McGill University
Math 325A: Differential Equations

LECTURE 1: INTRODUCTION

(Text: Sections 1.1, 1.2)

1 Definitions and Basic Concepts

1.1 Ordinary Differential Equation (ODE)

An equation involving the derivatives of an unknown function y of a single variable x over an interval $x \in (I)$.

1.2 Solution

Any function $y = f(x)$ which satisfies this equation over the interval (I) is called a solution of the ODE.

For example, $y = e^{2x}$ is a solution of the ODE

$$y' = 2y$$

and $y = \sin(x^2)$ is a solution of the ODE

$$xy'' - y' + 4x^3y = 0.$$

1.3 Order n of the DE

An ODE is said to be order n, if $y^{(n)}$ is the highest order derivative occurring in the equation. The simplest first order ODE is $y' = g(x)$.

The most general form of an n-th order ODE is

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

with F a function of $n+2$ variables $x, u_0, u_1, \ldots, u_n$. The equations

$$xy'' + y = x^3, \quad y' + y^2 = 0, \quad y''' + 2y' + y = 0$$

are examples of ODE's of second order, first order and third order respectively with respectively

$$F(x, u_0, u_1, u_2) = xu_2 + u_0 - x^3, \quad F(x, u_0, u_1) = u_1 + u_0^2, \quad F(x, u_0, u_1, u_2, u_3) = u_3 + 2u_1 + u_0.$$

1.4 Linear Equation:

If the function F is linear in the variables $u_0, u_1, \ldots, u_n$ the ODE is said to be **linear**. If, in addition, F is homogeneous then the ODE is said to be homogeneous. The first of the above examples above is linear are linear, the second is non-linear and the third is linear and homogeneous. The general n-th order linear ODE can be written

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

1.5 Homogeneous Linear Equation:

The linear DE is homogeneous, if and only if $b(x) \equiv 0$. Linear homogeneous equations have the important property that linear combinations of solutions are also solutions. In other words, if $y_1, y_2, \ldots, y_m$ are solutions and $c_1, c_2, \ldots, c_m$ are constants then

$$c_1 y_1 + c_2 y_2 + \cdots + c_m y_m$$

is also a solution.

1.6 Partial Differential Equation (PDE)

An equation involving the partial derivatives of a function of more than one variable is called PED. The concepts of linearity and homogeneity can be extended to PDE's. The general second order linear PDE in two variables x, y is

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = g(x,y).$$

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is a linear, homogeneous PDE of order 2. The functions $u = \log(x^2 + y^2)$, $u = xy$, $u = x^2 - y^2$ are examples of solutions of Laplace's equation. We will not study PDE's systematically in this course.

1.7 General Solution of a Linear Differential Equation

It represents the set of all solutions, i.e., the set of all functions which satisfy the equation in the interval (I). For example, the general solution of the differential equation $y' = 3x^2$ is $y = x^3 + C$ where C is an arbitrary constant. The constant C is the value of y at $x = 0$. This **initial condition** completely determines the solution. More generally, one easily shows that given a, b there is a unique solution y of the differential equation with $y(a) = b$. Geometrically, this means that the one-parameter family of curves $y = x^2 + C$ do not intersect one another and they fill up the plane $\mathbb{R}^2$.

1.8 A System of ODE's

$$\begin{aligned} y_1' &= G_1(x, y_1, y_2, \ldots, y_n) \\ y_2' &= G_2(x, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= G_n(x, y_1, y_2, \ldots, y_n) \end{aligned}$$

An n-th order ODE of the form $y^{(n)} = G(x, y, y', \ldots, y^{n-1})$ can be transformed in the form of the system of first order DE's. If we introduce dependant variables $y_1 = y, y_2 = y', \ldots, y_n = y^{n-1}$ we obtain the equivalent system of first order equations

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ &\vdots \\ y_n' &= G(x, y_1, y_2, \ldots, y_n). \end{aligned}$$

For example, the ODE $y'' = y$ is equivalent to the system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_1. \end{aligned}$$

In this way the study of n-th order equations can be reduced to the study of systems of first order equations. Some times, one called the latter as the **normal form** of the n-th order ODE. Systems of equations arise in the study of the motion of particles. For example, if $P(x, y)$ is the position of a particle of mass m at time t, moving in a plane under the action of the force field $(f(x, y), g(x, y)$, we have

$$\begin{aligned} m\frac{d^2x}{dt^2} &= f(x, y), \\ m\frac{d^2y}{dt^2} &= g(x, y). \end{aligned}$$

This is a second order system.

The general first order ODE in normal form is

$$y' = F(x, y).$$

If F and $\frac{\partial F}{\partial y}$ are continuous one can show that, given a, b, there is a unique solution with $y(a) = b$. Describing this solution is not an easy task and there are a variety of ways to do this. The dependence of the solution on initial conditions is also an important question as the initial values may be only known approximately.

The non-linear ODE $yy' = 4x$ is not in normal form but can be brought to normal form

$$y' = \frac{4x}{y}.$$

by dividing both sides by y.

2 The Approaches of Finding Solutions of ODE

2.1 Analytical Approaches

- Analytical solution methods: finding the exact form of solutions;
- Geometrical methods: finding the qualitative behavior of solutions;
- Asymptotic methods: finding the asymptotic form of the solution, which gives good approximation of the exact solution.

2.2 Numerical Approaches

- Numerical algorithms — numerical methods;
- Symbolic manipulators — Maple, MATHEMATICA, MacSyma.

This course mainly discuss the analytical approaches and mainly on analytical solution methods.