Some random thoughts about Cauchy's functional equation

In this lecture I will discuss a variation on Cauchy's functional equation

$$(*) \qquad f(x+y) = f(x) + f(y) \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

After reviewing the familiar fact that any measurable f which satisfies (*) must be linear, I will investigate what can be said when (*) is replaced by

$$(**) \qquad f(x+y) = f(x) + f(y) \quad \text{for Lebesgue almost every } (x,y) \in \mathbb{R}^2.$$

Borrowing ideas from probability theory, I will show that any measurable solution to (**) is Lebesgue almost everywhere equal to a linear function. If time permits, I will also show how the same ideas apply in more exotic settings.