Boundary behavior of analytic functions on $\mathbb{D}$

Javad Mashreghi

February 10, 2012

Outline

The class $\mathcal{H}ol(\mathbb{D})$

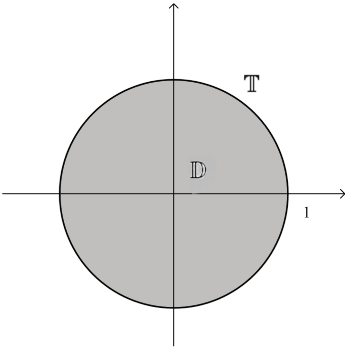


Figure: The open unit disc $\mathbb{D}$ and its boundary $\mathbb{T}$.

$$\mathcal{H}ol(\mathbb{D}) = \{f : f \text{ is analytic on } \mathbb{D}\}.$$

Six questions on boundary behavior

Question 1:

Let $f \in \mathcal{H}ol(\mathbb{D})$, and let $\zeta \in \mathbb{T}$.

Does f have an analytic continuation to an open neighborhood of ζ?

Six questions on boundary behavior

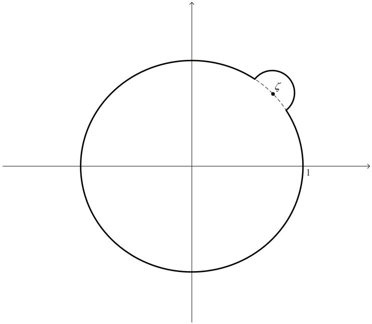


Figure: Analytic continuation across ζ.

Six questions on boundary behavior

Question 2:

Let $f \in \mathcal{H}ol(\mathbb{D})$, and let $\zeta \in \mathbb{T}$.

Does the (non-restrictive) limit

$$\lim_{\substack{z \longrightarrow \zeta \\ z \in \mathbb{D}}} f(z)$$

exit?

Six questions on boundary behavior

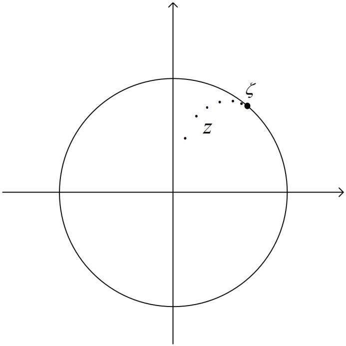


Figure: The (nonrestrictive) limit at ζ.

Six questions on boundary behavior

Question 3:

Let $f \in \mathcal{H}ol(\mathbb{D})$, and let $\zeta \in \mathbb{T}$.

Does the nontangential limit

$$f^{\triangleleft *}(\zeta) = \lim_{z \overset{\triangleleft}{\to} \zeta} f(z)$$

exit?

Six questions on boundary behavior

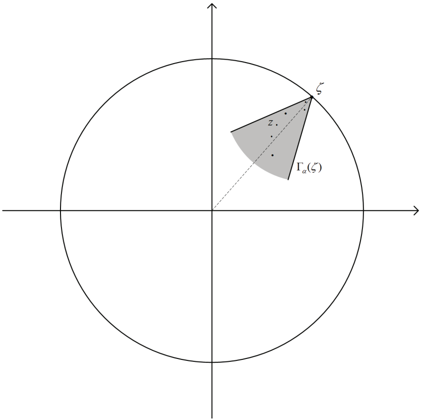


Figure: The nontangential limit at ζ.

Six questions on boundary behavior

Question 4:

Let $f \in \mathcal{H}ol(\mathbb{D})$, and let $\zeta \in \mathbb{T}$.

Does the radial limit

$$f^*(\zeta) = \lim_{r \to 1} f(r\zeta)$$

exit?

Six questions on boundary behavior

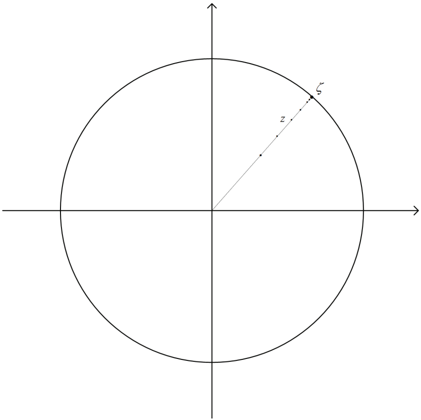


Figure: The radial limit at ζ.

Six questions on boundary behavior

Question 5:

Assuming that $f^{\triangleleft *}$ exists for almost all $\zeta \in \mathbb{T}$, can we recover $f(z)$, $z \in \mathbb{D}$, from $f^{\triangleleft *}$?

Six questions on boundary behavior

Question 5:

Assuming that $f^{\lhd *}$ exists for almost all $\zeta \in \mathbb{T}$, can we recover $f(z)$, $z \in \mathbb{D}$, from $f^{\lhd *}$?

Question 6:

Assuming that f^{*} exists for almost all $\zeta \in \mathbb{T}$, can we recover $f(z)$, $z \in \mathbb{D}$, from f^{*}?

Monsters!

Answer to Q1: No, $\mathbb{T}$ is a *natural boundary*, i.e. there is an analytic function on $\mathbb{D}$ which cannot be analytically extended to any larger domain.

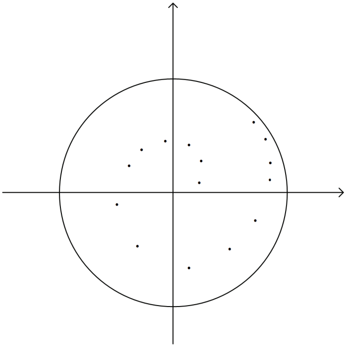
Weierstrass

Let $(z_n)_{n\geq 1}$ be any sequence in $\mathbb{D}$ such that $\lim_{n\to\infty} |z_n| = 1$, and each point of $\mathbb{T}$ is an accumulation point of the sequence $(z_n)_{n\geq 1}$, e.g.

$$z_n = (1-\varepsilon_n)\, e^{i\theta_n}, \qquad (\varepsilon_n \longrightarrow 0),$$

where $(\theta_n)_{n\geq 1}$ is an enumeration of $\mathbb{Q}$. Then there is a nonconstant analytic function f on $\mathbb{D}$ such that $f(z_n) = 0$.

Monsters!



By the uniqueness theorem for analytic functions, f cannot be analytically extended across any point of $\mathbb{T}$.

Monsters!

Blaschke (1915)

Let $(z_n)_{n\geq 1}$ any sequence in $\mathbb{D}$ such that

$$\sum_{n=1}^{\infty}(1-|z_n|) < \infty,$$

and each point of $\mathbb{T}$ is an accumulation point of the sequence $(z_n)_{n\geq 1}$. Then

$$B(z) = \prod_n \frac{|z_n|}{z_n}\,\frac{z_n - z}{1-\overline{z}_n\,z},$$

is a *bounded* analytic function on $\mathbb{D}$ with $f(z_n) = 0$.

Monsters!

Answer to Q2: Unrestricted limits of an analytic function may fail to exist even at *all* points of $\mathbb{T}$.

Lohwater–Piranian (1957), Littlewood (1927)

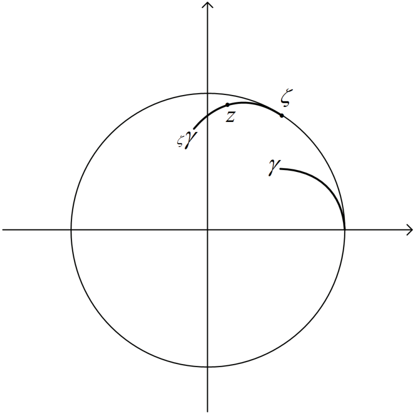
Let γ be a simple closed Jordan curve which is internally tangent to $\mathbb{T}$ at the point 1. Then there exists a bounded analytic function f on $\mathbb{D}$ such that

$$\lim_{\substack{z\longrightarrow\zeta \\ z\in\zeta\gamma}} f(z)$$

does not exist for any $\zeta\in\mathbb{T}$.

Remark: "almost everywhere" version is due to Littlewood.

Monsters!



Monsters!

Answer to Q3, Q4: Radial limits of an analytic function may fail to exist even at *all* points of $\mathbb{T}$.

Littlewood (1930)

Let $(a_n)_{n\geq 1}$ be a sequence on complex numbers such that

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1 \qquad \text{and} \qquad \sum_{n=1}^{\infty} |a_n|^2 = \infty.$$

Then for almost every choice of the signs $\varepsilon_n = \pm 1$, the function

$$f(z) = \sum_{n=1}^{\infty} \varepsilon_n \, a_n \, z^n$$

has a radial limit almost nowhere on $\mathbb{T}$.

Monsters!

Bagemihl–Seidel (1954), Rudin (1954)

For any continuous function φ on $\mathbb{D}$, and any set E of first category on $\mathbb{T}$, there is an analytic function f on $\mathbb{D}$ such that

$$\lim_{r\to 1} \big(f(r\zeta) - \varphi(r\zeta) \big) = 0$$

for all $\zeta \in E$.

Remark: There is a set E of first category such that $|E| = 2\pi$.

Monsters!

Maclane (1962)

There exists an analytic function f on $\mathbb{D}$ (even without any zeros and satisfying a certain growth condition) such that

$$\liminf_{r\to 1} |f(r\zeta)| = 0 \qquad \text{and} \qquad \limsup_{r\to 1} |f(r\zeta)| = \infty$$

for all $\zeta \in \mathbb{T}$.

Monsters!

Answer to Q6: No. But, there is hope!

Littlewood (1927)

There exists a nonzero analytic function f on $\mathbb{D}$ such that

$$f^*(\zeta) = \lim_{r\to 1} f(r\zeta) = 0, \qquad \text{(a.e. } \zeta \in \mathbb{T}).$$

Monsters!

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F. and M. Riesz (1916)

Let f be a bounded analytic function on $\mathbb{D}$. Assume that there is a set $E \subset \mathbb{T}$, with $|E| > 0$, such that

$$f^*(\zeta) = \lim_{r\to1} f(r\zeta) = 0, \qquad (\zeta \in E).$$

Then $f \equiv 0$.

Monster of monsters!!!

Frostman (1942)

There is a Blaschke product B such that

(i) For each $\zeta \in \mathbb{T}$,
$$\lim_{z\to\zeta} B(z) = \overline{\mathbb{D}}.$$

(ii) For each $\zeta \in \mathbb{T}$,
$$B^{\triangleleft *}(\zeta) = \lim_{z \overset{\triangleleft}{\to} \zeta} B(z)$$
exists and, moreover,
$$|B^{\triangleleft *}(\zeta)| = 1.$$

Positive results

Answer to Q5: In principle ‘Yes’.

Lusin–Privalov (1925)

Let f be an analytic function on $\mathbb{D}$ such that

$$f^{\lhd *}(\zeta) = \lim_{z \overset{\lhd}{\longrightarrow} \zeta} f(z) = 0$$

for all $\zeta \in E$, where E is a Borel subset of $\mathbb{T}$ with $|E| > 0$. Then

$$f \equiv 0.$$

Recovering f from $f^{\lhd *}$ is a more delicate problem.

Positive results

Fatou (1906)

Let f be a bounded analytic function on $\mathbb{D}$. Then

$$f^{\lhd *}(\zeta) = \lim_{z \overset{\lhd}{\to} \zeta} f(z)$$

exists for all $\zeta \in \mathbb{T}$. Moreover,

$$f(z) = \int_0^{2\pi} \frac{1-|z|^2}{|z-\zeta|^2} \, f^*(\zeta)\, dm(\zeta), \qquad (z \in \mathbb{D}).$$

Positive results

Lindelöf (1915)

Let f be a bounded analytic function on $\mathbb{D}$. Assume that

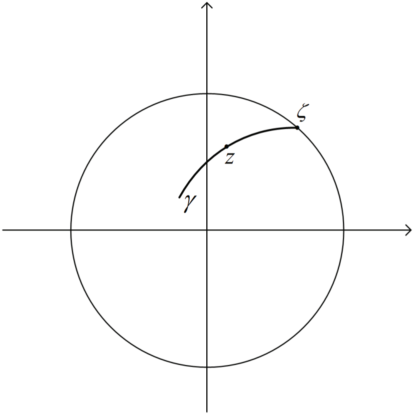
$$\lim_{\substack{z \longrightarrow \zeta \\ z \in \gamma}} f(z),$$

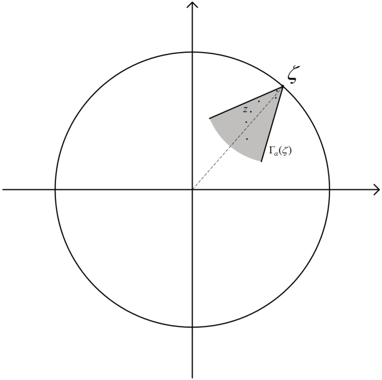
where γ is a curve inside $\mathbb{T}$ terminating at $\zeta \in \mathbb{T}$, exists. Then

$$f^{\lhd *}(\zeta) = \lim_{z \overset{\lhd}{\longrightarrow} \zeta} f(z)$$

exists.

Positive results





Positive results

Plessner (1927)

Let f be an analytic (even meromorphic) function on $\mathbb{D}$, and let $\zeta \in \mathbb{T}$. Then one of the following situations hold:

1. $f^{\lhd *}(\zeta)$ exists.
2. $f(\Gamma_\alpha(\zeta))$ is dense in the Riemann sphere for all α.

Definition

Put

$$\begin{aligned} m_p(f,r) &= \left(\int_0^{2\pi} |f(r\zeta)|^p \, dm(\zeta)\right)^{1/p}, \qquad (0 < p < \infty) \\ m_\infty(f,r) &= \max_{\zeta\in\mathbb{T}} |f(r\zeta)| \end{aligned}$$

and

$$\|f\|_p = \sup_{0<r<1} m_p(f,r).$$

Then

$$H^p = \{f : \|f\|_p < \infty\}.$$

The first brick

Hardy (1914)

Let f be an analytic function on $\mathbb{D}$, and let $0 < p \leq \infty$. Then the following hold:

1. $m_p(f,r)$ is an increasing function of r;
2. $\log m_p(f,r)$ is is a convex function of $\log r$.

Remark: The case $p = \infty$ is due to Hadamard and is known as the *Hadamard's three circle theorem*.

Boundary behavior

Riesz (1923), Smirnov (1929)

Let $f \in H^p$, $0 < p \le \infty$, $f \not\equiv 0$. Then f has a unique canonical factorization of the form $f = B\,S\,h$, where

$$B(z) = \gamma \prod_n \frac{|z_n|}{z_n} \, \frac{z_n - z}{1 - \bar{z}_n\, z},$$

is the Blaschke product formed with the zeros of f,

$$S(z) = \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\sigma(\zeta) \right),$$

is an inner function formed with the singular measure σ, and

$$h(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, \log|f^*(\zeta)| \, dm(\zeta) \right).$$

Boundary behavior

There are some results hidden in the previous theorem which are important by themselves:

1. the zeros of f satisfy the Blaschke condition

$$\sum_n (1 - |z_n|) < \infty;$$

2. $f^{\triangleleft *}(\zeta)$ exists for almost all $\zeta \in \mathbb{T}$;
3. $\log|f^*| \in L^1(\mathbb{T})$, i.e.

$$\int_{\mathbb{T}} \Big| \log|f^*(\zeta)| \Big| \, dm(\zeta) < \infty.$$

Boundary behavior

More comments:

1. The function $\Theta = BS$ is called the inner part of f. Generally speaking, any bounded function whose boundary values are unimodular almost everywhere on $\mathbb{T}$ is called an inner function. The theorem reveals that each inner function Θ has the unique decomposition $\Theta = BS$.
2. h is called the outer part of f.
3. given any $\varphi \geq 0$, $\varphi \in L^p(\mathbb{T})$, $\log \varphi \in L^1(\mathbb{T})$, we can construct the outer function

$$h(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta)\, dm(\zeta) \right) \in H^p.$$

Boundary behavior

Fatou–Riesz–Smirnov

Let $f \in H^1$. Then

$$f^*(\zeta) = \lim_{r \to 1} f(r\zeta)$$

exists for almost all $\zeta \in \mathbb{T}$ and, moreover,

$$f(z) = \int_0^{2\pi} \frac{1 - |z|^2}{|z - \zeta|^2} \, f^*(\zeta) \, dm(\zeta), \qquad (z \in \mathbb{D}).$$

Definition

Consider the forward shift operator

$$\begin{array}{rccc} S: & H^2 & \longrightarrow & H^2 \\ & f & \longmapsto & zf. \end{array}$$

Question: What are the ‘closed invariant’ subspaces of H^2, i.e. $M \subset H^2$ with

$$SM \subset M?$$

Definition

It is clear that if Θ is an inner function, then $M = \Theta H^2$ is a closed invariant subspace of H^2.

Beurling (1949)

Let M be closed invariant subspace of H^2. Then there exists a (unique) inner function Θ such that $M = \Theta H^2$.

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Beurling (1949)

Let M be closed invariant subspace of H^2. Then there exists a (unique) inner function Θ such that $M = \Theta H^2$.

Corollary

Let M be closed subspace of H^2. Then M is invariant under S^* if and only if there exists an inner function Θ such that $M = (\Theta H^2)^\perp$.

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Model subspace

$$K_\Theta = (\Theta H^2)^\perp.$$

Boundary behavior

A general principle:

Each $f \in K_\Theta$ is nice at $\zeta \in \mathbb{T} \iff \Theta$ is nice at $\zeta \in \mathbb{T}$.

What does this mean?!

Boundary behavior

Put $S_\Theta = P_\Theta S\, i_\Theta$:

$$K_\Theta \xrightarrow{i_\Theta} H^2 \xrightarrow{S} H^2 \xrightarrow{P_\Theta} K_\Theta.$$

Helson (1964)

Let Θ be an inner function, and let $\zeta \in \mathbb{T}$. Then the following are equivalent:

1. Θ has an analytic continuation across ζ.
2. Each $f \in K_\Theta$ has an analytic continuation across ζ.
3. The operator $I - \overline{\zeta}\, S_\Theta$ is invertible.

Boundary behavior

Ahern–Clark (1969)

Let $\Theta = BS$ be an inner function, and let $\zeta \in \mathbb{T}$. Then the following are equivalent:

1. $$\sum_n \frac{1 - |z_n|^2}{|1 - \overline{\zeta}\, z_n|^2} + \int_{\mathbb{T}} \frac{d\sigma(\tau)}{|1 - \overline{\zeta}\, \tau|^2} < \infty.$$
2. For each $f \in K_\Theta$, the nontangential limit $f^{\lhd *}(\zeta)$ exists.
3. The function $S_\Theta P_\Theta 1$ is in the range of operator $I - \overline{\zeta}\, S_\Theta$.

Boundary behavior

Ahern–Clark (1969)

Let $\Theta = BS$ be an inner function, let $N \geq 0$, and let $\zeta \in \mathbb{T}$. Then the following are equivalent:

1. $$\sum_n \frac{1-|z_n|^2}{|1-\overline{\zeta}\, z_n|^{2N+2}} + \int_{\mathbb{T}} \frac{d\sigma(\tau)}{|1-\overline{\zeta}\,\tau|^{2N+2}} < \infty.$$
2. For each $f \in K_\Theta$, and its derivative up to order N, have nontangential limits at ζ.
3. The function $S_\Theta^N P_\Theta 1$ is in the range of operator $(I - \overline{\zeta}\, S_\Theta)^{N+1}$.

Definition

Let $\varphi \in L^\infty(\mathbb{T})$. Then the Toeplitz operator T_φ is

$$H^2 \xrightarrow{i_+} L^2 \xrightarrow{M_\varphi} L^2 \xrightarrow{P_+} H^2,$$

i.e. $T_\varphi = P_+ \, M_\varphi \, i_+$.

In particular,

$$S = T_z$$

and

$$S^* = T_{\bar{z}}.$$

Definition

Theorem

Let $\varphi \in L^\infty(T)$. Then

$$\| T_\varphi \|_{H^2 \to H^2} = \| \varphi \|_{L^\infty(\mathbb{T})}.$$

Definition

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Let $\varphi \in L^\infty(T)$. Then

$$\| T_\varphi \|_{H^2 \to H^2} = \| \varphi \|_{L^\infty(\mathbb{T})}.$$

Corollary

Let $\varphi \in L^\infty(T)$, with $\|\varphi\|_\infty \leq 1$. Then

$$I - T_{\overline{\varphi}} T_\varphi \geq 0.$$

Definition

Let $b \in H^\infty$, b nonconstant, $||b||_\infty \leq 1$. Then

$$\mathcal{H}(b) = \mathcal{R}\Big((I - T_b T_{\bar{b}})^{1/2}\Big) = (I - T_b T_{\bar{b}})^{1/2} H^2,$$

endowed with the inner product

$$\Big\langle (I - T_b T_{\bar{b}})^{1/2} f,\ (I - T_b T_{\bar{b}})^{1/2} g \Big\rangle_{\mathcal{H}(b)} = \langle f, g \rangle_{H^2},$$

where $f \perp \ker(I - T_b T_{\bar{b}})$ and $g \perp \ker(I - T_b T_{\bar{b}})$.

Some properties

(i) $\mathcal{H}(b)$ is a reproducing kernel Hilbert space, with kernel

$$k_w^b(z) = \frac{1 - \overline{b(w)}\, b(z)}{1 - \bar{w}\, z}.$$

(ii) $\mathcal{H}(b)$ is boundedly inside H^2, i.e.

$$\|f\|_{H^2} \le \|f\|_{\mathcal{H}(b)}, \qquad (f \in \mathcal{H}(b)).$$

(iii) $\mathcal{H}(b)$ is invariant under S^*. Put $X_b = S^*|\mathcal{H}(b)$.

(iv) $\mathcal{H}(b)$ is a closed subspace of H^2 if and only if b is an inner.

(v) If $b = \Theta$, an inner function, then $\mathcal{H}(b) = K_\Theta$.

Boundary behavior

Fricain–M. (2008)

Let $b \in H^\infty$, b nonconstant, $\|b\|_\infty \leq 1$, and let $\zeta \in \mathbb{T}$. Then the following are equivalent:

1. $$\sum_n \frac{1-|z_n|^2}{|1-\overline{\zeta}\, z_n|^2} + \int_{\mathbb{T}} \frac{d\sigma(\tau)}{|1-\overline{\zeta}\,\tau|^2} + \int_{\mathbb{T}} \frac{-\log|b^*(\tau)|}{|1-\overline{\zeta}\,\tau|^2} dm(\tau) < \infty.$$
2. For each $f \in \mathcal{H}(b)$, the radial limit $f^*(\zeta)$ exists.
3. The function k_0^b is in the range of operator $I - \overline{\zeta} X_b^*$.

Boundary behavior

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Let $b \in H^\infty$, b nonconstant, $\|b\|_\infty \leq 1$, and let $\zeta \in \mathbb{T}$. Let $N \geq 0$. Then the following are equivalent:

1. $$\sum_n \frac{1-|z_n|^2}{|1-\overline{\zeta}\, z_n|^{2N+2}} + \int_{\mathbb{T}} \frac{d\sigma(\tau)}{|1-\overline{\zeta}\, \tau|^{2N+2}} + \int_{\mathbb{T}} \frac{-\log|b^*(\tau)|}{|1-\overline{\zeta}\, \tau|^{2N+2}} dm(\tau) < \infty.$$
2. For each $f \in \mathcal{H}(b)$, the radial limits of f and its derivatives up to order N at ζ exist.
3. The function $X_b^{*N} k_0^b$ is in the range of operator $(I - \overline{\zeta}\, X_b^*)^{N+1}$.

Our behavior!

Thank you.