Semi-classical analysis for magnetic Schrödinger operators and applications

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We would like to present some of these problems and their solutions.
This involves mathematically a fine analysis of the bottom of the spectrum for Schrödinger operators with magnetic fields.
The boundary condition (namely the Neumann condition) will play a basic role.
The recent results presented here are obtained in collaboration with A. Morame, S. Fournais, Y. Kordyukov, X. Pan .. or by my student N. Raymond.

Our main object of interest is the Neumann Laplacian with magnetic field.
We consider a magnetic field

$$\beta = \text{ curl } \mathbf{F}$$

on a regular domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) associated with a magnetic potental $\mathbf{F}$ (vector field on Ω), which (for normalization) satisfies :

$$\text{div } \mathbf{F} = 0 \,, \; \mathbf{F} \cdot N_{\partial\Omega} = 0 \,,$$

where $N(x)$ is the unit interior normal vector to $\partial\Omega$.

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where $N(x)$ is the unit interior normal vector to $\partial\Omega$.
We start from the closed quadratic form Q_B

$$W^{1,2}(\Omega) \ni u \mapsto Q_B(u) := \int_\Omega |(-i\nabla + B\mathbf{F})u(x)|^2 \, dx. \tag{1}$$

Let $\mathcal{H}^N(B\mathbf{F},\Omega)$ be the self-adjoint operator associated to Q_B.
$\mathcal{H}^N(B\mathbf{F},\Omega)$ is the differential operator $(-i\nabla + B\mathbf{F})^2$ with domain

$$\{u \in W^{2,2}(\Omega)\,:\, N\cdot\nabla u_{/\partial\Omega} = 0\}\,.$$

When Ω is bounded, the operator $\mathcal{H}^N(B\mathbf{F},\Omega)$ has compact resolvent and we introduce

$$\lambda_1^N(B\mathbf{F},\Omega) := \inf\ \mathrm{Spec}\ \mathcal{H}^N(B\mathbf{F},\Omega)\,. \tag{2}$$

One could also look at the Dirichlet realization $\mathcal{H}^D(B\mathbf{F},\Omega)$ with domain

$$\{u \in W^{2,2}(\Omega)\,:\, u_{/\partial\Omega} = 0\}\,.$$

and to the corresponding groundstate energy $\lambda_1^D(B\mathbf{F},\Omega)$.

Motivated by various questions we consider the three connected problems in the asymptotic $B \to +\infty$.

Pb 1 Find an accurate estimate of the groundstate energy $B \mapsto \lambda_1^N(B\mathbf{F}, \Omega)$.

Pb 2 Find where a corresponding groundstate is living.

Pb 3 Show that $B \mapsto \lambda_1^N(B\mathbf{F}, \Omega)$ for large B is monotonically increasing.

We will present results which are

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If time permits, we will discuss also applications of this monotonicity for the identification of the critical fields in superconductivity and present similar questions in the context of the theory of liquid crystals.

These results (initiated by Lu-Pan and Bernoff-Sternberg) are based on the analysis models with constant magnetic field β :

1. The case in $\mathbb{R}^d$

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2. The case in the half space $\mathbb{R}^d_+$
   - $$\inf \sigma(\mathcal{H}^N(B\mathbf{F}, \mathbb{R}^2_+)) = \Theta_0 B|\beta|\,,$$
   - $$\inf \sigma(\mathcal{H}^N(B\mathbf{F}, \mathbb{R}^3_+)) = \varsigma(\vartheta) B|\beta|\,,$$

     where ϑ is the angle between β and N,

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  where ϑ is the angle between β and N,

- $$\inf \sigma(\mathcal{H}^D(B\mathbf{F}, \mathbb{R}^d_+) = B|\beta| \,.$$

The main points are

- $$0 < \Theta_0 < 1\,. \tag{3}$$
- For the case $d = 3$
$$\Theta_0 = \varsigma(\frac{\pi}{2}) \leq \varsigma(\vartheta) \leq \varsigma(0) = 1\,. \tag{4}$$
- $\vartheta \mapsto \varsigma(\vartheta)$ is decreasing on $[0, \frac{\pi}{2}]$.

From this, we get

1. The bottom of the spectrum for Neumann in the half space is below the problem in $\mathbb{R}^d$.
2. In the $3D$ case, when we minimize over the β such that $|\beta| = 1$, the bottom of the spectrum is minimal when β is tangent to the boundary.

We introduce

$$b = \inf_{x \in \overline{\Omega}} |\beta(x)| \,, \tag{5}$$

$$b' = \inf_{x \in \partial\Omega} |\beta(x)| \,, \tag{6}$$

and, for $d = 2$,

$$b_2' = \Theta_0 \inf_{x \in \partial\Omega} |\beta(x)| \,, \tag{7}$$

and, for $d = 3$,

$$b_3' = \inf_{x \in \partial\Omega} |\beta(x)| \varsigma(\theta(x)) \tag{8}$$

Theorem 1 : rough asymptotics

$$\lambda_1^N(B\mathbf{F},\Omega) = B\min(b, b'_d) + o(B)\,, \tag{9}$$

$$\lambda_1^D(B\mathbf{F},\Omega) = Bb + o(B) \tag{10}$$

Particular case, if $|\beta(x)| = 1$, then

$$\min(b, b'_d) = b'_d = \Theta_0\,. \tag{11}$$

The consequences for Pb 2 are that a ground state is localized as $B \to +\infty$,

- for Dirichlet, at the points of $\overline{\Omega}$ where $|\beta(x)|$ is minimum,
- for Neumann,
  - if $b < b'_d$, at the points of Ω where $|\beta(x)|$ is minimum (no difference with Dirichlet).
  - if $b > b'_d$ at the points of $\partial\Omega$ where $|\beta(x)|\varsigma(\theta(x)$ is minimum.

In particular, if $|\beta(x)| = 1$, we are, for Neumann, in the second case, hence the groundstate is localized at the boundary.

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All the results of localization are obtained through Agmon estimates (as Helffer-Sjöstrand, Simon have done in the eighties for $-h^2\Delta + V$) .

If

$$b < \inf_{x \in \partial\Omega} |\beta(x)| \text{ for Dirichlet}$$

or if

$$b < b' \text{ for Neumann,}$$

the asymptotics are the same (modulo an exponentially small error).

If we assume in addition

Assumption A

- There exists a unique point $x_{min} \in \Omega$ such that $b = |\beta(x_{min})|$.
- This minimum is non degenerate.

we get in $2D$ (Helffer-Morame)

Theorem 2

$$\lambda_1^{D\ or\ N}(B\mathbf{F}) = bB + \Theta_{\frac{1}{2}} B^{\frac{1}{2}} + o(B^{\frac{1}{2}})\,. \tag{12}$$

where $\Theta_{\frac{1}{2}}$ is computed from the Hessian of β at the minimum.

The problem is still open (Helffer-Kordyukov W. in P.) in the $3D$ case.

There are also many results for the case when $b = 0$ (Montgomery, Helffer-Mohamed, Pan-Kwek, Aramaki, Helffer-Kordyukov ....).
The ground state is localized near the minimum.
When more than a minimum, tunneling can occur (Helffer-Sjöstrand, Helffer-Kordyukov).

We describe recent results of N. Raymond in the $2D$-case. This time we assume that

Assumption B

- $\Theta_0 b' := b_2' < b$
- $\partial\Omega \ni x \mapsto |\beta(x)|$ has a unique non degenerate minimum x_{min}.

Theorem 3

$$\lambda_1^N(B\mathbf{F}) = b_2' B + \Theta_{\frac{1}{2}} B^{\frac{1}{2}} + o(B^{\frac{1}{2}})\,. \tag{13}$$

where

$$\Theta_{\frac{1}{2}} = -\frac{k_0 + k_1}{2} C_1 - \Theta_0 \xi_0 \frac{\partial_\nu \beta}{b'} + \sqrt{3C_1}\,\Theta_0^{\frac{3}{4}} \sqrt{\alpha} \tag{14}$$

with

$$\alpha = \partial_{ss}^2 \beta / b'\,,$$

k_0 the curvature at the minimum, $k_1 = k_0 - \frac{\partial_\nu \beta}{b'}$, $C_1 > 0$ (spectral constant), all the derivatives being computed at x_{min}.

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Concerning Pb 2, the ground state is localized at the minimum.

The $3D$-case is open. For the applications, one can first start with the case when $|\beta(x)| = 1$. This is the case of interest in the theory of Liquid crystals.

In the constant magnetic field case, which plays a special role in superconductivity, one needs to go further !

In the two dimensional case, it was proved by DelPino-Felmer-Sternberg–Lu-Pan–Helffer-Morame the

Theorem 4

$$\lambda_1(B) = \Theta_0 B - C\widehat{k}_0 B^{\frac{1}{2}} + o(B^{\frac{1}{2}}) , \tag{15}$$

where $\widehat{k}_0$ is the maximal curvature of the boundary and $C > 0$ is a universal constant.

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Concerning Pb 2, we have localization at the points of maximal curvature.
Concerning Pb 3, this has been proved by Fournais-Helffer, who in addition get a complete expansion when the curvature has a non-degenerate maximum.

Let us consider the $(3D)$ situation.

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We assume

Assumption C1

The set of boundary points where β is tangent to $\partial\Omega$, i.e.

$$\Gamma := \{x \in \partial\Omega \,|\, \beta \cdot N(x) = 0\}, \tag{16}$$

is a regular submanifold of $\partial\Omega$:

$$k_n(x) := |d^T(\beta \cdot N)(x)| \neq 0 \,, \ \forall x \in \Gamma \,. \tag{17}$$

We also assume that

Assumption C2

The set of points where β is tangent to Γ is finite.

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These assumptions are rather generic and for instance satisfied for ellipsoids.

We have the following two-term asymptotics of $\lambda_1(B)$ (due to Helffer-Morame- Pan).

Theorem 5

If Ω and β satisfy C1-C2, then as $B \to +\infty$

$$\lambda_1^N(B) = \Theta_0 B + \widehat{\gamma}_0 B^{\frac{2}{3}} + \mathcal{O}(B^{\frac{2}{3}-\eta}),$$

for some $\eta > 0$.

In Formula

$$\lambda_1(B) = \Theta_0 B + \widehat{\gamma}_0 B^{\frac{2}{3}} + o(B^{\frac{2}{3}})\,. \tag{18}$$

$\widehat{\gamma}_0$ is defined by

$$\widehat{\gamma}_0 := \inf_{x\in\Gamma} \widetilde{\gamma}_0(x), \tag{19}$$

where

$$\widetilde{\gamma}_0(x) := 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}|k_n(x)|^{2/3}\Big(\delta_0 + (1-\delta_0)|T(x)\cdot\beta|^2\Big)^{1/3}\,. \tag{20}$$

Here $T(x)$ is the oriented, unit tangent vector to Γ at the point x, $\delta_0 \in]0,1[$ and $\widehat{\nu}_0$ are spectral quantities.

In almost all the previously mentioned statements, we can answer positively to Pb 3.

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We mentioned already this result for $d = 2$, which is also true in the case of the disc, although $\lambda_1(B)$ is somewhat oscillatory.

For $d = 3$, a recent result obtained in collaboration with S. Fournais is the

Theorem 6

Let $\Omega \subset \mathbb{R}^3$ and β satisfying Assumptions C1-C2.
Let us assume that

1. Γ has one component.
2. There exists $x \in \Gamma$ such that $\widetilde{\gamma}_0(x) > \widehat{\gamma}_0$.

Then the left and right derivatives $\lambda'_{1,\pm}$ exist and satisfy

$$\lim_{B\to\infty} \lambda'_{1,+}(B) = \lim_{B\to\infty} \lambda'_{1,-}(B) = \Theta_0. \tag{21}$$

The proof is a mixture of the following considerations :

- Existence of an expansion of the groundstate.
- Fine localization of a groundstate.
- Analytic perturbation theory.
- Particular structure of the hamiltonian which is quadratic in B.

The Ginzburg-Landau functional is given by

$$\mathcal{E}_{\kappa,H}[\psi,\mathbf{A}] = \int_\Omega \left\{ |\nabla_{\kappa H\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + \kappa^2 H^2 |\operatorname{curl}\mathbf{A} - \beta|^2 \right\} dx\,,$$

with

- Ω simply connected,
- $(\psi,\mathbf{A}) \in W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^3)$,
- $\beta = (0,0,1)$,
- $\nabla_{\mathbf{A}} = (\nabla + i\mathbf{A})$.

We fix the choice of gauge by imposing that

$$\operatorname{div}\mathbf{A} = 0 \quad \text{in } \Omega\,,\ \ \mathbf{A}\cdot\nu = 0 \quad \text{on } \partial\Omega\,.$$

The pair $(0, \mathbf{F})$ is called the Normal State.
A minimizer (ψ, A) for which ψ never vanishes will be called SuperConducting State.
In the other cases, one will speak about Mixed State.

The general question is to determine the topology of the subset in $\mathbb{R}^+ \times \mathbb{R}^+$ of the (κ, H) corresponding to minimizers belonging to each of these three situations.

Theorem 7

There exists κ_0 such that, $\forall \kappa \geq \kappa_0$, $(0, \mathbf{F})$ is NOT a global minimizer of $\mathcal{E}_{\kappa,H}$ iff $\lambda_1(\kappa H) < \kappa^2$.

Remark.

This make the monotonicity of λ_1 (for B large), which implies, for κ large, the existence of a unique H such that

$$\lambda_1(\kappa H) = \kappa^2 \,.$$

particularly interesting.

The energy for the model in Liquid Crystals can be written[1] as

$$\mathcal{E}[\psi,\mathbf{n}] = \int_\Omega \Big\{ |\nabla_{q\mathbf{n}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + K_1\,|\,\mathrm{div}\ \mathbf{n}|^2 + K_2\,|\mathbf{n}\cdot\ \mathrm{curl}\ \mathbf{n} + \tau|^2 + K_3\,|\mathbf{n}\times\ \mathrm{curl}\ \mathbf{n}|^2 \Big\}\,dx\,,$$

where :

- $\Omega \subset \mathbb{R}^3$ is the region occupied by the liquid crystal,
- ψ is a complex-valued function called the *order parameter*,
- $\mathbf{n}$ is a real vector field of unit length called *director field*,
- q is a real number called *wave number*,
- τ is a real number measuring the chiral pitch,
- $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ are called the *elastic coefficients*,
- $\kappa > 0$ depends on the material and on temperature.

The questions are then :

What is the minimum of the energy ?

What is the nature of the minimizers ?

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Of course the answer depends heavily on the various parameters !!

As in the theory of superconductivity, a special role will be played by the following critical points of the functional, i.e. the pairs

$$(0, \mathbf{n}) \ ,$$

where **n** should minimize the second part :

$$\int_\Omega \left\{ K_1 \,|\, \operatorname{div} \mathbf{n}|^2 + K_2 \,|\mathbf{n} \cdot \operatorname{curl} \mathbf{n} + \tau|^2 + K_3 \,|\mathbf{n} \times \operatorname{curl} \mathbf{n}|^2 \right\} dx \ .$$

These special solutions are called "nematic phases" and one is naturally asking if they are minimizers or local minimizers of the functional.

For $\tau > 0$, let us consider $\mathcal{C}(\tau)$ the set of the $\mathbb{S}^2$-valued vectors satisfying :

$$\operatorname{curl} \mathbf{n} = -\tau \mathbf{n} \,, \quad \operatorname{div} \mathbf{n} = 0 \,.$$

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It can be shown that $\mathcal{C}(\tau)$ consists of the vector fields $\mathbb{N}_\tau^Q$ such that, for some $Q \in SO(3)$,

$$\mathbb{N}_\tau^Q(x) \equiv Q\mathbb{N}_\tau(Q^t x)\,, \quad \forall x \in \Omega, \tag{22}$$

where

$$\mathbb{N}_\tau(y_1, y_2, y_3) = (\cos(\tau y_3), \sin(\tau y_3), 0)\,,\ \forall y \in \mathbb{R}^3\,. \tag{23}$$

As a consequence, if we denote by

$$C(K_1, K_2, K_3, \kappa, q, \tau) = \inf_{(\psi, \mathbf{n}) \in \mathbb{V}(\Omega)} \mathcal{E}[\psi, \mathbf{n}]\,,$$

the infimum of the energy over the natural maximal form domain of the functional, then

$$C(K_1, K_2, K_3, \kappa, q, \tau) \leq c(\kappa, q, \tau)\;, \tag{24}$$

where

$$c(\kappa, q, \tau) = \inf_{\mathbf{n} \in \mathcal{C}(\tau)} \inf_{\psi} \mathcal{G}_{q\mathbf{n}}(\psi) \tag{25}$$

and $\mathcal{G}_{q\mathbf{n}}(\psi)$ is the so called reduced Ginzburg-Landau functional.

Given a vector field $\mathbf{A}$, this functional is defined on $H^1(\Omega, \mathbb{C})$ by

$$\psi \mapsto \mathcal{G}_{\mathbf{A}}[\psi] = \int_\Omega \{|\nabla_{\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\}\, dx\,. \tag{26}$$

For convenience, we also write $\mathcal{G}_{\mathbf{A}}[\psi]$ as $\mathcal{G}[\psi, \mathbf{A}]$.
So we have

$$c(\kappa, q, \tau) = \inf_{\mathbf{n}\in\mathcal{C}(\tau), \psi\in H^1(\Omega,\mathbb{C})} \mathcal{G}[\psi, q\mathbf{n}]\,. \tag{27}$$

and

$$\mathcal{E}(\psi, \mathbf{n}) = \mathcal{G}[\psi, q\mathbf{n}]\,, \tag{28}$$

if

$$\mathbf{n} \in \mathcal{C}(\tau)\,.$$

We have seen that in full generality that

$$C(K_1, K_2, K_3, \kappa, q, \tau) \leq c(\kappa, q, \tau)\ . \tag{29}$$

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Conversely, it can be shown [BCLP, P2, HP2], that when the elastic parameters tend to $+\infty$, the converse is asymptotically true.

Proposition 1

$$\lim_{K_1, K_2, K_3 \to +\infty} C(K_1, K_2, K_3, \kappa, q, \tau) = c(\kappa, q, \tau)\ . \tag{30}$$

So $c(\kappa, q, \tau)$ is a good approximation for the minimal value of $\mathcal{E}$ for large K_j's.

We now examine the non-triviality of the minimizers realizing $c(\kappa, q, \tau)$.

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As for the Ginzburg-Landau functional in superconductivity, this question is closely related to the analysis of the lowest eigenvalue $\mu(q\mathbf{n})$ of the Neumann realization of the magnetic Schrödinger operator $-\nabla^2_{q\mathbf{n}}$ in Ω.

But the new point is that we will minimize over $\mathbf{n} \in \mathcal{C}(\tau)$. So we shall actually meet

$$\mu_*(q,\tau) = \inf_{\mathbf{n}\in\mathcal{C}(\tau)} \mu(q\mathbf{n}). \tag{31}$$

As a consequence of the analysis, we obtain that the transition from nematic phases to non-nematic phases (the so called smectic phases) is strongly related to the analysis of the solution of

$$1 - \kappa^{-2}\mu_*(q,\tau) = 0\,. \tag{32}$$

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This is a pure spectral problem concerning a family indexed by $\mathbf{n} \in \mathcal{C}(\tau)$ of Schrödinger operators with magnetic field $-\nabla^2_{q\mathbf{n}}$.

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In the analysis of (32), the monotonicity of $q \mapsto \mu_*(q,\tau)$ is an interesting open question (see Fournais-Helffer [FH3] in Surface Superconductivity).

We have proved with Pan that if τ stays in a bounded interval, then this quantity and $\mu_*(q,\tau)$ can be controlled when $\sigma \to +\infty$ where

$$\sigma = q\tau$$

which is in some sense the leading parameter in the theory.

A simpler question which is partially solved in Pan [P2] (with the help of [HM4]) and corresponds to the case $\tau = 0$ is the following :

A simpler question which is partially solved in Pan [P2] (with the help of [HM4]) and corresponds to the case $\tau = 0$ is the following :

Given a strictly convex open set, find the direction **h** of the constant magnetic field giving asymptotically as $\sigma \to +\infty$ the lowest energy for the Neumann realization in Ω of the Schrödinger operator with magnetic field $\sigma\, \mathbf{h}$.

When looking at the general problem, various problems occur. The magnetic field $-q\tau\mathbf{n}$ (corresponding when $\mathbf{n} \in \mathcal{C}(\tau)$ to the magnetic potential $q\mathbf{n}$) is no more constant, so one should extend the analysis of Helffer-Morame [HM4] to this case.

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A first analysis (semi-classical in spirit) gives, as $\sigma = q\tau \to +\infty$,

$$\mu_*(q,\tau) = \Theta_0(q\tau) + \mathcal{O}((q\tau)^{\frac{2}{3}}) \tag{33}$$

where the remainder is controlled uniformly for[2] $\tau \in]0, \tau_0]$.
The analysis of the second term is partially done by Pan (upper bound) and is a Work in Progress (with Pan) for the lower bound.

[2] This condition can be relaxed [Ray] at the price of a worse remainder.

The spectral analysis is based in particular on the analysis of the family

$$H(\xi) = D_t^2 + (t+\xi)^2 \ , \tag{34}$$

on the half-line (Neumann at 0) whose lowest eigenvalue $\mu(\xi)$ admits a unique minimum at $\xi_0 < 0$.

So our two universal constants attached to the problem on $\mathbb{R}^+$ can be now defined by :
The first one is

$$\Theta_0 = \mu(\xi_0)\,. \tag{35}$$

It corresponds to the bottom of the spectrum of the Neumann realization in $\mathbb{R}^2_+$ (with $B = 1$).
Note that

$$\Theta_0 \in]0,1[\,.$$

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Note that

$$\Theta_0 \in]0, 1[\,.$$

The second constant is

$$\delta_0 = \frac{1}{2}\mu''(\xi_0)\,, \tag{36}$$

When the assumptions are not satisfied, and that the magnetic field B vanishes. Other models should be consider. An interesting case is the case when B vanishes along a line. This model was proposed by Montgomery in connection with subriemannian geometry but this model appears also in the analysis of the dimension 3 case.

When the assumptions are not satisfied, and that the magnetic field B vanishes. Other models should be consider. An interesting case is the case when B vanishes along a line. This model was proposed by Montgomery in connection with subriemannian geometry but this model appears also in the analysis of the dimension 3 case.
More precisely, we meet the following family (depending on ρ) of quartic oscillators :

$$D_t^2 + (t^2 - \rho)^2 \,. \tag{37}$$

Denoting by $\nu(\rho)$ the lowest eigenvalue, Pan-Kwek have shown that there exists a unique minimum of $\nu(\rho)$ leading to a new universal constant

$$\hat{\nu}_0 = \inf_{\rho\in\mathbb{R}} \nu(\rho) \,. \tag{38}$$

Let us be more precise on how we can derive the monotonicity result from the known asymptotics of the groundstate energy and localization estimates for the groundstate itself.

Proof of Theorem 6

For simplicity, we assume that Γ is connected. Applying Kato's analytic perturbation theory to $\mathcal{H}(B)$ gives the first part.
Let $s_0 \in \Gamma$ be a point with $\widetilde{\gamma}(s_0) > \widehat{\gamma}_0$. Let $\widehat{\mathbf{A}}$ be the vector potential which is gauge equivalent to $\mathbf{A}$ to be chosen later.

Let $\widehat{Q}_B$ the quadratic form

$$W^{1,2}(\Omega) \ni u \mapsto \widehat{Q}_B(u) = \int_\Omega |-i\nabla u + B\widehat{\mathbf{A}}u|^2 dx\,,$$

and $\widehat{\mathcal{H}}(B)$ be the associated operator.
Then $\widehat{\mathcal{H}}(B)$ and $\mathcal{H}(B)$ are unitarily equivalent:
$\widehat{\mathcal{H}}(B) = e^{iB\phi}\mathcal{H}(B)e^{-iB\phi}$, for some ϕ independent of B.
With $\psi_1^+(\,\cdot\,;\beta)$ being a suitable choice of normalized groundstate, we get (by analytic perturbation theory applied to $\mathcal{H}(B)$ and the explicit relation between $\widehat{\mathcal{H}}(B)$ and $\mathcal{H}(B)$,

$$\begin{aligned} &\lambda_{1,+}'(B) \\ &= \langle \widehat{\mathbf{A}}\psi_1^+(\,\cdot\,;B)\,,\, p_{B\widehat{\mathbf{A}}}\psi_1^+(\,\cdot\,;B)\rangle \\ &\qquad +\langle p_{B\widehat{\mathbf{A}}}\psi_1^+(\,\cdot\,;B)\,,\, \widehat{\mathbf{A}}\psi_1^+(\,\cdot\,;B)\rangle\,. \end{aligned} \tag{39}$$

We now obtain for any $b > 0$,

$$\lambda'_{1,+}(B) = \frac{\widehat{Q}_{B+b}(\psi_1^+(\,\cdot\,;B)) - \widehat{Q}_B(\psi_1^+(\,\cdot\,;B))}{b} \tag{40}$$

$$- b \int_\Omega |\widehat{\mathbf{A}}|^2 \, |\psi_1^+(x;B)|^2 \, dx$$

$$\geq \frac{\lambda_1(B+b) - \lambda_1(B)}{b} - b \int_\Omega |\widehat{\mathbf{A}}|^2 \, |\psi_1^+(x;B)|^2 \, dx \,. \tag{41}$$

We choose $b := MB^{\frac{2}{3}-\eta}$, with η from (18) and $M > 0$ (to be taken arbitrarily large in the end). Then, using (18), (40) becomes

$$\begin{aligned} \lambda'_{1,+}(B) \quad & \geq \Theta_0 + \widehat{\gamma}_0 B^{-1/3} \tfrac{(1+b/B)^{2/3}-1}{b/B} \\ & - CM^{-1} - b \textstyle\int_\Omega |\widehat{\mathbf{A}}|^2 \, |\psi_1^+(x;B)|^2 \, dx \,, \end{aligned} \tag{42}$$

for some constant C independent of M, B.

If we can prove that we can find $\widehat{A}$ such that

$$B^{\frac{2}{3}} \int_\Omega |\widehat{\mathbf{A}}(x)\,|^2\, |\psi_1^+(x;B)|^2\, dx \leq C, \tag{43}$$

for some constant C independent of B, then we can take the limit $B \to \infty$ in (42) and obtain

$$\liminf_{B\to\infty} \lambda'_{1,+}(B) \geq \Theta_0 - CM^{-1}. \tag{44}$$

Since M was arbitrary this implies the lower bound for $\lambda'_{1,+}(B)$.
Applying the same argument to the derivative from the left, $\lambda'_{1,-}(B)$, we get (the inequality gets turned since $b<0$)

$$\limsup_{B\to\infty} \lambda'_{1,-}(B) \leq \Theta_0. \tag{45}$$

Since, by perturbation theory, $\lambda'_{1,+}(B) \leq \lambda'_{1,-}(B)$ for all B, we get (21).
Thus it remains to prove (43).

We can estimate

$$\begin{aligned}&\int_\Omega |\widehat{\mathbf{A}}|^2\, |\psi_1^+(x;B)|^2\, dx\\ &\leq C\int_{\Omega\left(\epsilon,s_0\right)}(t^2+r^4)|\psi_1^+(x;B)|^2\, dx\\ &\quad+\|\widehat{\mathbf{A}}\|_\infty^2\int_{\Omega\setminus\Omega\left(\epsilon,s_0\right)}|\psi_1^+(x;B)|^2\, dx\,.\end{aligned}$$

Combining with decay properties, we therefore find the existence of a constant $C>0$ such that :

$$\int_\Omega |\widehat{\mathbf{A}(\mathbf{x})}|^2\, |\psi_1^+(x;B)|^2\, dx \leq C\, B^{-1}\,, \tag{46}$$

which is stronger than the needed estimate (43).

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