Measure non-rigidity for linear cellular automata

Given two commuting transformations $\sigma, \Phi : X \to X$ acting on a compact metric space, what are the measures on X which are invariant under the action of σ and Φ? This general question includes the open problem posed by Furstenberg, which is to find the measures on the unit interval which are invariant under both $x \mapsto 2x \mod 1$ and $x \to 3x \mod 1$.

Let p be a prime number and let $\mathbb{F}_p$ be the field of cardinality p. A *linear cellular automaton* $\Phi : \mathbb{F}_p^{\mathbb{Z}} \to \mathbb{F}_p^{\mathbb{Z}}$ is an $\mathbb{F}_p$-linear map that commutes with the (left) shift map $\sigma : \mathbb{F}_p^{\mathbb{Z}} \to \mathbb{F}_p^{\mathbb{Z}}$. A famous linear cellular automaton is Ledrappier's, defined by $\Phi(x) = x + \sigma(x)$, where, in contrast to a symbolic version of Furstenberg's question, addition is performed "bitwise" and without carry.

For a linear cellular automaton Φ, examples of measures which are (Φ, σ)-invariant are the uniform measure on $\mathbb{F}_p^{\mathbb{Z}}$, and measures supported on a finite set. In work by Einsiedler from the early 2000's, if we recast linear cellular automata in the setting of *Markov subgroups*, we find a new family of nontrivial (σ, Φ)-invariant measures. In recent joint work with Eric Rowland, we find another family of of nontrivial (σ, Φ)-invariant measures, using constant length substitutions, and their characterisation by Christol. I will describe how we obtain these measures, and compare them to Einsiedler's construction.