McGill University
Math 325A: Differential Equations

LECTURE 11: SOLUTIONS FOR EQUATIONS WITH CONSTANTS COEFFICIENTS (I)

HIGHER ORDER DIFFERENTIAL EQUATIONS (III)

(Text: pp. 338-367, Chap. 6)

1 Introduction

In what follows, we shall first focus on the linear equations with constant coefficients:

$$L(y) = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x)$$

and present two different approaches to solve them.

2 The Method with Undetermined Parameters

To illustrate the idea, as a special case, let us first consider the 2-nd order Linear equation with the constant coefficients:

$$L(y) = ay'' + by' + cy = f(x). \tag{1}$$

The associate homogeneous equation is:

$$L(y) = ay'' + by' + cy = 0. \tag{2}$$

2.1 Basic Equalities (I)

We first give the following basic identities:

$$D(\mathrm{e}^{rx}) = r\mathrm{e}^{rx};\ D^2(\mathrm{e}^{rx}) = r^2\mathrm{e}^{rx};\ \cdots\ D^n(\mathrm{e}^{rx}) = r^n\mathrm{e}^{rx}. \tag{3}$$

To solve this equation, we assume that the solution is in the form $y(x) = \mathrm{e}^{rx}$, where r is a constant to be determined. Due to the properties of the exponential function e^{rx}:

$$y'(x) = ry(x);\ y''(x) = r^2 y(x);\ \cdots\ y^{(n)} = r^n y(x),$$

we can write

$$L\big(\mathrm{e}^{rx}\big) = \phi(r)\mathrm{e}^{rx}. \tag{4}$$

for any given (r, x), where

$$\phi(r) = ar^2 + br + c.$$

is called the characteristic polynomial. From (4) it is seen that the function e^{rx} satisfies the equation (1), namely $L\big(\mathrm{e}^{rx}\big) = 0$, as long as the constant r is the root of the characteristic polynomial, i.e. $\phi(r) = 0$. In general, the polynomial $\phi(r)$ has two roots $(r1, r_2)$: One can write

$$\phi(r) = ar^2 + br + c = a(r - r_1)(r - r_2).$$

Accordingly, the equation (2) has two solutions:

$$\{y_1(x) = \mathrm{e}^{r_1 x};\ y_2(x) = \mathrm{e}^{r_2 x}\}.$$

Two cases should be discussed separately.

2.2 Cases (I) ($r_1 > r_2$)

When $b^2 - 4ac > 0$, the polynomial $\phi(r)$ has two distinct real roots ($r_1 \neq r_2$). In this case, the two solutions, $y_(x); y_2(x)$ are different. The following linear combination is not only solution, but also the general solution of the equation:

$$y(x) = Ay_1(x) + By_2(x), \tag{5}$$

where A, B are arbitrary constants. To prove that, we make use of the fundamental theorem which states that if y, z are two solutions such that $y(0) = z(0) = y_0$ and $y'(0) = z'(0 = y_0')$ then $y = z$. Let y be any solution and consider the linear equations in A, B

$$\begin{aligned} Ay_1(0) + By_2(0) &= y(0), \\ Ay_1'(0) + By_2'(0) &= y'(0), \end{aligned}$$

or

$$\begin{aligned} A + B &= y_0, \\ Ar_1 + Br_2 &= y_0'. \end{aligned}$$

Due to $r_1 \neq r_2$, these conditions leads to the unique solution A, B. With this choice of A, B the solution $z = Ay_1 + By_2$ satisfies $z(0) = y(0)$, $z'(0) = y'(0)$ and hence $y = z$. Thus, (5) contains all possible solutions of the equation, so, it is indeed the general solution.

2.3 Cases (II) ($r_1 = r_2$)

When $b^2 - 4ac = 0$, the polynomial $\phi(r)$ has double root: $r_1 = r_2 = \frac{-b}{2a}$. In this case, the solution $y_1(x) = y_2(x) = \mathrm{e}^{r_1 x}$. Thus, for the general solution, one needs to derive another type of the second solution. For this purpose, one may use the **method of reduction of order**.

Let us look for a solution of the form $C(x)e^{r_1 x}$ with the undetermined function $C(x)$. By substituting the equation, we derive that

$$L\Big(C(x)e^{r_1 x}\Big) = C(x)\phi(r_1)e^{r_1 x} + a\Big[C''(x) + 2r_1 C'(x)\Big]e^{r_1 x} + bC'(x)e^{r_1 x} = 0.$$

Noting that

$$\phi(r_1) = 0; \quad 2ar_1 + b = 0,$$

we get

$$C''(x) = 0$$

or

$$C(x) = Ax + B,$$

where A, B are arbitrary constants. Thus, we solution:

$$y(x) = (Ax + B)\mathrm{e}^{r_1 x}, \tag{6}$$

is a two parameter family of solutions consisting of the linear combinations of the two solutions $y_1 = e^{r_1 x}$ and $y_2 = xe^{r_1 x}$. It is also the general solution of the equation. The proof is similar to that given for the case (I) based on the fundamental theorem of existence and uniqueness. Let y be any solution and consider the linear equations in A, B

$$\begin{aligned} Ay_1(0) + By_2(0) &= y(0), \\ Ay_1'(0) + By_2'(0) &= y'(0), \end{aligned}$$

or

$$\begin{aligned} A &= y(0), \\ Ar_1 + B &= y'(0). \end{aligned}$$

these conditions leads to the unique solution $A = y(0), B = y'(0) - r_1 y(0)$. With this choice of A, B the solution $z = Ay_1 + By_2$ satisfies $z(0) = y(0)$, $z'(0) = y'(0)$ and hence $y = z$. Thus, (6) contains all possible solutions of the equation, so, it is indeed the general solution. The approach presented in this subsection is applicable to any higher order equations with constant coefficients.

Example 1. Consider the linear DE $y'' + 2y' + y = x$. Here $L(y) = y'' + 2y' + y$. A particular solution of the DE $L(y) = x$ is $y_p = x - 2$. The associated homogeneous equation is

$$y'' + 2y'' + y = 0.$$

The characteristic polynomial

$$\phi(r) = r^2 + 2r + 1 = (r+1)^2$$

has double roots $r_1 = r_2 = -1$. Thus the general solution of the DE

$$y'' + 2y' + y = x$$

is $y = Axe^{-x} + Be^{-x} + x - 2$.

This equation can be solved quite simply without the use of the fundamental theorem if we make essential use of operators.

2.4 Cases (III) ($r_{1,2} = \lambda \pm \mathrm{i}\mu$)

When $b^2 - 4ac < 0$, the polynomial $\phi(r)$ has two conjugate complex roots $r_{1,2} = \lambda \pm \mathrm{i}\mu$. We have to define the complex number,

$$\mathrm{i}^2 = -1; \qquad \mathrm{i}^3 = -\mathrm{i}; \qquad \mathrm{i}^4 = 1; \qquad \mathrm{i}^5 = \mathrm{i}, \cdots$$

and define and complex function with the Taylor series:

$$\mathrm{e}^{\mathrm{i}x} = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} + \mathrm{i} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos x + \mathrm{i} \sin x. \tag{7}$$

From the definition, it follows that

$$\mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x \mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x (\cos y + \mathrm{i} \sin y).$$

and

$$D(\mathrm{e}^{rx}) = r\mathrm{e}^x, \qquad D^n(\mathrm{e}^{rx}) = r^n \mathrm{e}^x$$

where r is a complex number. So that, the basic equalities are now extended to the case with complex number r. Thus, we have the two complex solutions:

$$y_1(x) = \mathrm{e}^{r_1 x} = \mathrm{e}^{\lambda x}(\cos \mu x + \mathrm{i} \sin \mu x), \quad y_2(x) = \mathrm{e}^{r_2 x} = \mathrm{e}^{\lambda x}(\cos \mu x - \mathrm{i} \sin \mu x)$$

with a proper combination of these two solutions, one may derive two real solutions:

$$\tilde{y}_1(x) = \mathrm{e}^{\lambda x} \cos \mu x, \quad \tilde{y}_2(x) = \mathrm{e}^{\lambda x} \sin \mu x$$

and the general solution:

$$y(x) = \mathrm{e}^{\lambda x}(A \cos \mu x + B \sin \mu x).$$