Almost-orthogonality: almost as good as orthogonality

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Abstract.

A set of functions $\{\psi_\gamma\}_{\gamma\in\Gamma} \subset L^2(\mathbf{R}^d,\mu)$ is called *almost-orthogonal* if there is a finite R so that, for all finite subsets $\mathcal{F} \subset \Gamma$ and all linear combinations $\sum_{\gamma\in\mathcal{F}} \lambda_\gamma \psi_\gamma$,

$$\left\| \sum_{\gamma\in\mathcal{F}} \lambda_\gamma \psi_\gamma \right\|_{L^2(\mathbf{R}^d,\mu)} \leq R \left(\sum_{\gamma\in\mathcal{F}} |\lambda_\gamma|^2 \right)^{1/2}. \tag{1}$$

By duality, $\{\psi_\gamma\}_{\gamma\in\Gamma}$ satisfies (1) if and only if, for all $f \in L^2(\mathbf{R}^d,\mu)$,

$$\left(\sum_{\Gamma} |\langle f, \psi_\gamma\rangle_\mu|^2 \right)^{1/2} \leq R\|f\|_{L^2(\mathbf{R}^d,\mu)},$$

where $\langle\cdot,\cdot\rangle_\mu$ is the inner product in $L^2(\mathbf{R}^d,\mu)$. The set $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is called a *frame* for $L^2(\mathbf{R}^d,\mu)$ if there are positive, finite constants A and B so that, for all $f \in L^2(\mathbf{R}^d,\mu)$,

$$A\|f\|_{L^2(\mathbf{R}^d,\mu)} \leq \left(\sum_{\Gamma} |\langle f, \psi_\gamma\rangle_\mu|^2 \right)^{1/2} \leq B\|f\|_{L^2(\mathbf{R}^d,\mu)}.$$

Frames are used to build coordinate systems for $L^2(\mathbf{R}^d,\mu)$. In a sense, an almost-orthogonal set is halfway to being a frame.

After looking at some familiar orthogonal and almost-orthogonal sets, we will explore the extent to which almost-orthogonality of a family $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is preserved when we change the measure. We will make use of a clever, non-trivial condition (apparently due to Yves Meyer) which, for certain, natural families $\{\psi_\gamma\}_{\gamma\in\Gamma}$ and measures μ, is equivalent to almost-orthogonality in $L^2(\mathbf{R}^d,\mu)$. The condition has an interesting connection to the celebrated $T1$ theorem of David and Journé.

We intend to give no detailed proofs in this talk, but only sketch the main ideas.