Bistable Waves for Differential-Difference Equations with Inhomogeneous Diffusion

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One-Day Workshop on Lattice, Delay

and Functional Differential Equations

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Bistable Waves in Discrete Inhomogeneous Media

Collaborators

- Tony Humphries, McGill University
- Erik Van Vleck, University of Kansas

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Contents

- Introduction to the problem and history
- Derivation of solutions
- Propagation failure

Bistable Equation with Inhomogeneous Diffusion

$$\dot{u}_j = \alpha_{j+1}(u_{j+1} - u_j) - \alpha_j(u_j - u_{j-1}) - f(u_j)$$

for

$$\alpha_j = \begin{cases} \beta_j & -m \leq j \leq m \\ \alpha & |j| > m \end{cases}$$

Diffusion coefficients vary on the lattice to give a region of decreased excitability.

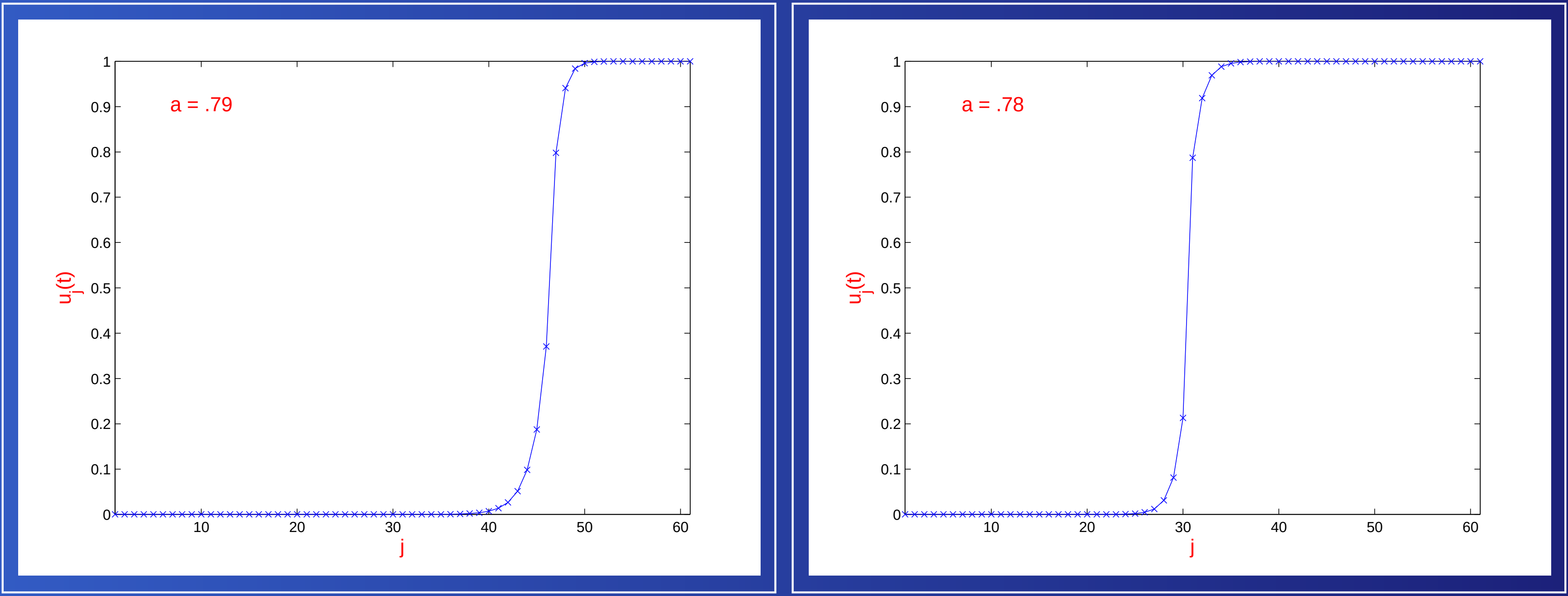
The nonlinearity

$$f(u) = u(u-a)(u-1)$$

is the derivative of a double-well potential W(u).

Numerical Simulations for the Evolution Equation

For the case of a single defect
$$\alpha_j = \begin{cases} 0.6 & j = 30 \\ 1 & j \neq 30 \end{cases}$$



A slightly slower wave is stopped by the defect.

Bistable Waves in Discrete Homogeneous Media

Cahn, Mallet-Paret, and Van Vleck (1998) derive an analytic solution for $\alpha_j = \alpha$ for all $j \in \mathbb{Z}$.

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Traveling Wave ansatz

$$u_j(t) = \phi(\xi) \qquad \text{for} \qquad \xi = j - ct$$

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$$-c\phi'(\xi) = \alpha(\phi(\xi+1) - 2\phi(\xi) + \phi(\xi-1)) - f(\phi(\xi))$$

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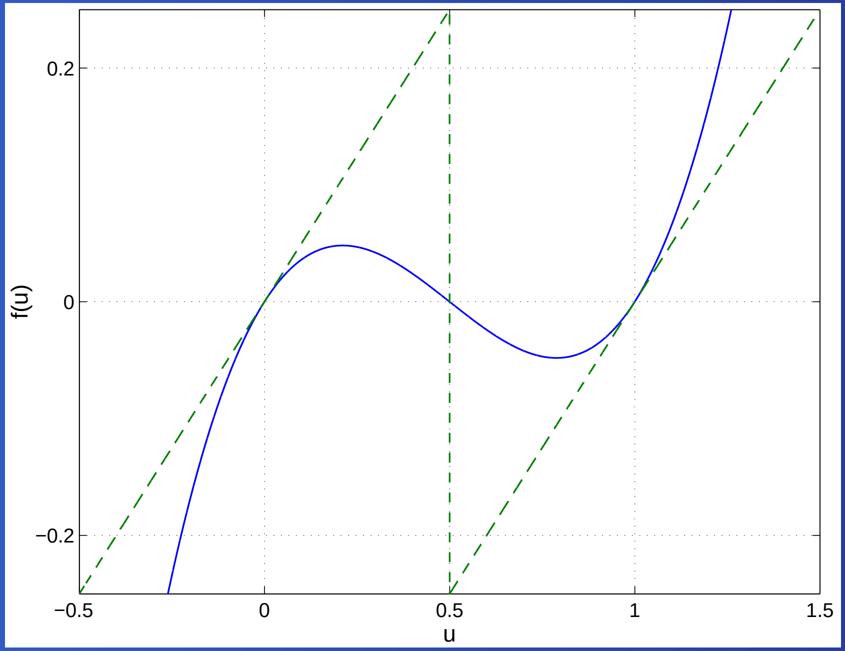
Seek solutions with $\phi(-\infty) = 0$, $\phi(\infty) = 1$ and $\phi(0) = a$ with

$$\phi(\xi) < a \quad \text{for} \quad \xi < 0 \qquad \text{and} \qquad \phi(\xi) > a \quad \text{for} \quad \xi > 0.$$

McKean's Caricature of the Cubic



$$f(\phi) = \begin{cases} d(\phi - 1), & \phi > a, \\ d[\phi - 1, \phi], & \phi = a, \\ d\phi, & \phi < a. \end{cases}$$

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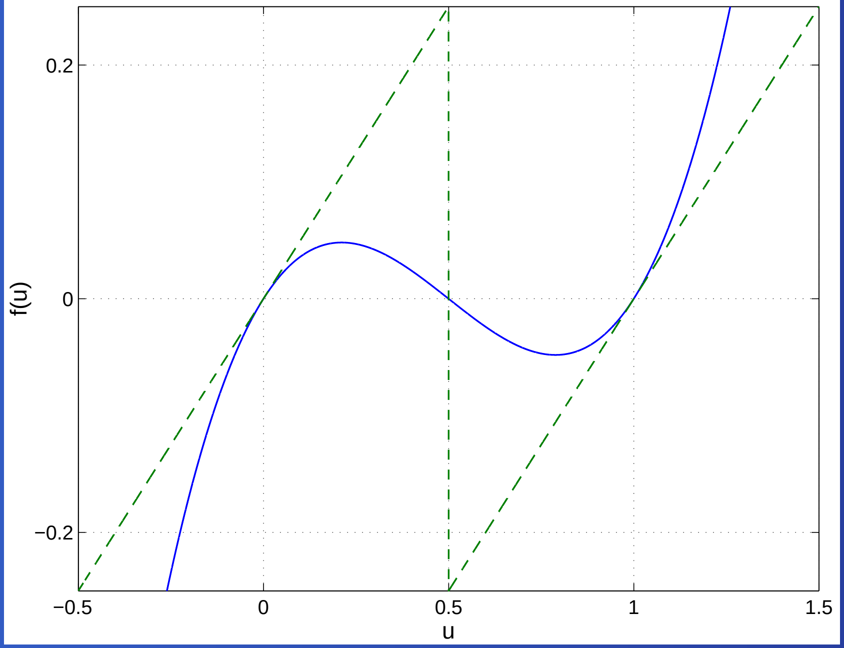

$$f(\phi) = \begin{cases} d(\phi - 1), & \phi > a, \\ d[\phi - 1, \phi], & \phi = a, \\ d\phi, & \phi < a. \end{cases}$$

or $$f(\phi(\xi)) = \phi(\xi) - h(\phi(\xi) - a) = \phi(\xi) - h(\xi),$$

where h is the Heaviside function.

An Analytic Solution via Fourier Transform

Using the Fourier transform

$$\hat{\phi}(s) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}s\xi}\phi(\xi)ds$$

gives the solution

$$\Phi(\xi) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \frac{A(s)\sin(s\xi)}{s(A(s)^2 + c^2 s^2)}ds + \frac{c}{\pi}\int_0^{\infty} \frac{\cos s\xi}{A(s)^2 + c^2 s^2}ds$$

where

$$A(s) = 1 + 2\alpha(1 - \cos(s)).$$

Detuning Parameter a vs. Wave Speed c

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- a fixed $\Longrightarrow$ the wave form and c will depend on the position of the wave in the media.

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Ansatz for Inhomogeneous Diffusion

Traveling Wave ansatz

$$u_j(t) = \phi(\xi;\xi^*) \qquad \text{for} \qquad \xi = j - ct$$

and for some $\xi^* \in \mathbb{R}$ gives

$$-c\phi'(\xi) = \alpha_j(\phi(\xi+1) - \phi(\xi)) + \alpha_{j-1}(\phi(\xi-1) - \phi(\xi)) - f(\phi(\xi))$$

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Seek solutions with

$$\phi(-\infty; \xi^*) = 0, \qquad \phi(\infty; \xi^*) = 1, \qquad \phi(\xi^*; \xi^*) = a(\xi^*)$$

and

$$\begin{aligned} \phi(\xi; \xi^*) < a(\xi^*) \quad &\text{for} \quad \xi < \xi^* \\ \phi(\xi; \xi^*) > a(\xi^*) \quad &\text{for} \quad \xi > \xi^*. \end{aligned}$$

Significance of ξ^*

- Each ξ^* defines a new problem.

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- Variation of ξ^* allows one to translate the wave with respect to the defect.
- The result is

$$a(\xi^*) = \phi(\xi^*;\xi^*) \equiv \Gamma(\xi^*, \alpha_j, c).$$

Case: A Single Defect

If $\alpha_j = \alpha$ for $j \neq 0$, and $\gamma = \alpha_0 - \alpha$, then

$$
\begin{aligned}
-c\phi'(\xi) &= \alpha(\phi(\xi+1) - 2\phi(\xi) + \phi(\xi-1)) - f(\phi(\xi)) \\
&+ \gamma\delta(\xi-\xi_0)(\phi(\xi+1) - \phi(\xi)) \\
&+ \gamma\delta(\xi-\xi_1)(\phi(\xi-1) - \phi(\xi))
\end{aligned}
$$

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$$\begin{aligned} -c\phi'(\xi) &= \alpha(\phi(\xi+1) - 2\phi(\xi) + \phi(\xi-1)) - f(\phi(\xi)) \\ &+ \gamma\delta(\xi-\xi_0)(\phi(\xi+1) - \phi(\xi)) \\ &+ \gamma\delta(\xi-\xi_1)(\phi(\xi-1) - \phi(\xi)) \end{aligned}$$

Fourier transform implies

$$\phi(\xi;\xi^*) = \Phi(\xi-\xi^*) + \gamma(\phi(\xi_0;\xi^*) - \phi(\xi_1;\xi^*))(F(\xi-\xi_1) - F(\xi-\xi_0))$$

for

$$F(x) = \frac{2}{\pi}\int_0^\infty \frac{A(s)\cos(sx) - cs\sin(sx)}{s(A(s)^2 + c^2s^2)}ds.$$

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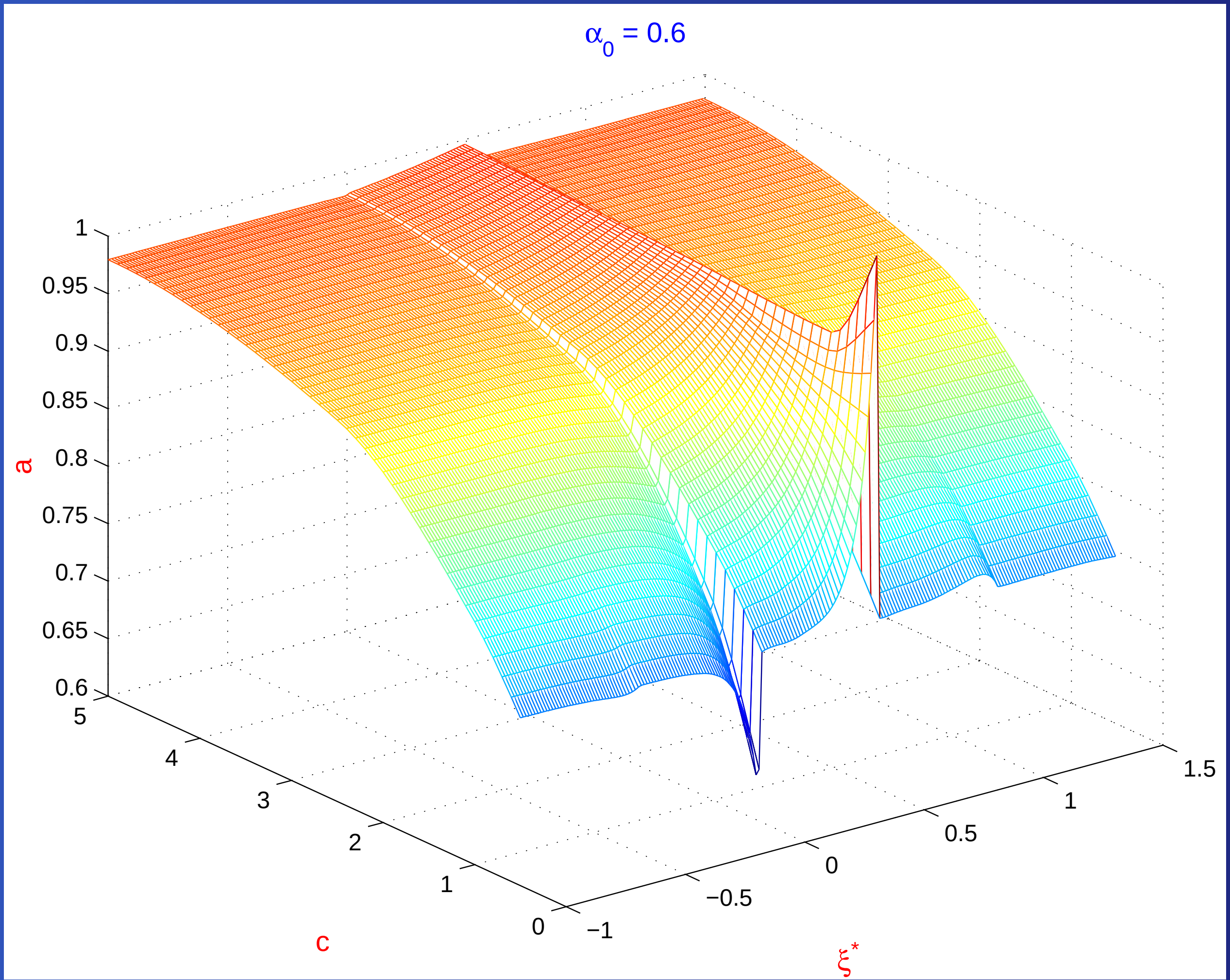
$a(c, \xi^*)$



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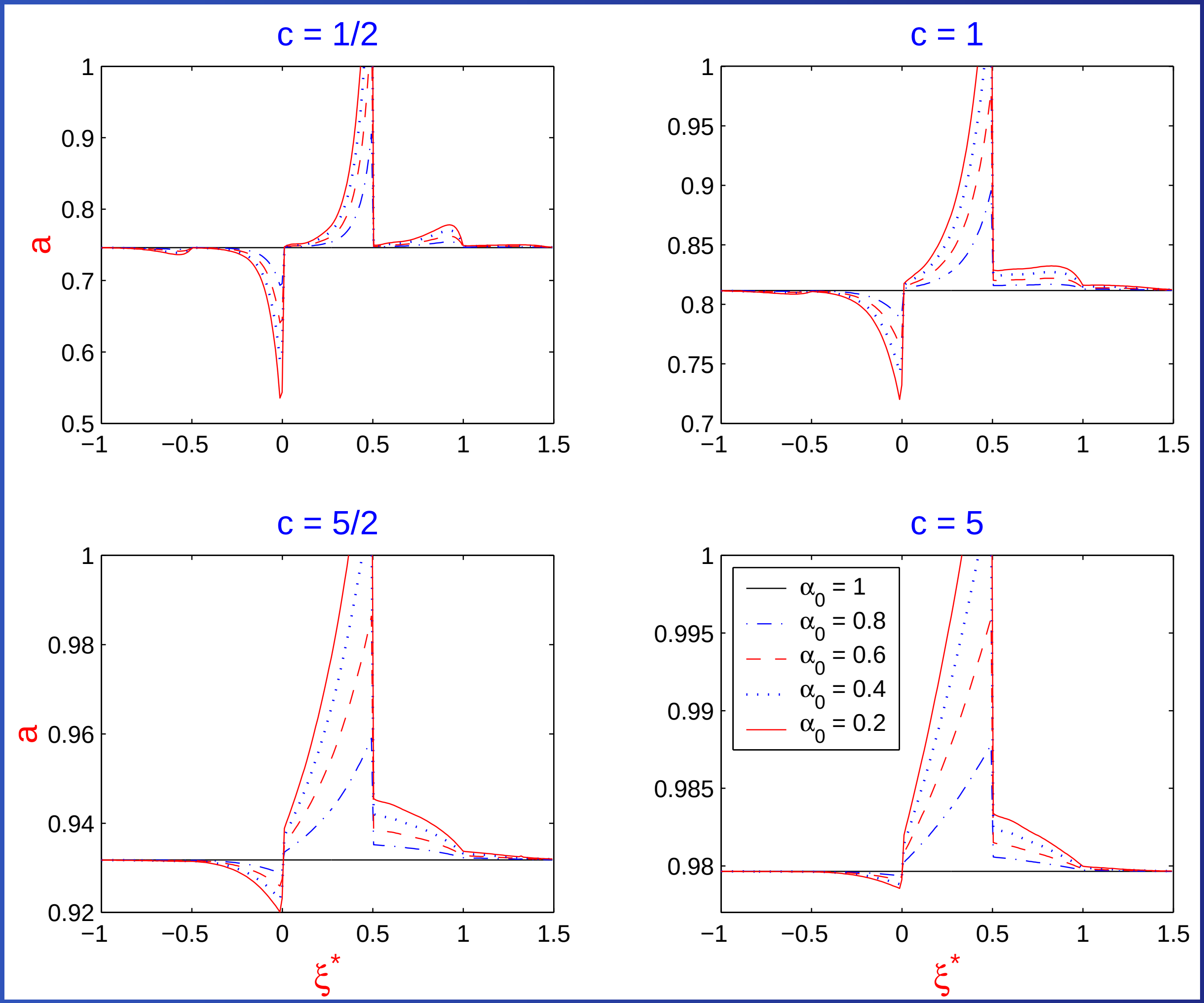
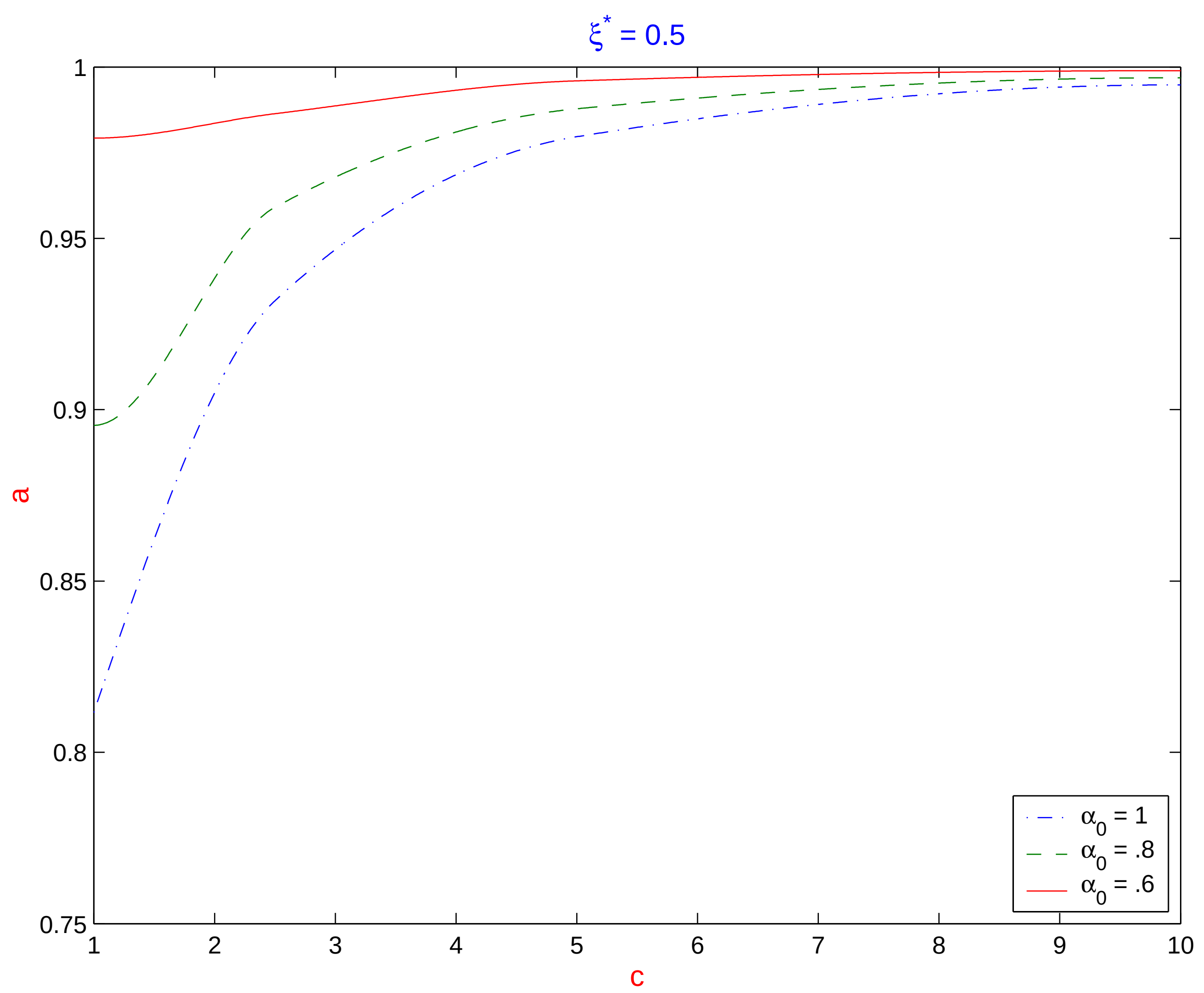
$a(\xi^*)$



$a(c)$



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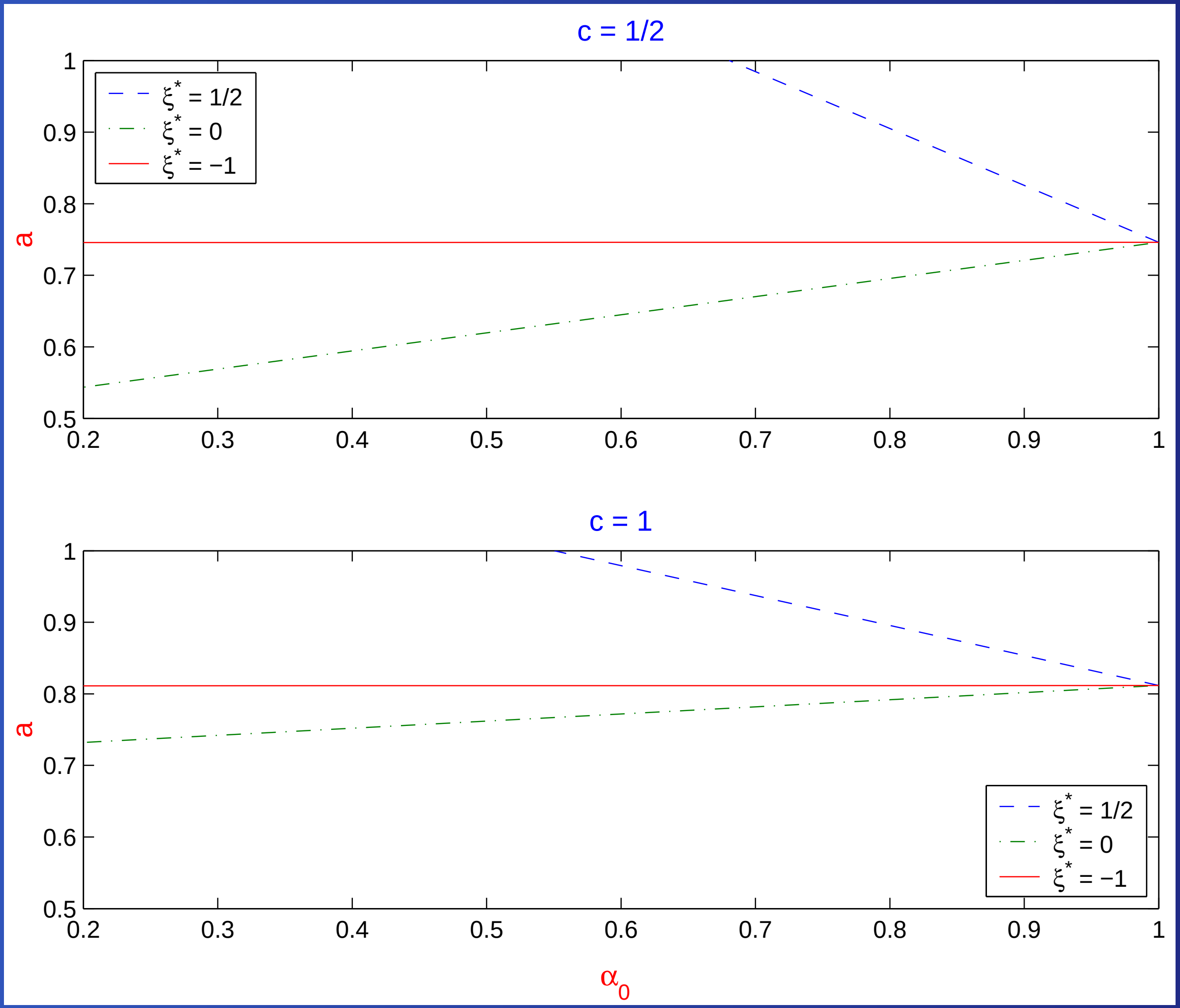
$a(\alpha_0)$



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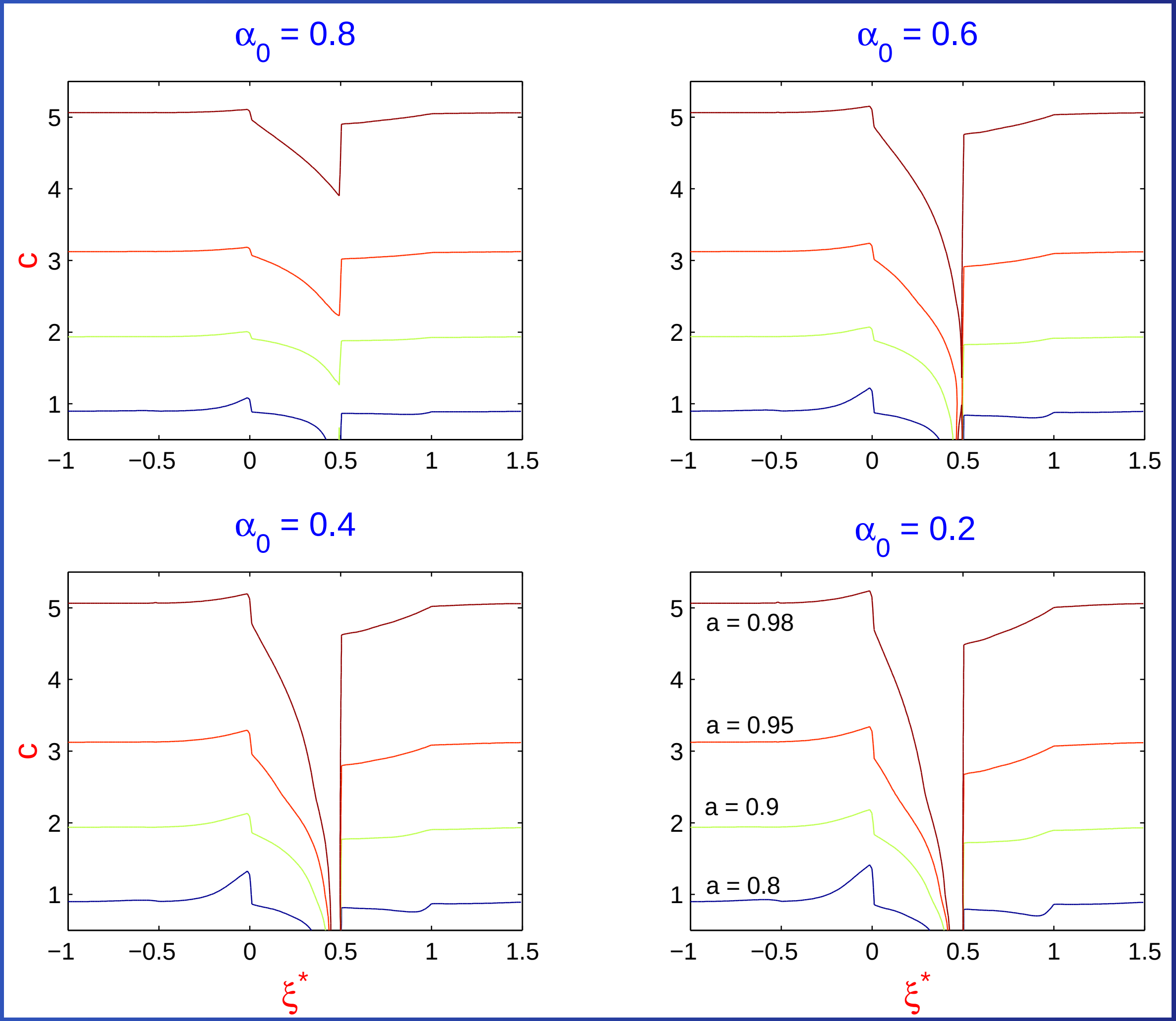
$c(\xi^*)$



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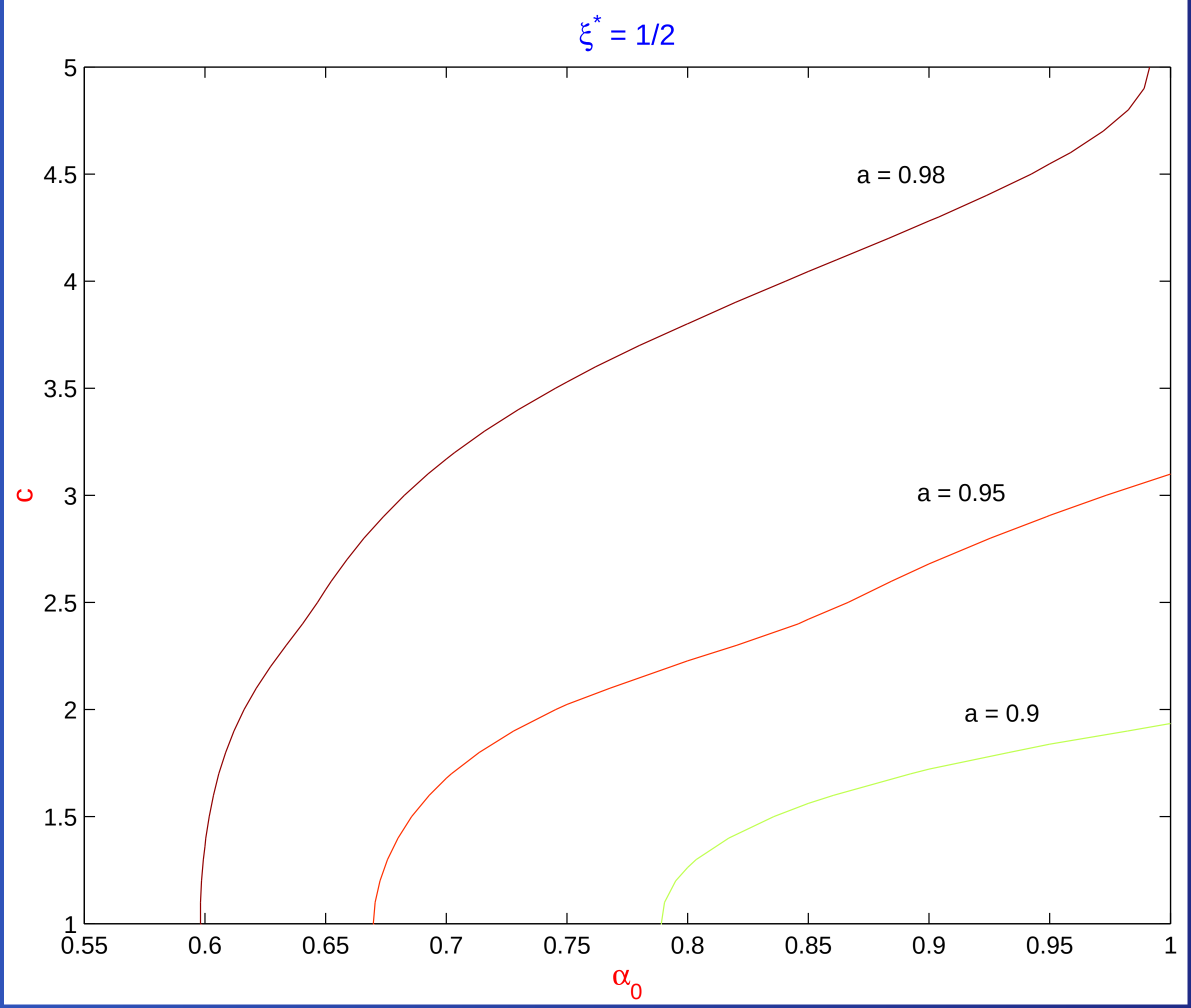
$c(\alpha_0)$



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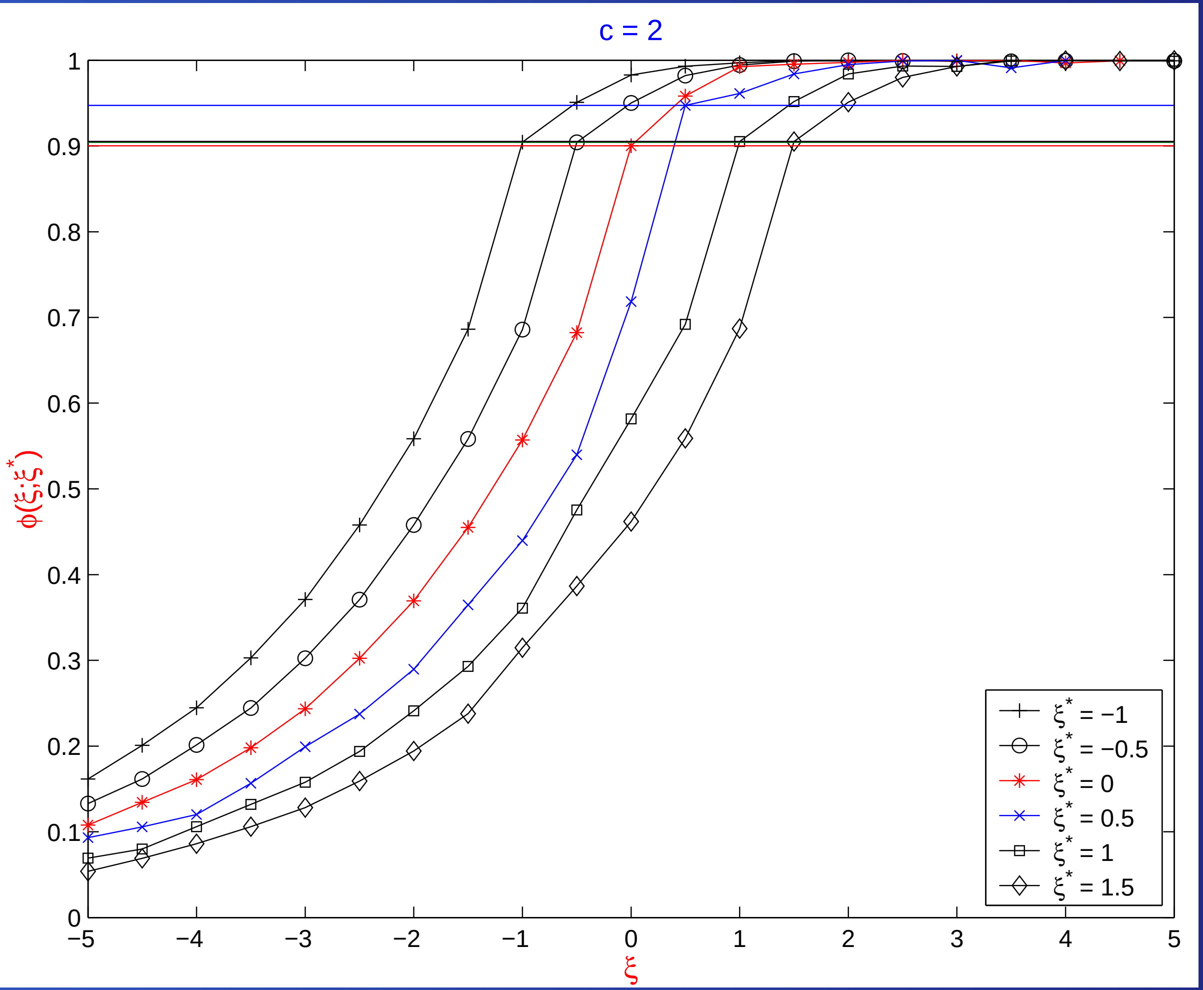
Wave Forms $\phi(\xi;\xi^*)$ for $\alpha_0 = 0.6$



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