McGill University
Math 325A: Differential Equations

LECTURE 17: SERIES SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS (II)

(Text: Chap. 8)

1 Introduction

In this lecture we investigate series solutions for the general linear DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x),$$

where the functions $a_1, a_2, \ldots, a_n, b$ are analytic at $x = x_0$. If $a_0(x_0) \neq 0$ the point $x = x_0$ is called an **ordinary point** of the DE. In this case, the solutions are analytic at $x = x_0$ since the normalized DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = q(x),$$

where $p_i(x) = a_i(x)/a_0(x), q(x) = b(x)/a_0(x)$, has coefficient functions which are analytic at $x = x_0$. If $a_0(x_0) = 0$, the point $x = x_0$ is said to be a **singular point** for the DE. If k is the multiplicity of the zero of $a_0(x)$ at $x = x_0$ and the multiplicities of the other coefficient functions at $x = x_0$ is as big then, on cancelling the common factor $(x - x_0)^k$ for $x \neq x_0$, the DE obtained holds even for $x = x_0$ by continuity, has analytic coefficient functions at $x = x_0$ and $x = x_0$ is an ordinary point. In this case the singularity is said to be **removable**. For example, the DE $xy'' + \sin(x)y' + xy = 0$ has a removable singularity at $x = 0$.

2 Series Solutions near a Regular Singular Point

In general, the solution of a linear DE in a neighborhood of a singularity is extremely difficult. However, there is an important special case where this can be done. For simplicity, we treat the case of the general second order homogeneous DE

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad (x > x_0),$$

with a singular point at $x = x_0$. Without loss of generality we can, after possibly a change of variable $x - x_0 = t$, assume that $x_0 = 0$. We say that $x = 0$ is a **regular singular point** if the normalized DE

$$y'' + p(x)y' + q(x)y = 0, \quad (x > 0),$$

is such that $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$. A necessary and sufficient condition for this is that

$$\lim_{x\to 0} xp(x) = p_0, \quad \lim_{x\to 0} x^2q(x) = q_0$$

exist and are finite. In this case

$$xp(x) = p_0 + p_1x + \cdots + p_nx^n + \cdots, \quad x^2q(x) = q_0 + q_1x + \cdots + q_nx^n + \cdots$$

and the given DE has the same solutions as the DE

$$x^2y'' + x(xp(x))y' + x^2q(x)y = 0.$$

This DE is an Euler DE if $xp(x) = p_0$, $x^2q(x) = q_0$. This suggests that we should look for solutions of the form

$$y = x^r\left(\sum_{n=0}^{\infty} a_nx^n\right) = \sum_{n=0}^{\infty} a_nx^{n+r},$$

with $a_0 \neq 0$. Substituting this in the DE gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} + \left(\sum_{n=0}^{\infty} p_nx^n\right)\left(\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}\right) + \left(\sum_{n=0}^{\infty} q_nx^n\right)\left(\sum_{n=0}^{\infty} a_nx^{n+r}\right) = 0$$

which, on expansion and simplification, becomes

$$a_0F(r)x^r + \sum_{n=1}^{\infty}\Big\{F(n+r)a_n + \big[(n+r-1)p_1 + q_1\big]a_{n-1} + \cdots + (rp_n + q_n)a_0\Big\}x^{n+r} = 0,$$

where $F(r) = r(r-1) + p_0r + q_0$. Equating coefficients to zero, we get

$$r(r-1) + p_0r + q_0 = 0, \tag{1}$$

the **indicial equation**, and

$$F(n+r)a_n = -\big((n+r-1)p_1 + q_1\big)a_{n-1} - \cdots - (rp_n + q_n)a_0 \tag{2}$$

for $n \geq 1$. The indicial equation (1) has two roots: r_1, r_2. Three cases should be discussed separately.

2.1 Case (I): The roots $(r_1 - r_2 \neq N)$

Two roots don't differ by an integer. In this case, the above recursive equation (2) determines a_n uniquely for $r = r_1$ and $r = r_2$. If $a_n(r_i)$ is the solution for $r = r_i$ and $a_0 = 1$, we obtain the linearly independent solutions

$$y_1 = x^{r_1}\left(\sum_{n=0}^{\infty} a_n(r_1)x^n\right), \quad y_2 = x^{r_2}\left(\sum_{n=0}^{\infty} a_n(r_2)x^n\right).$$

It can be shown that the radius of convergence of the infinite series is the distance to the singularity of the DE nearest to the singularity $x = 0$. If $r_1 - r_2 = N \geq 0$, the above recursion equations can be solved for $r = r_1$ as above to give a solution

$$y_1 = x^{r_1}\left(\sum_{n=0}^{\infty} a_n(r_1)x^n\right).$$

A second linearly independent solution can then be found by reduction of order.

However, the series calculations can be quite involved and a simpler method exists which is based on solving the recursion equation for $a_n(r)$ as a ratio of polynomials of r. This can always be done since $F(n+r)$ is not the zero polynomial for any $n \geq 0$. If $a_n(r)$ is the solution with $a_0(r) = 1$ and we let

$$y = y(x,r) = x^r \left(\sum_{n=0}^{\infty} a_n(r) x^n \right). \tag{3}$$

Thus, we have the following equality with two variables (x, r):

$$x^2 y'' + x^2 p(x) y' + x^2 q(x) y = a_0 F(r) x^r = (r - r_1)(r - r_2) x^r. \tag{4}$$

2.2 Case (II): The roots $(r_1 = r_2)$

In this case, from the equality (4) we get

$$x^2 y'' + x^2 p(x) y' + x^2 q(x) y = (r - r_1)^2 x^r.$$

Differentiating this equation with respect to r, we get

$$x^2 \left(\frac{\partial y}{\partial r} \right)'' + x^2 p(x) \left(\frac{\partial y}{\partial r} \right)' + x^2 q(x) \frac{\partial y}{\partial r} = 2(r - r_1) + (r - r_1)^2 x^r \ln(x).$$

Setting $r = r_1$, we find that

$$y_2 = \frac{\partial y}{\partial r}(x, r_1) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n(r_1) x^n \right) \ln(x) + x^{r_1} \sum_{n=0}^{\infty} a_n'(r_1) x^n = y_1 \ln(x) + x^{r_1} \sum_{n=0}^{\infty} a_n'(r_1) x^n,$$

where $a_n'(r)$ is the derivative of $a_n(r)$ with respect to r. This is a second linearly independent solution. Since this solution is unbounded as $x \to 0$, any solution of the given DE which is bounded as $x \to 0$ must be a scalar multiple of y_1.

Case (III): The roots $(r_1 - r_2 = N > 0)$

For this case, we let $z(x,r) = (r - r_2) y(x,r)$. Thus, from the equality (4) we get

$$x^2 z'' + x^2 p(x) z' + x^2 q(x) z = (r - r_1)(r - r_2)^2 x^r.$$

Differentiating this equation with respect to r, we get

$$x^2 \left(\frac{\partial z}{\partial r} \right)'' + x^2 p(x) \left(\frac{\partial z}{\partial r} \right)' + x^2 q(x) \frac{\partial z}{\partial r} = (r - r_2)\big[(r - r_2) + 2(r - r_1)\big] x^r + (r - r_1)(r - r_2)^2 x^r \ln(x).$$

Setting $r = r_2$, we see that $y_2 = \frac{\partial z}{\partial r}(x, r_2)$ is a solution of the given DE. Letting $b_n(r) = (r - r_2) a_n(r)$, we have

$$F(n+r) b_n(r) = -\big[(n + r - 1) p_1 + q_1\big] b_{n-1}(r) - \cdots - (r p_n + q_n) b_0(r) \tag{5}$$

and

$$y_2 = \lim_{r \to r_2} \left(x^r \ln(x) \sum_{n=0}^{\infty} b_n(r) x^n + x^r \sum_{n=0}^{\infty} b_n'(r) x^n \right). \tag{6}$$

Note that $a_n(r_2) \neq 0$, for $n = 1, 2, \dots N-1$. Hence, we have

$$b_0(r_2) = b_1(r_2) = b_2(r_2) = \cdots = b_{N-1}(r_2) = 0.$$

However, $a_N(r_2) = \infty$, as $F(r_2 + N) = F(r_1) = 0$. Hence, we have

$$b_N(r_2) = \lim_{r \to r_2} (r - r_2) a_n(r) = a < \infty,$$

subsequently,

$$\lim_{r \to r_2} x^r \ln(x) b_N(r) x^N = a x^{r_1} \ln(x).$$

Furthermore,

$$\begin{aligned} F(N+1+r_2) b_{N+1}(r_2) &= F(1+r_1) b_{N+1}(r_2) \\ &= -\big(r_1 p_1 + q_1\big) b_N(r_2) - \cdots - (r_2 p_{N+1} + q_{N+1}) b_0(r_2) \\ &= -\big(r_1 p_1 + q_1\big) b_N(r_2) \end{aligned} \tag{7}$$

Thus,

$$b_{N+1}(r_2) = \frac{\big(r_1 p_1 + q_1\big)}{F(1+r_1)} b_N(r_2) = a \frac{\big(r_1 p_1 + q_1\big)}{F(1+r_1)} = a a_1(r_1). \tag{8}$$

Similarly, we have

$$\begin{aligned} F(N+2+r_2) b_{N+2}(r_2) &= F(2+r_1) b_{N+2}(r_2) \\ &= -\big[(1+r_1) p_1 + q_1\big] b_{N+1}(r_2) - (r_1 p_2 + q_2) b_N(r_2) \\ &\quad - \cdots - (r_2 p_{N+2} + q_{N+2}) b_0(r_2) \\ &= -a \big[(1+r_1) p_1 + q_1\big] a_1(r_1) - a\big(r_1 p_2 + q_2\big), \end{aligned} \tag{9}$$

then we obtain

$$b_{N+2}(r_2) = -a \frac{\big[(1+r_1) p_1 + q_1\big] a_1(r_1) + \big(r_1 p_2 + q_2\big)}{F(2+r_1)} = a a_2(r_1). \tag{10}$$

In general, we can write

$$b_{N+k}(r_2) = a a_k(r_1). \tag{11}$$

Substituting the above results to (6), we finally derive

$$\begin{aligned} y_2 &= a x^{r_1} \left(\sum_{n=0}^{\infty} a_n(r_1) x^n \right) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n'(r_2) x^n \right) \\ &= a y_1 \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n'(r_2) x^n \right). \end{aligned} \tag{12}$$

This gives a second linearly independent solution.

The above method is due to Frobenius and is called the **Frobenius method**.

Example 1. The DE $2xy'' + y' + 2xy = 0$ has a regular singular point at $x = 0$ since $xp(x) = 1/2$ and $x^2q(x) = x^2$. The indicial equation is

$$r(r-1) + \frac{1}{2}r = r(r - \frac{1}{2}).$$

The roots are $r_1 = 1/2$, $r_2 = 0$ which do not differ by an integer. We have

$$(r+1)(r+\frac{1}{2})a_1 = 0,$$
$$(n+r)(n+r-\frac{1}{2})a_n = -a_{n-2} \quad \text{for } n \geq 2,$$

so that $a_n = -2a_{n-2}/(r+n)(2r+2n-1)$ for $n \geq 2$. Hence $0 = a_1 = a_3 = \cdots a_{2n+1}$ for $n \geq 0$ and

$$a_2 = -\frac{2}{(r+2)(2r+3)}a_0, \quad a_4 = -\frac{2}{(r+4)(2r+7)}a_2 = \frac{2^2}{(r+2)(r+4)(2r+3)(2r+7)}a_0.$$

It follows by induction that

$$a_{2n} = (-1)^{2n}\frac{2^n}{(r+2)(r+4)\cdots(r+2n)(2r+3)(2r+4)\cdots(2r+2n-1)}a_0.$$

Setting, $r = 1/2,\ 0$, $a_0 = 1$, we get

$$y_1 = \sqrt{x}\sum_{n=0}^{\infty}\frac{x^{2n}}{(5\cdot 9\cdots(4n+1))n!}, \quad y_2 = \sum_{n=0}^{\infty}\frac{x^{2n}}{(3\cdot 7\cdot\cdots(4n-1))n!}.$$

The infinite series have an infinite radius of convergence since $x = 0$ is the only singular point of the DE.

Example 2. The DE $xy'' + y' + y = 0$ has a regular singular point at $x = 0$ with $xp(x) = 1$, $x^2q(x) = x$. The indicial equation is

$$r(r-1) + r = r^2 = 0.$$

This equation has only one root $x = 0$. The recursion equation is

$$(n+r)^2 a_n = -a_{n-1}, \quad n \geq 1.$$

The solution with $a_0 = 1$ is

$$a_n(r) = (-1)^n\frac{1}{(r+1)^2(r+2)^2\cdots(r+n)^2}.$$

setting $r = 0$ gives the solution

$$y_1 = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(n!)^2}.$$

Taking the derivative of $a_n(r)$ with respect to r we get, using $a_n'(r) = a_n(r)\frac{d}{dr}\ln\left[a_n(r)\right]$ (logarithmic differentiation),we get

$$a_n'(r) = \left(\frac{2}{r+1} + \frac{2}{r+2} + \cdots + \frac{2}{r+n}\right)a_n(r)$$

so that

$$a_n'(0) = 2(-1)^n \frac{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}}{(n!)^2}.$$

Therefore a second linearly independent solution is

$$y_2 = y_1 \ln(x) + 2 \sum_{n=1}^{\infty} (-1)^n \frac{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}}{(n!)^2} x^n.$$

The above series converge for all x. Any bounded solution of the given DE must be a scalar multiple of y_1.