)$ we have to divide both sides of the equation by $a_0(x)$. This is possible only for those x where $a_0(x) \neq 0$. After possibly shrinking I we assume that $a_0(x) \neq 0$ on (I). So our equation has the form (standard form)

$$y' + p(x)y = q(x)$$

with $p(x) = a_1(x)/a_0(x)$ and $q(x) = b(x)/a_0(x)$, both continuous on (I). Solving for y' we get the normal form for a linear first order ODE, namely

$$y' = q(x) - p(x)y.$$

1.1 Linear homogeneous equation

Let us first consider the simple case: $q(x) = 0$, namely,

$$\frac{\mathrm{d}y}{\mathrm{dx}} + p(x)y = 0.$$

With the chain law of derivative, one may write

$$\frac{y'(x)}{y} = \frac{\mathrm{d}}{\mathrm{d}x} \ln \left[y(x)\right] = -p(x),$$

integrating both sides, we derive

$$\ln y(x) = -\int p(x)\mathrm{d}x + C,$$

or

$$y = C_1 \mathrm{e}^{-\int p(x)\mathrm{d}x},$$

where C, as well as $C_1 = \mathrm{e}^C$, is arbitrary constant.

1.2 Linear inhomogeneous equation

We now consider the general case:

$$\frac{\mathrm{d}y}{\mathrm{dx}} + p(x)y = q(x).$$

We multiply the both sides of our differential equation with a factor $\mu(x) \neq 0$. Then our equation is equivalent (has the same solutions) to the equation

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x).$$

We wish that with a properly chosen function $\mu(x)$,

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \frac{\mathrm{d}}{\mathrm{d}x}\big[\mu(x)y(x)\big].$$

For this purpose, the function $\mu(x)$ must has the property

$$\mu'(x) = p(x)\mu(x), \tag{1}$$

and $\mu(x) \neq 0$ for all x. By solving the linear homogeneous equation (1), one obtain

$$\mu(x) = e^{\int p(x)\mathrm{d}x}. \tag{2}$$

With this function, which is called an integrating factor, our equation is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[\mu(x)y(x)\big] = \mu(x)q(x), \tag{3}$$

Integrating both sides, we get

$$\mu(x)y = \int \mu(x)q(x)\mathrm{d}x + C$$

with C an arbitrary constant. Solving for y, we get

$$y = \frac{1}{\mu(x)}\int \mu(x)q(x)\mathrm{d}x + \frac{C}{\mu(x)} = y_P(x) + y_H(x) \tag{4}$$

as the general solution for the general linear first order ODE

$$y' + p(x)y = q(x).$$

In solution (4), the first part, $y_P(x)$, is a particular solution of the inhomogeneous equation, while the second part, $y_H(x)$, is the general solution of the associate homogeneous solution. Note that for any pair of scalars a, b with a in (I), there is a unique scalar C such that $y(a) = b$. Geometrically, this means that the solution curves $y = \phi(x)$ are a family of non-intersecting curves which fill the region $I \times \mathbb{R}$.

Example 1: $y' + xy = x$. This is a linear first order ODE in standard form with $p(x) = q(x) = x$. The integrating factor is

$$\mu(x) = e^{\int x\mathrm{d}x} = e^{x^2/2}.$$

Hence, after multiplying both sides of our differential equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{x^2/2}y) = xe^{x^2/2}$$

which, after integrating both sides, yields

$$e^{x^2/2}y = \int xe^{x^2/2}\mathrm{d}x + C = e^{x^2/2} + C.$$

Hence the general solution is $y = 1 + Ce^{-x^2/2}$. The solution satisfying the initial condition $y(0) = 1$ is $y = 1$ and the solution satisfying $y(0) = a$ is $y = 1 + (a-1)e^{-x^2/2}$.

Example 2: $xy' - 2y = x^3 \sin x$,
$(x > 0)$. We bring this linear first order equation to standard form by dividing by x. We get

$$y' + \frac{-2}{x}y = x^2 \sin x.$$

The integrating factor is

$$\mu(x) = e^{\int -2\mathrm{d}x/x} = e^{-2\ln x} = 1/x^2.$$

After multiplying our DE in standard form by $1/x^2$ and simplifying, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(y/x^2) = \sin x$$

from which $y/x^2 = -\cos x + C$ and $y = -x^2 \cos x + Cx^2$. Note that the later are solutions to the DE $xy' - 2y = x^3 \sin x$ and that they all satisfy the initial condition $y(0) = 0$. This non-uniqueness is due to the fact that $x = 0$ is a singular point of the DE.

2 Separable Equations.

The first order ODE $y' = f(x, y)$ is said to be separable if $f(x, y)$ can be expressed as a product of a function of x times a function of y. The DE then has the form $y' = g(x)h(y)$ and, dividing both sides by $h(y)$, it becomes

$$\frac{y'}{h(y)} = g(x).$$

Of course this is not valid for those solutions $y = y(x)$ at the points where $\phi(x) = 0$. Assuming the continuity of g and h, we can integrate both sides of the equation to get

$$\int \frac{y'(x)}{h\big[y(x)\big]}\mathrm{d}x = \int g(x)\mathrm{d}x + C.$$

Assume that

$$H(y) = \int \frac{\mathrm{d}y}{h(y)},$$

By chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}H\big[y(x)\big] = H'(y)y'(x) = \frac{1}{h\big[y(x)\big]}y'(x),$$

hence

$$H\big[y(x)\big] = \int \frac{y'(x)}{h\big[y(x)\big]}\mathrm{d}x = \int g(x)\mathrm{d}x + C.$$

Therefore,

$$\int \frac{\mathrm{d}y}{h(y)} = H(y) = \int g(x)\mathrm{d}x + C,$$

gives the implicit form of the solution. It determines the value of y implicitly in terms of x.

Example 1: $y' = \frac{x-5}{y^2}$.

To solve it using the above method we multiply both sides of the equation by y^2 to get

$$y^2 y' = (x-5).$$

Integrating both sides we get $y^3/3 = x^2/2 - 5x + C$. Hence,

$$y = \left[3x^2/2 - 15x + C_1\right]^{1/3}.$$

Example 2: $y' = \frac{y-1}{x+3}$ $(x > -3)$. By inspection, $y = 1$ is a solution. Dividing both sides of the given DE by $y - 1$ we get

$$\frac{y'}{y-1} = \frac{1}{x+3}.$$

This will be possible for those x where $y(x) \neq 1$. Integrating both sides we get

$$\int \frac{y'}{y-1}\mathrm{d}x = \int \frac{\mathrm{d}x}{x+3} + C_1,$$

from which we get $\ln|y-1| = \ln(x+3) + C_1$. Thus $|y-1| = e^{C_1}(x+3)$ from which $y-1 = \pm e^{C_1}(x+3)$. If we let $C = \pm e^{C_1}$, we get

$$y = 1 + C(x+3)$$

which is a family of lines passing through $(-3, 1)$; for any (a, b) with $b \neq 0$ there is only one member of this family which passes through (a, b). Since $y = 1$ was found to be a solution by inspection the general solution is

$$y = 1 + C(x+3),$$

where C can be any scalar.

Example 3: $y' = \frac{y \cos x}{1+2y^2}$. Transforming in the standard form then integrating both sides we get

$$\int \frac{(1+2y^2)}{y}\mathrm{d}y = \int \cos x \, \mathrm{d}x + C,$$

from which we get a family of the solutions:

$$\ln|y| + y^2 = \sin x + C,$$

where C is an arbitrary constant. However, this is not the general solution of the equation, as it does not contains, for instance, the solution: $y = 0$. With I.C.: y(0)=1, we get $C = 1$, hence, the solution:

$$\ln|y| + y^2 = \sin x + 1.$$