Measure non-rigidity for linear cellular automata

Given two commuting transformations $\sigma, \Phi : X \to X$ acting on a compact metric space, what are the measures on $X$ which are invariant under the action of $\sigma$ and $\Phi$? This general question includes the open problem posed by Furstenberg, which is to find the measures on the unit interval which are invariant under both $x \mapsto 2x \mod 1$ and $x \mapsto 3x \mod 1$.

Let $p$ be a prime number and let $\mathbb{F}_p$ be the field of cardinality $p$. A linear cellular automaton $\Phi : \mathbb{F}_p^\mathbb{Z} \to \mathbb{F}_p^\mathbb{Z}$ is an $\mathbb{F}_p$-linear map that commutes with the (left) shift map $\sigma : \mathbb{F}_p^\mathbb{Z} \to \mathbb{F}_p^\mathbb{Z}$. A famous linear cellular automaton is Ledrappier’s, defined by $\Phi(x) = x + \sigma(x)$, where, in contrast to a symbolic version of Furstenberg’s question, addition is performed “bitwise” and without carry.

For a linear cellular automaton $\Phi$, examples of measures which are $(\Phi, \sigma)$-invariant are the uniform measure on $\mathbb{F}_p^\mathbb{Z}$, and measures supported on a finite set. In work by Einsiedler from the early 2000’s, if we recast linear cellular automata in the setting of Markov subgroups, we find a new family of nontrivial $(\sigma, \Phi)$-invariant measures. In recent joint work with Eric Rowland, we find another family of nontrivial $(\sigma, \Phi)$-invariant measures, using constant length substitutions, and their characterisation by Christol. I will describe how we obtain these measures, and compare them to Einsiedler’s construction.