McGill University
Math 325A: Differential Equations

LECTURE 12: SOLUTIONS FOR EQUATIONS WITH CONSTANTS COEFFICIENTS (II)

HIGHER ORDER DIFFERENTIAL EQUATIONS (IV)

(Text: pp. 338-367, Chap. 6)

1 Introduction

We adopt the differential operator D and write the linear equation in the following form:

$$L(y) = (a_0 D^{(n)} + a_1 D^{(n-1)} + \cdots + a_n)y = P(D)y = b(x).$$

2 The Method with Differential Operator

2.1 Basic Equalities (II).

We may prove the following basic identity of differential operators: for any scalar a,

$$\begin{aligned} (D-a) \ &= e^{ax} D e^{-ax} \\ (D-a)^n &= e^{ax} D^n e^{-ax} \end{aligned} \tag{1}$$

where the factors e^{ax}, e^{-ax} are interpreted as linear operators. This identity is just the fact that

$$\frac{dy}{dx} - ay = e^{ax}\left(\frac{d}{dx}(e^{-ax}y)\right).$$

The formula (1) may be extensively used in solving the type of linear equations under discussion. Let write the equation (**??**) with the differential operator in the following form:

$$L(y) = (aD^2 + bD + c)y = \phi(D)y = 0, \tag{2}$$

where

$$\phi(D) = (aD^2 + bD + c)$$

is a polynomial of D. We now re-consider the cases above-discussed with the previous method.

2.2 Cases (I) ($b^2 - 4ac > 0$)

The polynomial $\phi(r)$ have two distinct real roots $r_1 > r_2$. Then, we can factorize the polynomial $\phi(D) = (D - r_1)(D - r_2)$ and re-write the equation as:

$$L(y) = (D - r_1)(D - r_2)y = 0.$$

letting

$$z = (D - r_2)y,$$

in terms the basic equalities, we derive

$$(D - r_1)z = \mathrm{e}^{r_1} D\mathrm{e}^{-r_1} z = 0,$$

$$\mathrm{e}^{r_2} z = A, \quad z = A\mathrm{e}^{r_1}.$$

Furthermore, from

$$(D - r_2)y = \mathrm{e}^{r_2} D\mathrm{e}^{-r_2} y = z = A\mathrm{e}^{r_1},$$

we derive

$$D\big(\mathrm{e}^{-r_2} y\big) = z = A\mathrm{e}^{r_1 - r_2}$$

and

$$y = \tilde{A}\mathrm{e}^{r_1} + B\mathrm{e}^{r_2},$$

where $\tilde{A} = \frac{A}{(r_1 - r_2)}$, B are arbitrary constants. It is seen that, in general, to solve the equation

$$L(y) = (D - r_1)(D - r_2) \cdots (D - r_n)y = 0,$$

where $r_i \neq r_j, (i \neq j)$, one can first solve each factor equations

$$(D - r_i)y_i = 0, \qquad (i = 1, 2, \cdots, n)$$

separately. The general solution can be written in the form:

$$y(x) = y_1(x) + y_2(x) + \cdots + y_n(x).$$

2.3 Cases (II) ($b^2 - 4ac = 0$)

. The polynomial $\phi(r)$ have double real roots $r_1 = r_2$. Then, we can factorize the polynomial $\phi(D) = (D - r_1)^2$ and re-write the equation as:

$$L(y) = (D - r_1)^2 y = 0.$$

In terms the basic equalities, we derive

$$(D - r_1)^2 y = \mathrm{e}^{r_1 x} D^2 \mathrm{e}^{-r_1 x} y = 0,$$

we derive

$$D\big(\mathrm{e}^{-r_2 x} y\big) = z = A\mathrm{e}^{(r_1 - r_2)x}$$

hence,

$$D^2\big(\mathrm{e}^{-r_1 x} y\big) = 0.$$

One can solve

$$\big(\mathrm{e}^{-r_1 x} y\big) = A + Bx,$$

or

$$y = (A + Bx)\mathrm{e}^{r_1 x}.$$

In general, for the equation,

$$L(y) = (D - r_1)^n y = 0.$$

we have the general solution:

$$y = \big(A_1 + A_2 x + \cdots + A_n x^{n-1}\big)\mathrm{e}^{r_1 x}.$$

So, we may write

$$\ker((D - a)^n) = \{(a_0 + a_x + \cdots + a_{n-1} x^{n-1})e^{ax} \mid a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}\}.$$

2.4 Cases (III) ($b^2 - 4ac < 0$)

The polynomial $\phi(r)$ have two complex conjugate roots $r_{1,2} = \lambda \pm \mathrm{i}\mu$. Then, we can factorize the polynomial $\phi(D) = (D-\lambda)^2 + \mu^2$, and re-write the equation as:

$$L(y) = \big((D-\lambda)^2 + \mu^2\big)y = 0. \tag{3}$$

Let us consider the special case first:

$$L(z) = \big(D^2 + \mu^2\big)z = 0.$$

From the formulas:

$$D(\cos\mu x) = -\mu \sin x, \qquad D(\sin x) = \mu \cos x,$$

it follows that

$$z(x) = A\cos\mu x + B\sin\mu x.$$

To solve for $y(x)$, we re-write the equation (3) as

$$\big(\mathrm{e}^{\lambda x} D^2 \mathrm{e}^{-\lambda x} + \mu^2\big)y = 0.$$

Then

$$D^2(\mathrm{e}^{-\lambda x}y) + \mu^2 \mathrm{e}^{-\lambda x} y = \big(D^2 + \mu^2\big)\mathrm{e}^{-\lambda x} y = 0.$$

Thus, we derive

$$\mathrm{e}^{-\lambda x} y(x) = A\cos\mu x + B\sin\mu x,$$

or

$$y(x) = \mathrm{e}^{\lambda x}\big(A\cos\mu x + B\sin\mu x\big). \tag{4}$$

One may also consider case (I) with the complex number r_1, r_2 and obtain the complex solution:

$$y(x) = \mathrm{e}^{\lambda x}\big(A\mathrm{e}^{\mathrm{i}\mu x} + B\mathrm{e}^{-\mathrm{i}\mu x}\big). \tag{5}$$

2.5 Theorems

In summary, it can be proved that the following results hold:

2.5.1 $\ker\big((D-a)^m\big) = \mathrm{span}(e^{ax}, xe^{ax}, \ldots, x^{m-1}e^{ax})$

It means that $\big((D-a)^m\big)y = 0$ has a set of fundamental solutions:

$$\Big\{e^{ax}, xe^{ax}, \ldots, x^{m-1}e^{ax}\Big\}$$

2.5.2 $\ker\big((D-a)^2 + b^2)^m\big) = \mathrm{span}\big(e^{ax}f(x), xe^{ax}f(x), \ldots, x^{m-1}e^{ax}f(x)\big)$, $f(x) = \cos(bx)$ or $\sin(bx)$

It means that $\big((D-a)^2 + b^2)^m\big)y = 0$ has a set of fundamental solutions:

$$\Big\{e^{ax}f(x), xe^{ax}f(x), \ldots, x^{m-1}e^{ax}f(x)\Big\},$$

where $f(x) = \cos(bx)$ or $\sin(bx)$.

2.5.3 $\ker(P(D)Q(D)) = \ker(P(D)) + \ker(Q(D)) = \{y_1 + y_2 \mid y_1 \in \ker(P(D)),\ y_2 \in \ker(Q(D))\}$

If $P(X), Q(X)$ are two polynomials with constant coefficients that have no common roots.

Example 1. By using the differential operation method, one can easily solve some inhomogeneous equations. For instance, let us reconsider the example 1. One may write the DE $y'' + 2y' + y = x$ in the operator form as

$$(D^2 + 2D + I)(y) = x.$$

The operator $(D^2 + 2D + I) = \phi(D)$ can be factored as $(D + I)^2$. With (1), we derive that

$$(D + I)^2 = \left(e^{-x} D e^x\right)\left(e^{-x} D e^x\right) = e^{-x} D^2 e^x.$$

Consequently, the DE $(D + I)^2(y) = x$ can be written $e^{-x} D^2 e^x(y) = x$ or

$$\frac{d^2}{dx}(e^x y) = x e^{-x}$$

which on integrating twice gives

$$e^x y = x e^{-x} - 2e^{-x} + Ax + B, \quad y = x - 2 + Axe^{-x} + Be^{-x}.$$

We leave it to the reader to prove that

$$\ker((D - a)^n) = \{(a_0 + a_x + \cdots + a_{n-1} x^{n-1}) e^{ax} \mid a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}\}.$$

Example 2. Now consider the DE $y'' - 3y' + 2y = e^x$. In operator form this equation is

$$(D^2 - 3D + 2I)(y) = e^x.$$

Since $(D^2 - 3D + 2I) = (D - I)(D - 2I)$, this DE can be written

$$(D - I)(D - 2I)(y) = e^x.$$

Now let $z = (D - 2I)(y)$. Then $(D - I)(z) = e^x$, a first order linear DE which has the solution $z = xe^x + Ae^x$. Now $z = (D - 2I)(y)$ is the linear first order DE

$$y' - 2y = xe^x + Ae^x$$

which has the solution $y = e^x - xe^x - Ae^x + Be^{2x}$. Notice that $-Ae^x + Be^{2x}$ is the general solution of the associated homogeneous DE $y'' - 3y' + 2y = 0$ and that $e^x - xe^x$ is a particular solution of the original DE $y'' - 3y' + 2y = e^x$.

Example 3. Consider the DE

$$y'' + 2y' + 5y = \sin(x)$$

which in operator form is $(D^2 + 2D + 5I)(y) = \sin(x)$. Now

$$D^2 + 2D + 5I = (D + I)^2 + 4I$$

and so the associated homogeneous DE has the general solution

$$Ae^{-x}\cos(2x) + Be^{-x}\sin(2x).$$

All that remains is to find a particular solution of the original DE. We leave it to the reader to show that there is a particular solution of the form $C\cos(x) + D\sin(x)$.

Example 4. Solve the initial value problem

$$y''' - 3y'' + 7y' - 5y = 0, \quad y(0) = 1, y'(0) = y''(0) = 0.$$

The DE in operator form is $(D^3 - 3D^2 + 7D - 3)(y) = 0$. Since

$$\phi(r) = r^3 - 3r^2 + 7r - 5 = (r-1)(r^2 - 2r + 5) = (r-1)\big[(r-1)^2 + 4\big],$$

we have

$$\begin{aligned} L(y) &= (D^3 - 3D^2 + 7D - 3)(y) \\ &= (D-1)\big[(D-1)^2 + 4\big](y) \\ &= \big[(D-1)^2 + 4\big](D-1)(y) \\ &= 0. \end{aligned} \tag{6}$$

From here, it is seen that the solutions for

$$(D-1)(y) = 0, \tag{7}$$

namely,

$$y(x) = c_1 e^x, \tag{8}$$

and the solutions for

$$\big[(D-1)^2 + 4\big](y) = 0, \tag{9}$$

namely,

$$y(x) = c_2 e^x \cos(2x) + c_3 e^x \sin(2x), \tag{10}$$

must be the solutions for our equation (6). Thus, we derive that the following linear combination

$$y = c_1 e^x + c_2 e^x \cos(2x) + c_3 e^x \sin(2x), \tag{11}$$

must be the solutions for our equation (6). In solution (11), there are three arbitrary constants (c_1, c_2, c_3). One can prove that this solution is the general solution, which covers all possible solutions of (6). For instance, given the I.C.'s: $y(0) = 1, y'(0) = 0, y''(0) = 0$, from (11), we can derive

$$c_1 + c_2 = 1,$$

$$c_1 + c_2 + 2c_3 = 0,$$

$$c_1 - 3c_2 + 4c_3 = 0,$$

and find $c_1 = 5/4, c_2 = -1/4, c_3 = -1/2$.