McGill University
Math 325A: Differential Equations

LECTURE 10: HIGHER ORDER DIFFERENTIAL EQUATIONS (II)

(Text: pp. 338-367, Chap. 4, 6)

1 Introduction

In this lecture we give an introduction to several methods for solving higher order differential equations. Most of what we say will apply to the linear case as there are relatively few non-numerical methods for solving nonlinear equations. There are two important cases however where the DE can be reduced to one of lower degree.

1.1 Case (I)

DE has the form:

$$y^{(n)} = f(x, y', y'', \ldots, y^{(n-1)})$$

where on the right-hand side the variable y does not appear. In this case, setting $z = y'$ leads to the DE

$$z^{(n-1)} = f(x, z, z', \ldots, z^{(n-2)})$$

which is of degree $n-1$. If this can be solved then one obtains y by integration with respect to x.

For example, consider the DE $y'' = (y')^2$. Then, setting $z = y'$, we get the DE $z' = z^2$ which is a separable first order equation for z. Solving it we get $z = -1/(x+C)$ or $z = 0$ from which $y = -\log(x+C) + D$ or $y = C$. The reader will easily verify that there is exactly one of these solutions which satisfies the initial condition $y(x_0) = y_0,\ y'(x_0) = y'_0$ for any choice of x_0, y_0, y'_0 which confirms that it is the general solution since the fundamental theorem guarantees a unique solution.

1.2 Case (II)

DE has the form:

$$y^{(n)} = f(y, y', y'', \ldots, y^{(n-1)})$$

where the independent variable x does not appear explicitly on the right-hand side of the equation. Here we again set $z = y'$ but try for a solution z as a function of y. Then, using the fact that $\frac{d}{dx} = z\frac{d}{dy}$, we get the DE

$$\left(z\frac{d}{dy}\right)^{n-1}(z) = f\left(y, z, z\frac{dz}{dy}, \ldots, (z\frac{d}{dy})^n(z)\right)$$

which is of degree $n-1$. For example, the DE $y'' = (y')^2$ is of this type and we get the DE

$$z\frac{dz}{dy} = z^2$$

which has the solution $z = Ce^y$. Hence $y' = Ce^y$ from which $-e^{-y} = Cx + D$. This gives $y = -\log(-Cx - D)$ as the general solution which is in agreement with what we did previously.

2 Linear Equations

2.1 Basic Concepts and General Properties

Let us now go to linear equations. The general form is

$$L(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x).$$

The function L is called a *differential operator*. The characteristic feature of L is that

$$L(a_1y_1 + a_2y_2) = a_1L(y_1) + a_2L(y_2).$$

Such a function L is what we call a *linear operator*. Moreover, if

$$L_1(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y$$

$$L_2(y) = b_0(x)y^{(n)} + b_1(x)y^{(n-1)} + \cdots + b_n(x)y$$

and $p_1(x), p_2(x)$ are functions of x the function $p_1L_1 + p_2L_2$ defined by

$$\begin{aligned}(p_1L_1 + p_2L_2)(y) &= p_1(x)L_1(y) + p_2(x)L_2(y)\\ &= \left[a_0(x) + p_2(x)b_0(x)\right]y^{(n)} + \cdots \left[p_1(x)a_n(x) + p_2(x)b_n(x)\right]y\end{aligned}$$

is again a linear differential operator. An important property of linear operators in general is the *distributive law*:

$$L(L_1 + L_2) = LL_1 + LL_2, \quad (L_1 + L_2)L = L_1L + L_2L.$$

The linearity of equation implies that for any two solutions y_1, y_2 the difference $y_1 - y_2$ is a solution of the associated homogeneous equation $L(y) = 0$. Moreover, it implies that any linear combination $a_1y_1 + a_2y_2$ of solutions y_1, y_2 of $L(y) = 0$ is again a solution of $L(y) = 0$. The solution space of $L(y) = 0$ is also called the **kernel** of L and is denoted by $\ker(L)$. It is a subspace of the vector space of real valued functions on some interval I. If y_p is a particular solution of $L(y) = b(x)$, the general solution of $L(y) = b(x)$ is

$$\ker(L) + y_p = \{y + y_p \mid L(y) = 0\}.$$

The differential operator $L(y) = y'$ may be denoted by D. The operator $L(y) = y''$ is nothing but $D^2 = D \circ D$ where $\circ$ denotes composition of functions. More generally, the operator $L(y) = y^{(n)}$ is D^n. The identity operator I is defined by $I(y) = y$. By definition $D^0 = I$. The general linear n-th order ODE can therefore be written

$$\Big[a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_n(x)I\Big](y) = b(x).$$

3 Basic Theory of Linear Differential Equations

In this lecture we will develop the theory of linear differential equations. The starting point is the fundamental existence theorem for the general n-th order ODE $L(y) = b(x)$, where

$$L(y) = D^n + a_1(x)D^{n-1} + \cdots + a_n(x).$$

We will also assume that $a_0(x), a_1(x), \ldots, a_n(x), b(x)$ are continuous functions on the interval I.

3.1 Basics of Linear Vector Space

3.1.1 Isomorphic Linear Transformation

From the fundamental theorem, it is known that for any $x_0 \in I$, the initial value problem

$$L(y) = b(x) \quad y(x_0) = d_1, y'(x_0) = d_2, \ldots, y^{(n-1)}(x_0) = d_n$$

has a unique solution for any $d_1, d_2, \ldots, d_n \in \mathbb{R}$.

Thus, if V is the solution space of the associated homogeneous DE $L(y) = 0$, the transformation

$$T : V \to \mathbb{R}^n,$$

defined by $T(y) = (y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0))$, is linear transformation of the vector space V into $\mathbb{R}^n$ since

$$T(ay + bz) = aT(y) + bT(z).$$

Moreover, the fundamental theorem says that T is one-to-one ($T(y) = T(z) \implies y = z$) and onto (every $d \in \mathbb{R}^n$ is of the form $T(y)$ for some $y \in V$). A linear transformation which is one-to-one and onto is called an **isomorphism**. Isomorphic vector spaces have the same properties.

3.1.2 Dimension and Basis of Vector Space

We call the vector space being n-dimensional with the notation by $\dim(V) = n$. This means that there exists a sequence of elements: $y_1, y_2, \ldots, y_n \in V$ such that every $y \in V$ can be uniquely written in the form

$$y = c_1y_1 + c_2y_2 + \ldots c_ny_n$$

with $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Such a sequence of elements of a vector space V is called a **basis** for V. In the context of DE's it is also known as a **fundamental set**. The number of elements in a basis for V is called the dimension of V and is denoted by $\dim(V)$. If

$$e_1 = (1, 0, \ldots, 0),\ e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$$

is the standard basis of $\mathbb{R}^n$ and y_i is the unique $y_i \in V$ with $T(y_i) = e_i$ then $y_1, y_2, \ldots, y_n$ is a basis for V. This follows from the fact that

$$T(c_1y_1 + c_2y_2 + \cdots + c_ny_n) = c_1T(y_1) + c_2T(y_2) + \cdots + c_nT(y_n).$$

3.1.3 (*) Span and Subspace

A set of vectors $v_1, v_2, \cdots, v_n$ in a vector space V is said to **span** or **generate** V if every $v \in V$ can be written in the form

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

with $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Obviously, not any set of n vectors can span the vector space V. It will be seen that $\{v_1, v_2, \cdots, v_n\}$ span the vector space V, if and only if they are linear independent. The set

$$S = \text{span}(v_1, v_2, \ldots, v_n) = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n \mid c_1, c_2, \ldots, c_n \in \mathbb{R}\}$$

consisting of all possible linear combinations of the vectors $v_1, v_2, \ldots, v_n$ form a **subspace** of V, which may be also called the **span** of $\{v_1, v_2, \ldots, v_n\}$. Then $V = \text{span}(v_1, v_2, \ldots, v_n)$ if and only if $v_1, v_2, \ldots, v_n$ spans V.

3.1.4 Linear Independency

The vectors $v_1, v_2, \ldots, v_n$ are said to be **linearly independent** if

$$c_1 v_1 + c_2 v_2 + \ldots c_n v_n = 0$$

implies that the scalars $c_1, c_2, \ldots, c_n$ are all zero. A basis can also be characterized as a linearly independent generating set since the uniqueness of representation is equivalent to linear independence. More precisely,

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = c_1' v_1 + c_2' v_2 + \cdots + c_n' v_n$$

implies

$$c_i = c_i' \qquad \text{for all } i,$$

if and only if $v_1, v_2, \ldots, v_n$ are linearly independent.

As an example of a linearly independent set of functions consider

$$\cos(x), \cos(2x), \cos(3x).$$

To prove their linear independence, suppose that c_1, c_2, c_3 are scalars such that

$$c_1 \cos(x) + c_2 \cos(2x) + c_3 \sin(3x) = 0$$

for all x. Then setting $x = 0, \pi/2, \pi$, we get

$$\begin{aligned} c_1 + c_2 &= 0, \\ -c_2 - c_3 &= 0, \\ -c_1 + c_2 &= 0 \end{aligned}$$

from which $c_1 = c_2 = c_3 = 0$.

An example of a linearly dependent set would be $\sin^2(x), \cos^2(x), \cos(2x)$ since

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

implies that $\cos(2x) + \sin^2(x) + (-1)\cos^2(x) = 0$.

3.2 Wronskian of n-functions

Another criterion for linear independence of functions involves the Wronskian.

3.2.1 Definition

If $y_1, y_2, \ldots, y_n$ are n functions which have derivatives up to order $n-1$ then the Wronskian of these functions is the determinant

$$W = W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

If $W(x_0) \neq 0$ for some point x_0, then $y_1, y_2, \ldots, y_n$ are linearly independent. This follows from the fact that $W(x_0)$ is the determinant of the coefficient matrix of the linear homogeneous system of equations in $c_1, c_2, \ldots, c_n$ obtained from the dependence relation

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

and its first $n-1$ derivatives by setting $x = x_0$.

For example, if $y_1 = \cos(x), \cos(2x), \cos(3x)$ we have

$$W = \begin{vmatrix} \cos(x) & \cos(2x) & \cos(3x) \\ -\sin(x) & -2\sin(2x) & -3\sin(3x) \\ -\cos(x) & -4\cos(2x) & -9\cos(3x) \end{vmatrix}$$

and $W(\pi/4)) = -8$ which implies that y_1, y_2, y_3 are linearly independent. Note that $W(0) = 0$ so that you cannot conclude linear dependence from the vanishing of the Wronskian at a point. This is not the case if $y_1, y_2, \ldots, y_n$ are solutions of an n-th order linear homogeneous ODE.

3.2.2 Theorem 1

The the Wronskian of n solutions of the n-th order linear ODE $L(y) = 0$ is subject to the following first order ODE:

$$\frac{dW}{dx} = -a_1(x)W,$$

with solution

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a_1(t)dt}.$$

From the above it follows that the Wronskian of n solutions of the n-th order linear ODE $L(y) = 0$ is either identically zero or vanishes nowhere.

3.2.3 Theorem 2

If $y_1, y_2, \ldots, y_n$ are solutions of the linear ODE $L(y) = 0$, the following are equivalent:

1. $y_1, y_2, \ldots, y_n$ is a basis for the vector space $V = \ker(L)$;
2. $y_1, y_2, \ldots, y_n$ are linearly independent;
3. $^{(*)}$ $y_1, y_2, \ldots, y_n$ span V;
4. $y_1, y_2, \ldots, y_n$ generate $\ker(L)$;
5. $W(y_1, y_2, \ldots, y_n) \neq 0$ at some point x_0;
6. $W(y_1, y_2, \ldots, y_n)$ is never zero.

Proof. The equivalence of $1, 2, 3$ follows from the fact that $\ker(L)$ is isomorphic to $\mathbb{R}^n$. The rest of the proof follows from the fact that if the Wronskian were zero at some point x_0 the homogeneous system of equations

$$\begin{aligned} c_1 y_1(x_0) + c_1 y_2(x_0) + \cdots + c_n y_n(x_0) &= 0 \\ c_1 y_1'(x_0) + c_1 y_2'(x_0) + \cdots + c_n y_n'(x_0) &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)}(x_0) + c_1 y_2^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) &= 0 \end{aligned}$$

would have a non-zero solution for $c_1, c_2, \ldots, c_n$ which would imply that

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

and hence that $y_1, y_2, \ldots, y_n$ are not linearly independent. **QED**

From the above, we see that to solve the n-th order linear DE $L(y) = b(x)$ we first find linear n independent solutions $y_1, y_2, \ldots, y_n$ of $L(y) = 0$. Then, if y_P is a particular solution of $L(y) = b(x)$, the general solution of $L(y) = b(x)$ is

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_P.$$

The initial conditions $y(x_0) = d_1, y'(x_0) = d_2, \ldots, y_n^{(n-1)}(x_0) = d_n$ then determine the constants $c_1, c_2, \ldots, c_n$ uniquely.