Perturbed Haar function expansions

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Abstract

The Haar functions \( \{ \phi_I \} \) form a complete orthonormal system for \( L^2(\mathbb{R}) \) and an unconditional basis for \( L^p(\mathbb{R}) \) (\( 1 < p < \infty \)): \( \forall f \in L^p(\mathbb{R}) \),

\[
f = \sum_I \langle f, \phi_I \rangle \phi_I,
\]

where \( \phi_I \) is the set of dyadic intervals, \( \langle f, g \rangle := \int f \overline{g} \, dx \) is the usual inner product, and the series converges unconditionally in \( L^p \). The Haar functions are readily generalized to a family \( \phi_{I,k} \) defined on \( \mathbb{R}^d \), with the analogous convergence result holds.

We explore the stability of (1) when the \( \phi_I \) are subjected to arbitrary, close-to-the-identity, (local) affine changes of variable. In one dimension this means replacing each \( \phi_I \) by, respectively, \( \tilde{\phi}_I \) and \( \phi^*_I \),

\[
\begin{align*}
\tilde{\phi}_I(x) &:= \phi_I(\tilde{a}_I(x-x_I + \tilde{y}_I \ell(I)) + x_I) \\
\phi^*_I(x) &:= \phi_I(a^*_I(x-x_I + y^*_I \ell(I)) + x_I),
\end{align*}
\]

where \( x_I \) is \( I \)'s center, \( \ell(I) \) is its length, and \( \{ a_I \}_{I \in \mathcal{D}_1}, \{ a^*_I \}_{I \in \mathcal{D}_1}, \{ y_I \}_{I \in \mathcal{D}_1}, \) and \( \{ y^*_I \}_{I \in \mathcal{D}_1} \) are sets of real numbers satisfying

\[
\sup_{I \in \mathcal{D}_1} \max (|1 - \tilde{a}_I|, |1 - a^*_I|, |y_I|, |y^*_I|) \leq \eta
\]

for a fixed \( 0 \leq \eta < 1/2 \). We show that, for all \( f \in L^2(\mathbb{R}) \),

\[
\left\| f - \sum_I \langle f, \tilde{\phi}_I \rangle \phi^*_I \right\|_2 \leq C \eta^{1/2} \| f \|_2,
\]

with \( C \) an absolute constant, and

\[
\left\| f - \sum_I \langle f, \tilde{\phi}_I \rangle \phi^*_I \right\|_p \leq C \eta^{1/2} \| f \|_p
\]

for all \( f \in L^p(\mathbb{R}) \), with \( C = C(p) \) for \( 2 < p < \infty \); and the analogous results, with more complicated notations, in \( L^p(\mathbb{R}^d) \) (\( 2 \leq p < \infty \)). (The “missing” \( * \) in the second inequality is not a typo.)