Model Problems and Truncation of Advanced-Retarded FDEs arising in Lattice Travelling Wave Problems

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Workshop on Analysis and Computation of

Lattice, Delay and Functional Differential Equations

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Model Problems and Truncation of Advanced-Retarded FDEs

Advanced-Retarded Functional Differential Equations arise in a wide range of applications, recently receiving attention because travelling wave solutions to lattice differential equations are defined by FDE boundary value problems on an unbounded domain. The presence of advances as well as delays complicates the analysis of these problems. As a first step before computing a numerical solution the problem is usually approximated on a bounded domain. This truncation can be done in several ways, but the process is not well studied or understood. We will discuss the issues that arise, including the need for and construction of good model test problems with known solutions.

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Lattice Differential Equations

A typical LDE has the form

$$\dot{u}_i = g_i(\{u_j\}_{j\in\Lambda}), \quad i \in \Lambda.$$

$\Lambda \subset \mathbb{R}^n$ is a lattice; a discrete subset of $\mathbb{R}^n$, finite or infinite number of points, regular spatial structure

- $u_i(t)$ for each $i \in \Lambda$ may be scalar or vector
- Continuous in time, discrete in space
- In this talk we restrict attention to 1D lattices for simplicity of explanation

Leading Edge Model

Discrete Nagumo Equation

$$\dot{u}_i = (u_{i+1} - 2u_i + u_{i-1}) - f(u_i)$$

Models leading edge behaviour of pulse. Two examples:

Leading Edge Model
Discrete Nagumo Equation

$$\dot{u}_i = (u_{i+1} - 2u_i + u_{i-1}) - f(u_i)$$

Models leading edge behaviour of pulse. Two examples:

1. Cubic nonlinearity

$$f(u) = \beta u(u-a)(u-1)$$

[Travelling Wave Movie] [Standing Wave Movie]

2. McKean's caricature of cubic

$$f(u) = \begin{cases} \beta(u-1), & u > a, \\ \beta[u-1, u], & u = a, \\ \beta u, & u < a. \end{cases}$$

[Travelling Wave Movie] [Standing Wave Movie]

Functional Differential Equation Reduction
Travelling Waves for Discrete Nagumo Equation

$$\dot{u}_i = (u_{i+1} - 2u_i + u_{i-1}) - f(u_i)$$

- Travelling Wave ansatz $u_i(t) = \varphi(i - ct) = \varphi(\xi)$ gives

$$-c\varphi'(\xi) = \varphi(\xi + 1) - 2\varphi(\xi) + \varphi(\xi - 1) - f(\varphi(\xi))$$

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- $i \in \mathbb{Z}$ but $\xi = i - ct \in \mathbb{R}$ is time-like and $\varphi : \mathbb{R} \to \mathbb{R}$.

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- Both nonlinearities have three constant solutions $\varphi \equiv 0$, a and 1. Seek solutions with $\varphi(-\infty) = 0$, $\varphi(\infty) = 1$.

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- $\varphi(\xi - 1) =$ delay, $\varphi(\xi + 1) =$ advance.
- Both nonlinearities have three constant solutions $\varphi \equiv 0$, a and 1. Seek solutions with $\varphi(-\infty) = 0$, $\varphi(\infty) = 1$.
- TW ansatz "reduces" LDE to an FDE (cf TW ansatz reduces PDE to ODE)

Linear Advanced-Retarded FDEs

Consider linear FDE:

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - \beta\varphi(\xi)$$

admits solutions of form $\varphi(\xi) = e^{\lambda\xi}$

where $0 = c\lambda + e^{\lambda} - 2 + e^{-\lambda} - \beta = c\lambda + 2\cosh\lambda - (2+\beta).$

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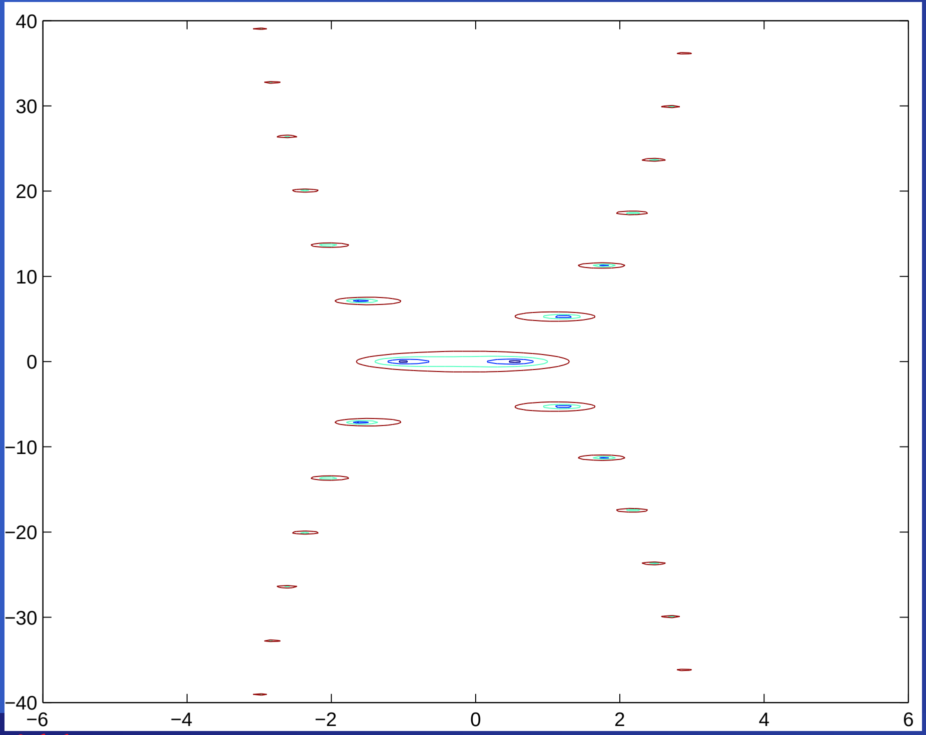

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- One positive real and one negative real λ.
- Infinitely many complex λ with $Re(\lambda) < 0$ and with $Re(\lambda) > 0$.
- As $|\lambda| \to \infty$ eigenvalues lie on $Re(\lambda) = \pm\ln|c\lambda|$.
- $Re(\lambda) \to \pm\infty$ as $|\lambda| \to \infty$.

Linear Advanced-Retarded FDEs

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where $\quad 0 = c\lambda + e^{\lambda} - 2 + e^{-\lambda} - \beta = c\lambda + 2\cosh\lambda - (2+\beta).$

- Bi-infinite sums of eigenfunctions define solutions.
- A solution with infinitely many eigenfunctions with $Re(\lambda) > 0$ will have faster than exponential growth forwards in time.
- A solution with infinitely many eigenfunctions with $Re(\lambda) < 0$ will have faster than exponential growth backwards in time.
- 0 is a saddle point with infinite dimensional stable and unstable manifolds.
- Not well-posed as IVP.

Nonlinear FDE BVP
Existence and Uniqueness

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - \beta\varphi(\xi)(\varphi(\xi)-a)(\varphi(\xi)-1)$$

$$\varphi(-\infty) = 0, \quad \varphi(\infty) = 1.$$

- [ZINNER 1991]: Uniqueness and Stability of Monotonic TWs
- [ZINNER 1992]: Existence of Monotonic TWs for β large.
- Zinner's theory covers larger class of f. More recent extensions of theory to wider class of problems, in particular work of [MALLET-PARET 1999A],[MALLET-PARET 1999B].
- Of interest in this and more general problems is what happens when β is not sufficiently large.

Functional Differential Equation
Truncated Boundary Value Problem

To solve

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - f(\varphi(\xi)), \quad \varphi(-\infty) = 0, \ \varphi(\infty) = 1$$

numerically must truncate to finite interval:

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - f(\varphi(\xi)), \quad \xi \in [T_-, T_+]$$

- What are suitable boundary terms ?
- To evaluate $\varphi'(\xi)$ for $\xi \in [T_-, T_+]$ need $\varphi(\xi)$ defined for $\xi \in [T_- - 1, T_+ + 1]$ and ideally we want it defined for $\xi \in (-\infty, \infty)$.
- We consider 6 possibilities:

Functional Differential Equation
Truncated Boundary Value Problem

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - f(\varphi(\xi)), \quad \xi \in [T_-, T_+]$$

Boundary Conditions/Functions:

1. Dirichlet
2. Neumann
3. Dominant Characteristic value
4. Dominant Characteristic value $+$ Nonlinear Correction
5. Projected BCs
6. Dominant Real part Characteristic value

Functional Differential Equation
Truncated Boundary Value Problem

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - f(\varphi(\xi)), \quad \xi \in [T_-, T_+]$$

1. Dirichlet BCs

- Since $f(0) = f(1) = 0$ and

$$\lim_{\xi\to-\infty} \varphi(\xi) = 0, \qquad \lim_{\xi\to\infty} \varphi(\xi) = 1,$$

  we could try

$$\varphi(T_-) = 0,\ \varphi(T_+) = 1$$
$$\varphi(\xi) = 0,\ \xi < T_- \text{ and } \varphi(\xi) = 1,\ \xi > T_+.$$

- Many people do this !
- Solution does not always converge as $|T_-|, |T_+| \to \infty$!!!

Functional Differential Equation
Truncated Boundary Value Problem

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - f(\varphi(\xi)), \quad \xi \in [T_-, T_+]$$

2. Neumann

- Since $\lim_{\xi\to-\infty}\varphi(\xi) = 0$, $\lim_{\xi\to\infty}\varphi(\xi) = 1$, we could try

$$\varphi'(T_-) = 0, \quad \varphi'(T_+) = 0$$

$\varphi(\xi) = \varphi(T_-)$, $\xi < T_-$ and $\varphi(\xi) = \varphi(T_+)$, $\xi > T_+$.

- Constant 'solutions' for $\xi < T_-$ and $\xi > T_+$ are not actually constant solutions of original equation.
- But if $\varphi(T_-) \approx 0$ and $\varphi(T_+) \approx 1$ close, then $|\varphi(T_-)|$ and $|\varphi(T_+) - 1|$ give a measure of the approximation error.

Functional Differential Equation
Truncated Boundary Value Problem

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - f(\varphi(\xi)), \quad \xi \in [T_-, T_+]$$

3. Linearization: For $\xi \leq T_-$ consider the linearization

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - f'(0)\varphi(\xi)$$

which has monotonic solutions of the form $\varphi(\xi) = e^{\lambda\xi}$ where $\lambda > 0$ given by $0 = c\lambda + 2\cosh\lambda - (2+\beta)$. Now

$$\varphi(\xi) = \varphi(T_-)e^{\lambda(\xi - T_-)}, \quad \xi \in (-\infty, T_-].$$

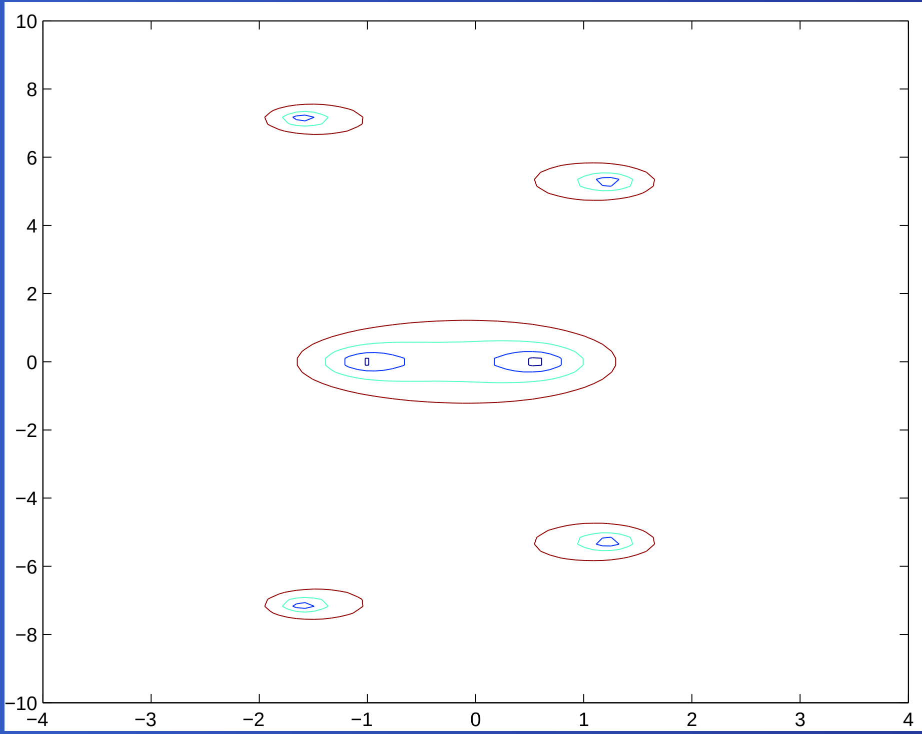
And BC: $\varphi(T_-) = ke^{\lambda T_-} \quad \Longrightarrow \quad \lambda\varphi(T_-) + \varphi'(T_-) = 0.$

Is this such a good idea ??

Functional Differential Equation
Truncated Boundary Value Problem

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - f(\varphi(\xi)), \quad \xi \in [T_-, T_+]$$

3. Linearization: $0 = c\lambda + 2\cosh\lambda - (2+\beta)$:



Functional Differential Equation
Truncated Boundary Value Problem

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - f(\varphi(\xi)), \quad \xi \in [T_-, T_+]$$

3. Linearization: $0 = c\lambda + 2\cosh\lambda - (2+\beta)$:

- Since $Re(\lambda_c) > \lambda > 0$ will be okay for $T_- \ll 0$.
- 1D approx to linear stable manifold better than previous 0d approximations
- infinitely many stable directions missed......

Nonlinear Nagumo Problem
Truncated Boundary Value Problem

Travelling Wave Equation:

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - f(\varphi(\xi)), \quad \xi \in [T_-, T_+]$$

Phase condition: $\varphi(0) = a$

Boundary Conditions, LHS:

λ_+ is positive root of $0 = c\lambda_+ + 2\cosh(\lambda_+) - 2 - f'(0).$

$$\varphi(\xi) = \varphi(T_-)e^{\lambda_+(\xi - T_-)}, \xi \in (-\infty, T_-]$$

$$\lambda_+(c)\varphi(T_-) + \varphi'(T_-) = 0$$

RHS has delays and advances.

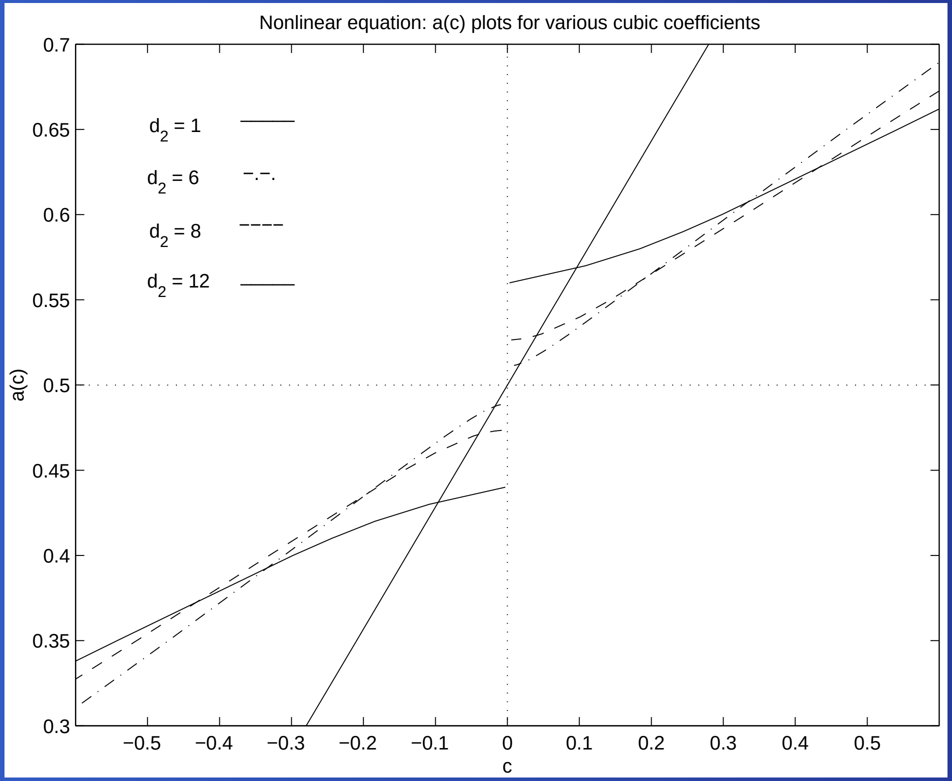
Solve these equations numerically using a mixed-type DDE collocation code written for the purpose [ABELL, ET AL 2004], (built on colmod [CASH ET AL 1995]).

Nonlinear Nagumo Equation

a-c curves

$$\dot{u}_i = u_{i+1} - 2u_i + u_{i-1} - \beta u_i(u_i - a)(u_i - 1), \quad \beta > 0$$

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - \beta\varphi(\xi)(\varphi(\xi)-a)(\varphi(\xi)-1),$$



β small $\Longrightarrow$
$c = 0 \iff a = 1/2$???

β large $\Longrightarrow$
$c = 0$ for growing range of a:
$=$ Propagation Failure

Nonlinear Nagumo Equation

Evolution of Wave Profile for $\beta = 1$ and $\beta = 8$.

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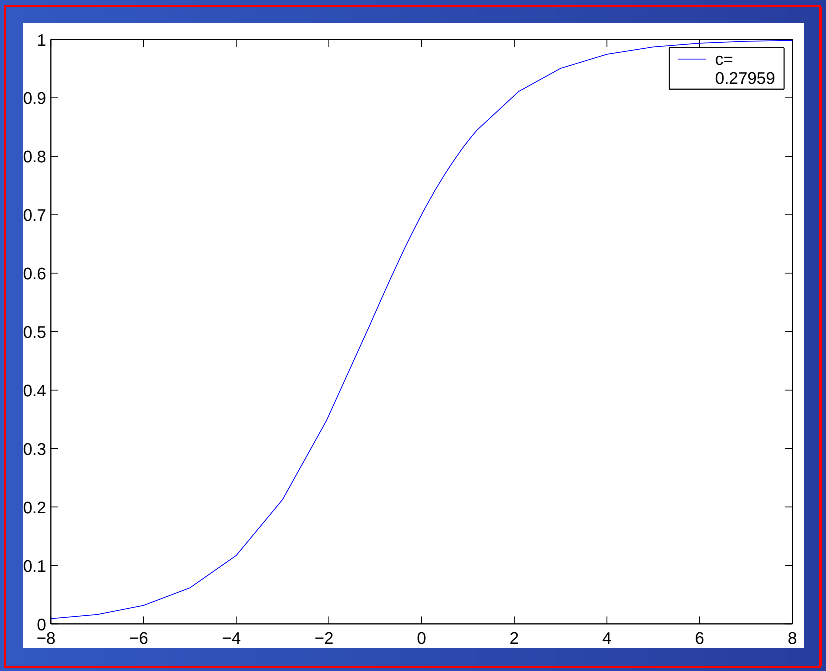

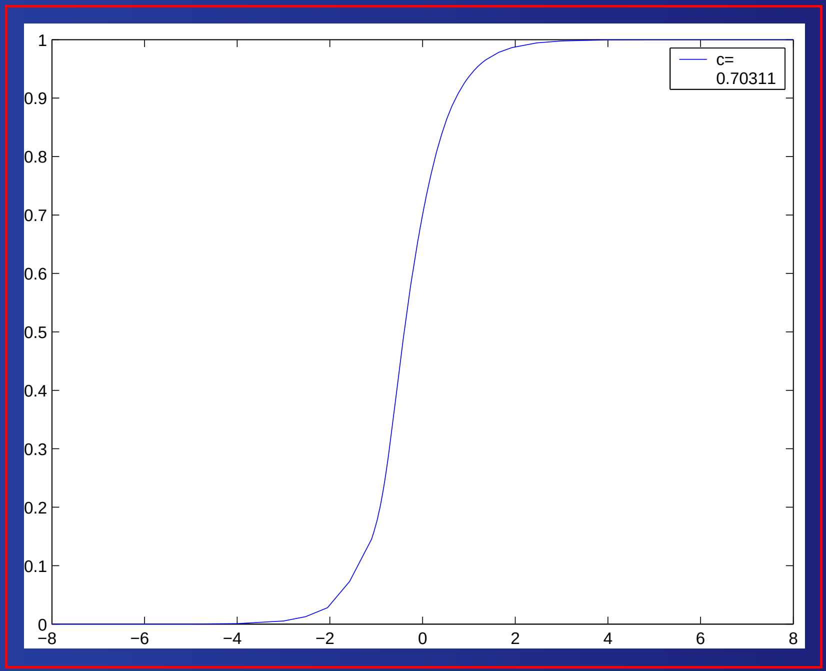

- Consider evolution of wave profile as $c \to 0$

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Nonlinear Nagumo Equation

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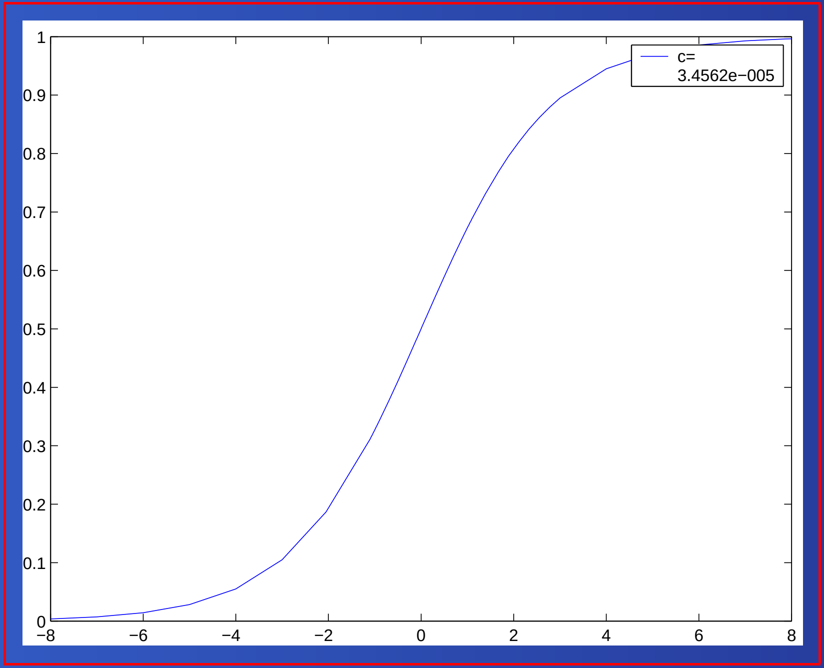

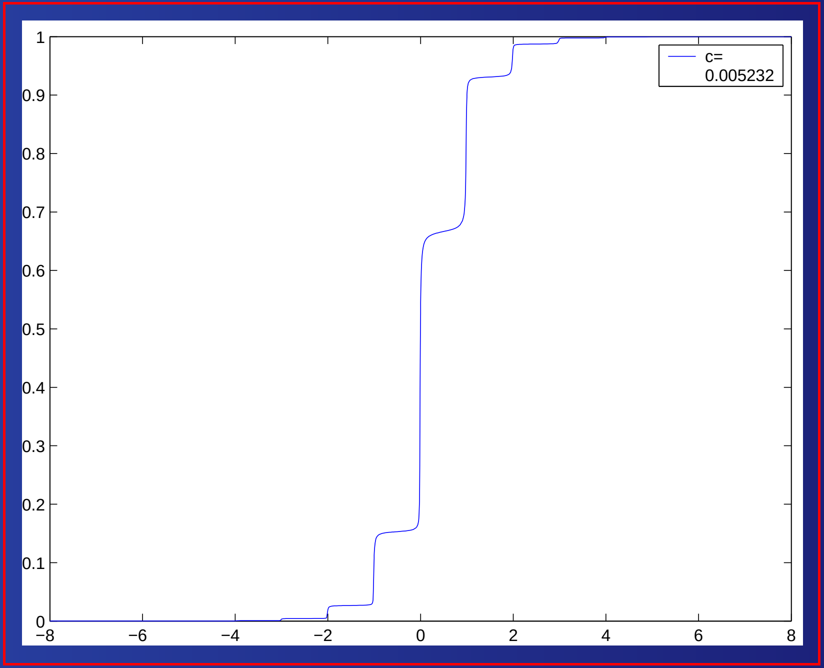

- Consider evolution of wave profile as $c \to 0$
- TW equation becomes a difference equation
- Step profile explains this

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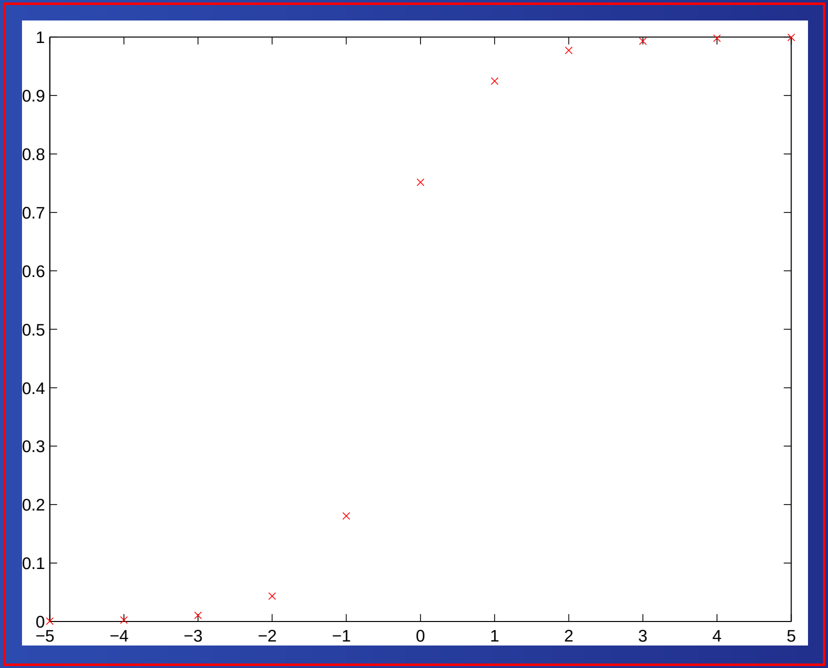
Propagation Failure & Standing Waves

$c = 0$: A difference Equation

$$0 = -c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - \beta f(\varphi(\xi))$$

$$0 = \dot{u}_i = u_{i+1} - 2u_i + u_{i-1} - \beta f(u_i)$$



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Propagation Failure & Standing Waves

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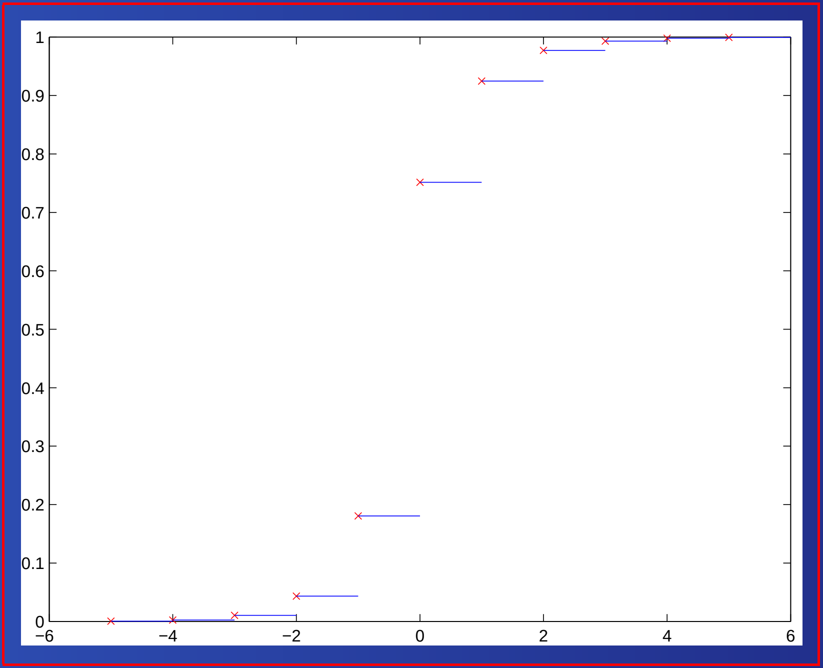
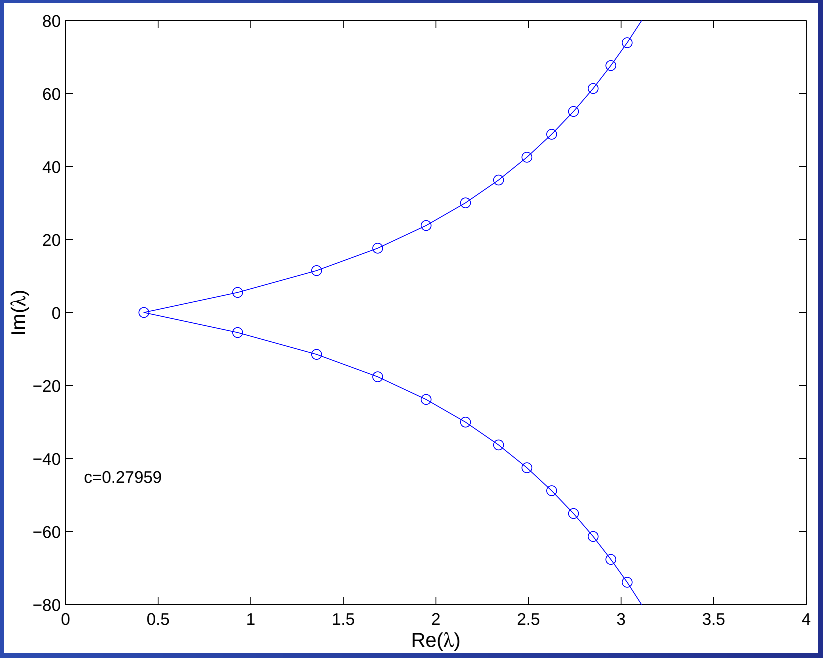


- Solution of Difference Equation defines solution of Functional Difference Equation.

Evolution of Characteristic Equation Roots



- Step function, does not resemble $e^{\lambda\xi}$
- Check assumption that $Re(\lambda_c) > \lambda > 0$ in limit as $c \rightarrow 0$.
- Compute roots by approximating the infinitesimal generator, using approach of [BREDA,MASET,VERMIGLIO].

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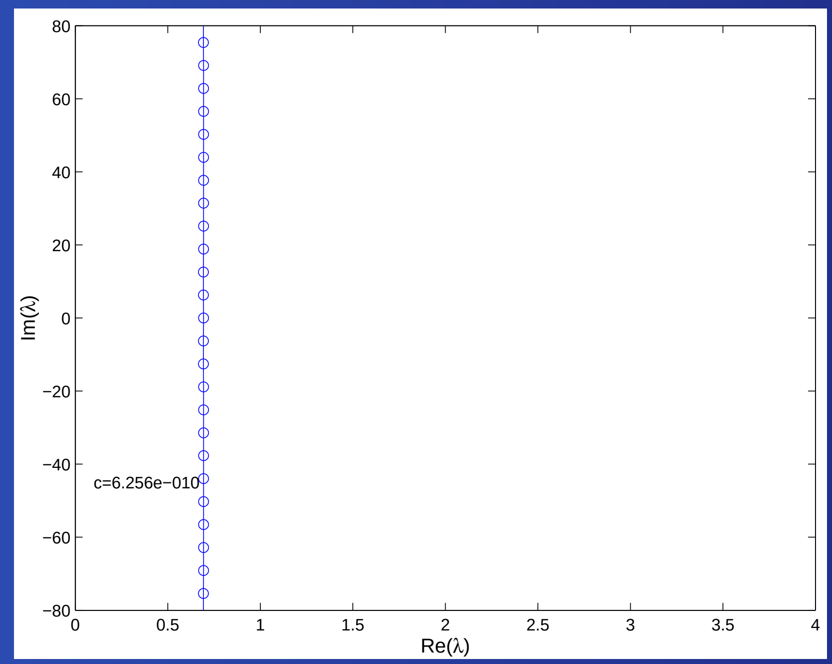
Evolution of Characteristic Equation Roots



- $0 = c\lambda + 2\cosh\lambda - (2+\beta)$:
- No dominant λ as $c \to 0$
- Smooth approximation $e^{\lambda\xi}$ to tail of wave not valid

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Functional Differential Equation

Truncated Boundary Value Problem

More boundary conditions:

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - f(\varphi(\xi)), \quad \xi \in [T_-, T_+]$$

4. Nonlinear Correction

- [CHI,BELL,HASSARD,1986] and other authors apply nonlinear corrections to the linear term
- Seemingly pointless: leading error terms are linear !

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Functional Differential Equation
Truncated Boundary Value Problem

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5. Projection BCs: Project into invariant subspace

- [BEYN 1990],[FRIEDMAN,DOEDEL 1991],[DEMMEL,DIECI,FRIEDMAN 2000],...
- State of the art for ODEs
- Difficult in our case, as all manifolds ∞-dim.

$c = 0$: Approximating Steps

Can compute eigenfunctions so could use more of them. Would that work ?

Linearized equation with $c = 0$:

$$0 = -c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - \beta f'(0)$$

Characteristic Equation:

$$0 = -c\lambda = 2\cosh\lambda - (2+K)$$

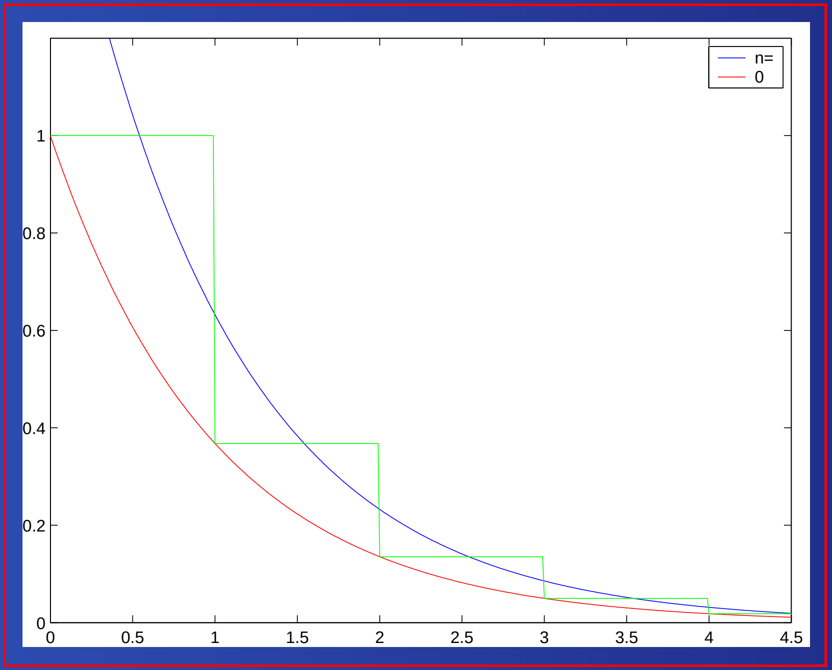
has roots

$$\lambda_n = \alpha + 2n\pi i, \quad n \in \mathbb{Z}.$$

Lets approximate step function $e^{\lambda_0 \lfloor x \rfloor}$ using these characteristic functions.

$c = 0$: Approximating Steps

$$e^{\lambda_0 \lfloor x \rfloor} = \sum_{n=-\infty}^{\infty} \alpha_n e^{\lambda_n x} \approx \sum_{n=-N}^{N} \alpha_n e^{\lambda_n x}, \qquad \alpha_n = \frac{1 - e^{-\lambda_0}}{\lambda_n}.$$



How good is the approximation ?

$c = 0$: Approximating Steps

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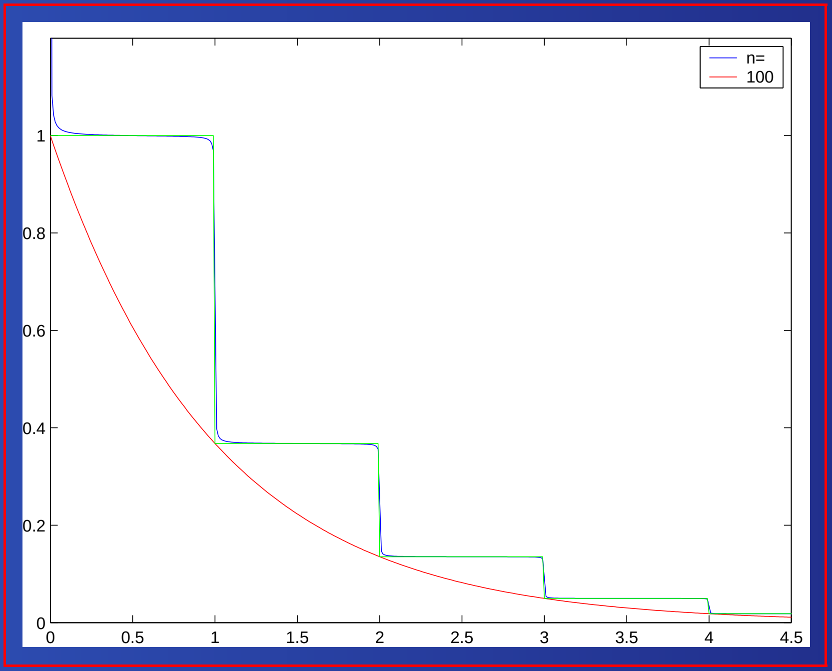


Need 201 dimension approx to manifold ! Try something else.

Boundary Conditions:
6. Dominant Real-part of Characteristic Value

- Smooth exponential function $\varphi(\xi) = e^{\lambda\xi}$ satisfies:

$$\varphi'(\xi) - \lambda\varphi(\xi) = 0, \qquad \varphi(\xi+1) = e^{\lambda}\varphi(\xi),$$

- Step exponential function $\varphi(\xi) = e^{\lambda\lfloor\xi\rfloor}$ satisfies:

$$\varphi'(\xi) = 0,\ a.e. \qquad \varphi(\xi+1) = e^{\lambda}\varphi(\xi).$$

- This suggests new boundary functions/conditions

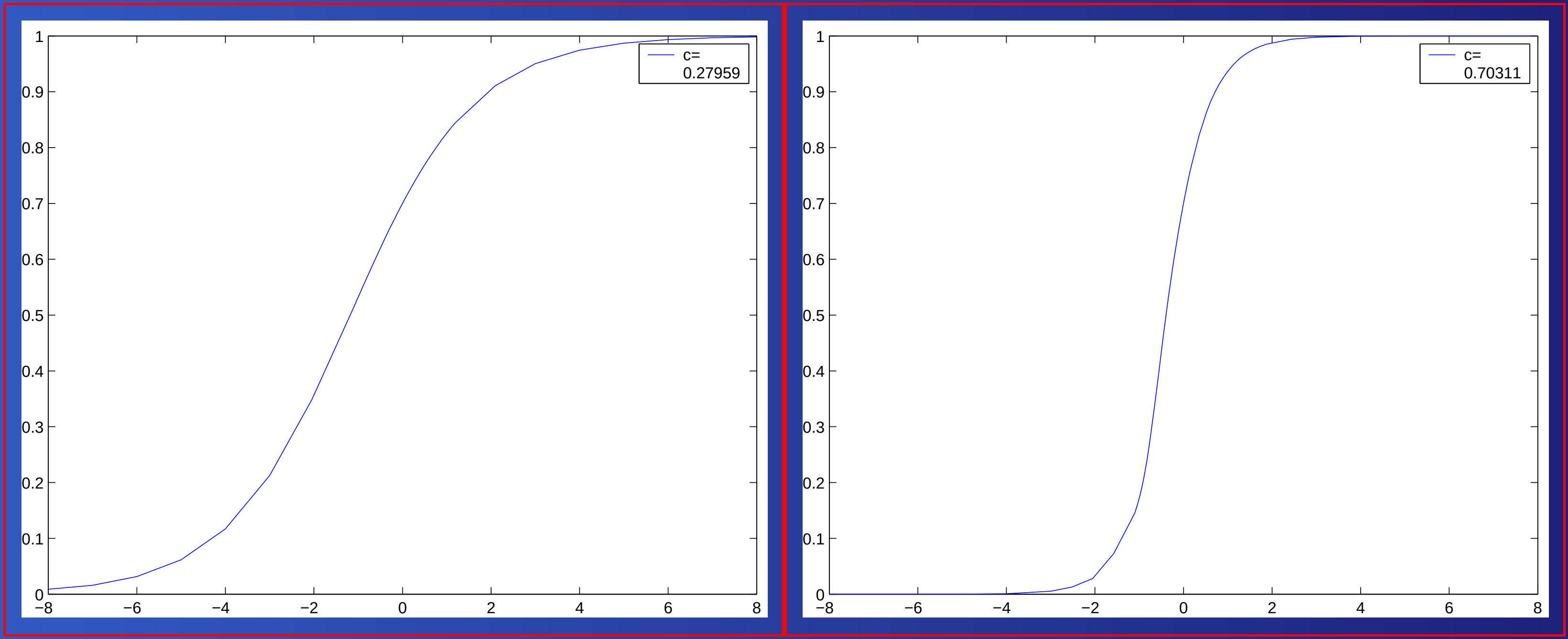
$$\varphi(\xi) = e^{-\lambda}\varphi(\xi+1), \quad \xi \in [T_- - 1, T_-].$$

And for continuity BC: $\qquad \varphi(T_-) = e^{-\lambda}\varphi(T_- + 1).$

- These apply equally well for smooth and step tails.

Simulations with New Boundary Conditions

$\beta = 1$ and $\beta = 8$



Simulations with New Boundary Conditions

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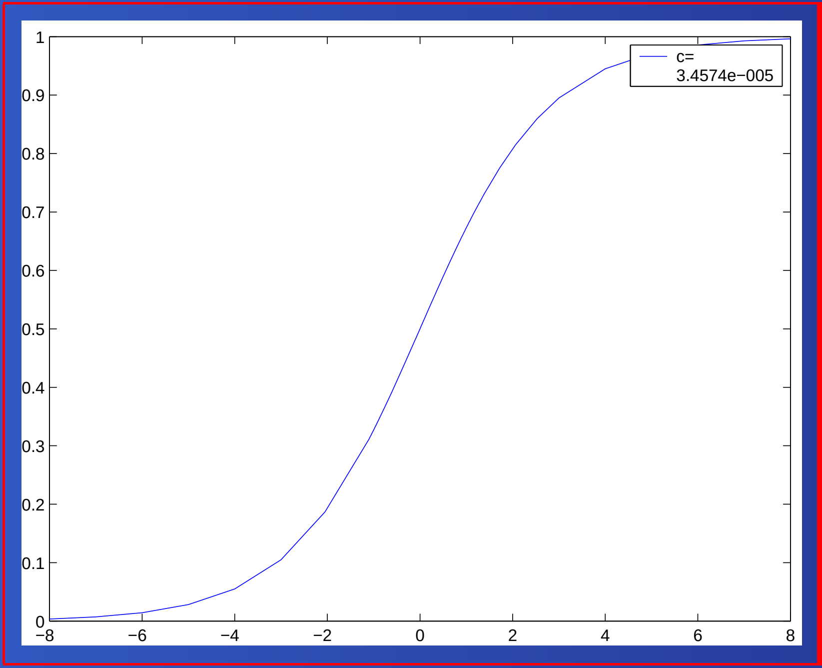
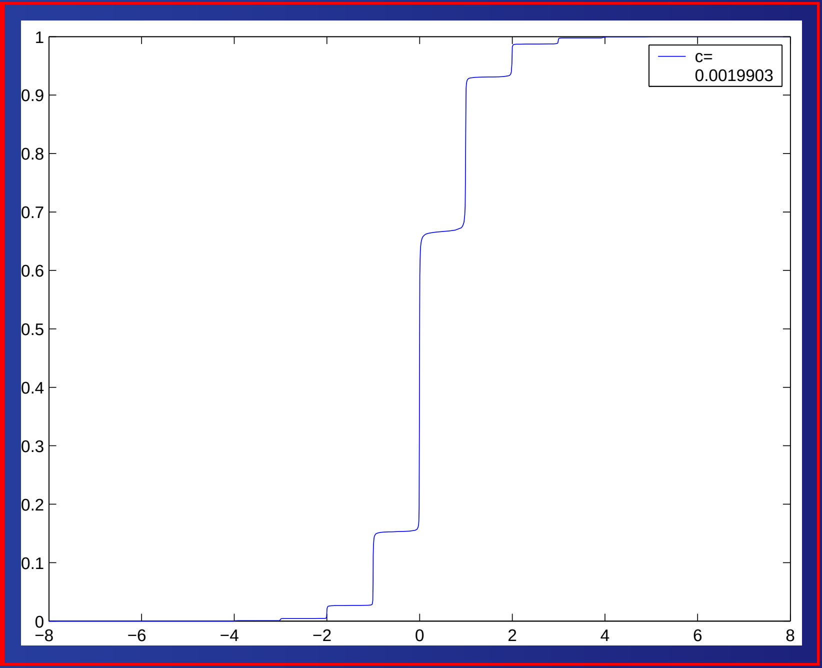




- For smooth solution as $c \to 0$, new BCs as good as old.
- For step solutions as $c \to 0$, new BCs allow computation to smaller c and obtain flatter steps.
- Why not always steps? Is this computation better?

Test Problem 1

McKean's caricature of the cubic

McKean's caricature of cubic

$$f(\varphi) = \begin{cases} \beta(\varphi - 1), & \varphi > a, \\ \beta[\varphi - 1, \varphi], & \varphi = a, \\ \beta\varphi, & \varphi < a. \end{cases}$$

Test Problem 1

McKean's caricature of the cubic

McKean's caricature of cubic

$$f(\varphi) = \begin{cases} \beta(\varphi - 1), & \varphi > a, \\ \beta[\varphi - 1, \varphi], & \varphi = a, \\ \beta\varphi, & \varphi < a. \end{cases} = \varphi - H(\varphi - a),$$

where H is heaviside function.

Test Problem 1

McKean's caricature of the cubic

McKean's caricature of cubic

$$f(\varphi) = \varphi - H(\varphi - a),$$

where H is heaviside function.

For monotonic travelling wave solutions $\varphi(\xi)$ $\exists!\xi$: $\varphi(\xi) = a$. Wlog let $\varphi(0) = a$. Then

$$f(\varphi(\xi)) = \varphi(\xi) - H(\varphi(\xi) - a) = \varphi(\xi) - H(\xi) = \text{linear.}$$

Travelling wave equation in this case is

$$-c\varphi'(\xi) = \varphi(\xi + 1) - 2\varphi(\xi) + \varphi(\xi - 1) - \varphi(\xi) + H(\xi)$$

Can be studied using transforms etc.

Test Problem 1

McKean's caricature of the cubic

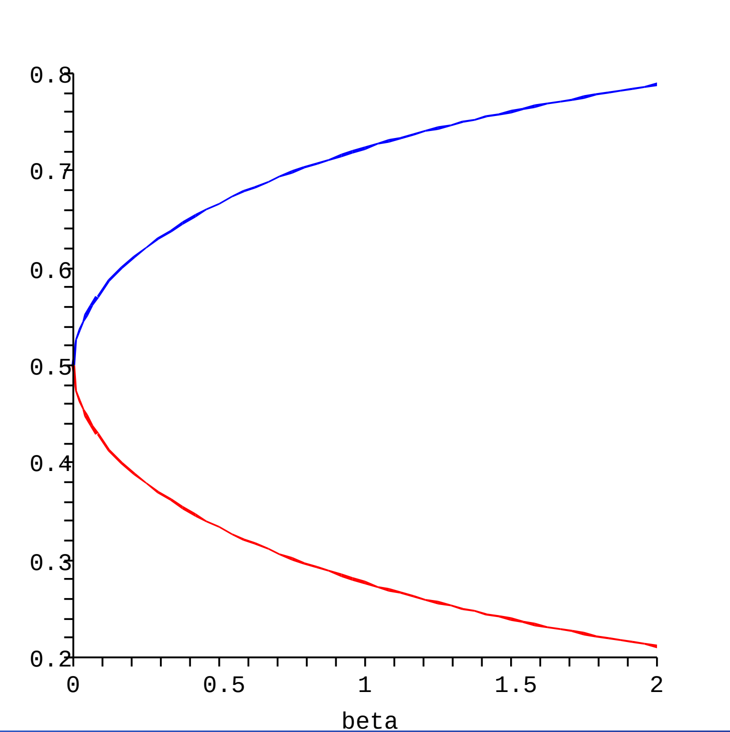
McKean's caricature of cubic

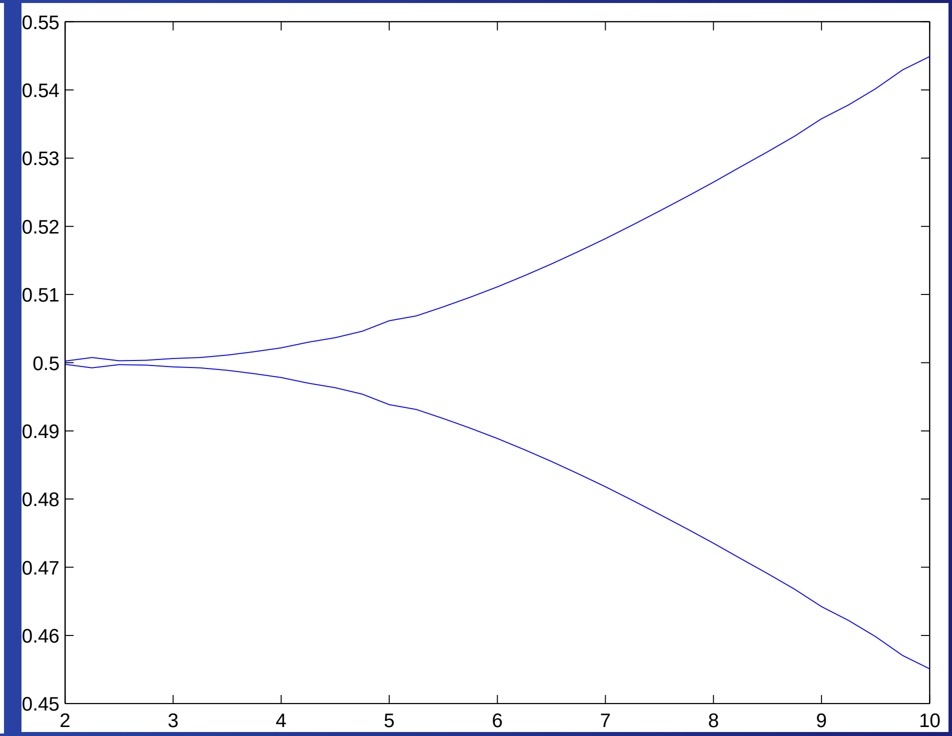
$$\varphi(\xi) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{A(s)\sin s\xi}{s(A(s)^2 + c^2 s^2)} ds + \frac{c}{\pi}\int_0^\infty \frac{\cos s\xi}{A(s)^2 + c^2 s^2} ds$$

where $A(s) = 1 + 2\alpha(1 - \cos s)$, and phase condition $\varphi(0) = a$, relates a to c.

- Convergence of integrals subtle especially as $c \to 0$, and good numerical evaluation is not a fun problem.
- Like cubic f has 'zeros' at 0, a and 1,
- But for McKean '$f'(a) = -\infty$' completely changes propagation failure characteristics
- $f'(0) = f'(1)$ leads to symmetry $\varphi(-\xi) = 1 - \varphi(\xi)$ for standing waves $c = 0$.

Propagation Failure
McKean and Cubic Nonlinearities





- McKean: $\forall \beta > 0\ \exists \varepsilon > 0 : c = 0$ for $a \in [1/2 - \varepsilon, 1/2 + \varepsilon]$
- Cubic: Only true for β sufficiently large, or range of a exponentially small for small β ??

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Standing Waves $c = 0$
As Hamiltonian Discretizations

$$0 = \dot{u}_i = u_{i+1} - 2u_i + u_{i-1} - \beta f(u_i, a)$$

Let $h = \sqrt{\beta}$ and $v_{j+1} = (u_{j+1} - u_j)/h$. Then

$$u_{j+1} = u_j - hv_{j+1}, \qquad v_{j+1} = v_j + hf(u_j, a)$$

Which is symplectic Euler applied to the Hamiltonian system

$$\dot{v} = -H_u(u, v, a), \qquad \dot{u} = H_v(u, v, a)$$

where

$$H(u, v, a) = \frac{v^2}{2} - W(u, a), \quad W'(u, a) = f(u, a)$$

The standing wave is a numerical heteroclinic connection between $(0, 0)$ and $(1, 0)$.

Standing Waves $c = 0$
As Hamiltonian Discretizations

In general

- ODE will have heteroclinic connection for isolated parameter value
- Expect heteroclinic connection for discretization for nearby parameter value [BEYN,1990],[DOEDEL&FRIEDMAN,1990]
- In general stable-unstable manifold intersection for discrete map will be transversal; so heteroclinic orbit will persist over (exponentially) small parameter range [FIEDLER & SCHEURLE, 1996]
- So we should expect to see propagation failure for β small too.

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Test Problem 2

The tanh Solution

Approach dates at least to [CHI,BELL,HASSARD,1986]
Choose solution:

$$\varphi(\xi) = \frac{1}{2}(1 + \tanh \xi) = \frac{e^{2\xi}}{1 + e^{2\xi}} \quad \Rightarrow \quad e^{2\xi} = \frac{\varphi(\xi)}{1 - \varphi(\xi)}.$$

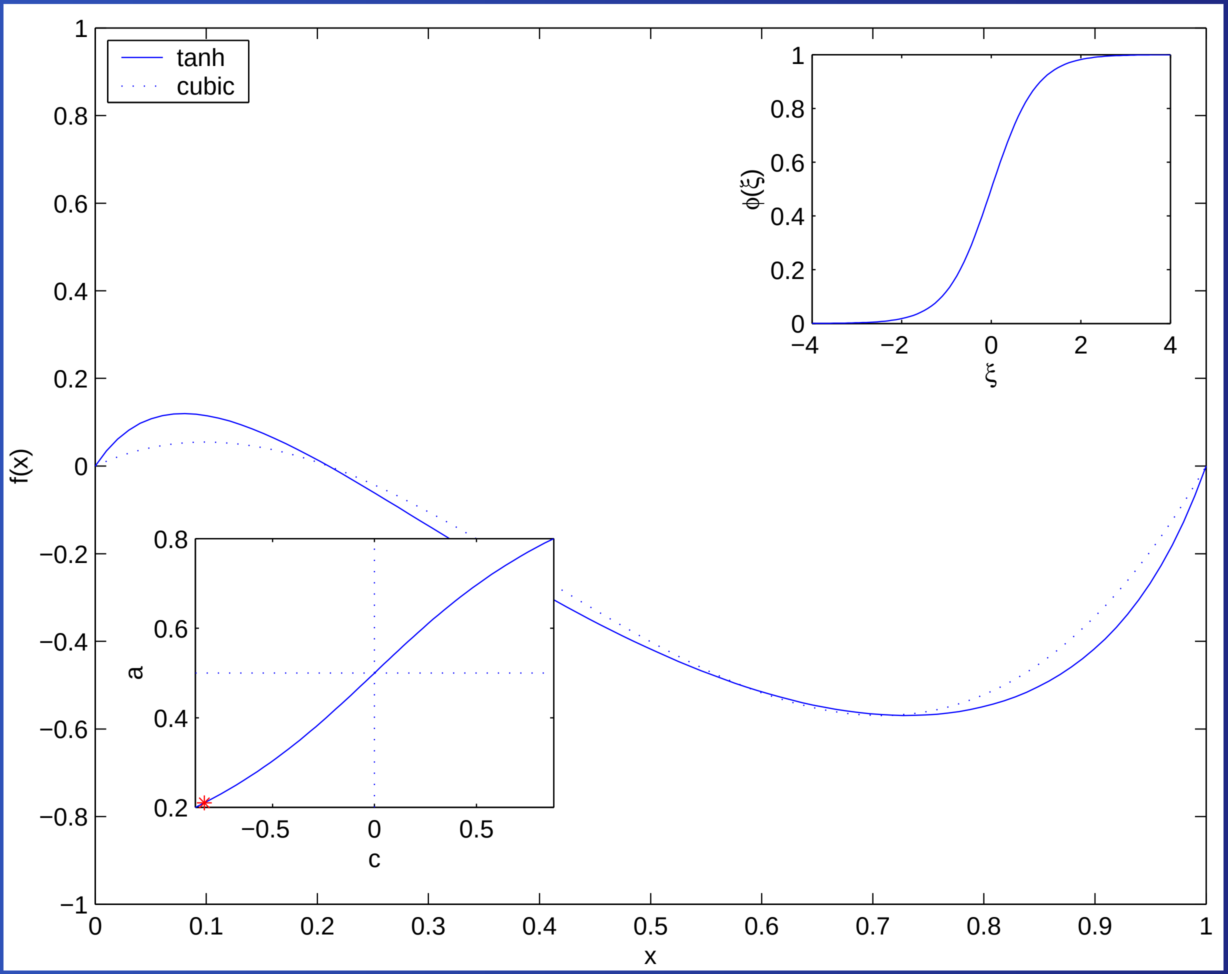
Now

$$-c\varphi'(\xi) = \varphi(\xi + 1) - 2\varphi(\xi) + \varphi(\xi - 1) - \beta f(\varphi(\xi)) \Rightarrow$$

$$\beta f(\varphi) = 2c\varphi(1 - \varphi) + \frac{\varphi}{\varphi + e^{2}(1 - \varphi)} - 2\varphi + \frac{\varphi}{\varphi + e^{-2}(1 - \varphi)}$$

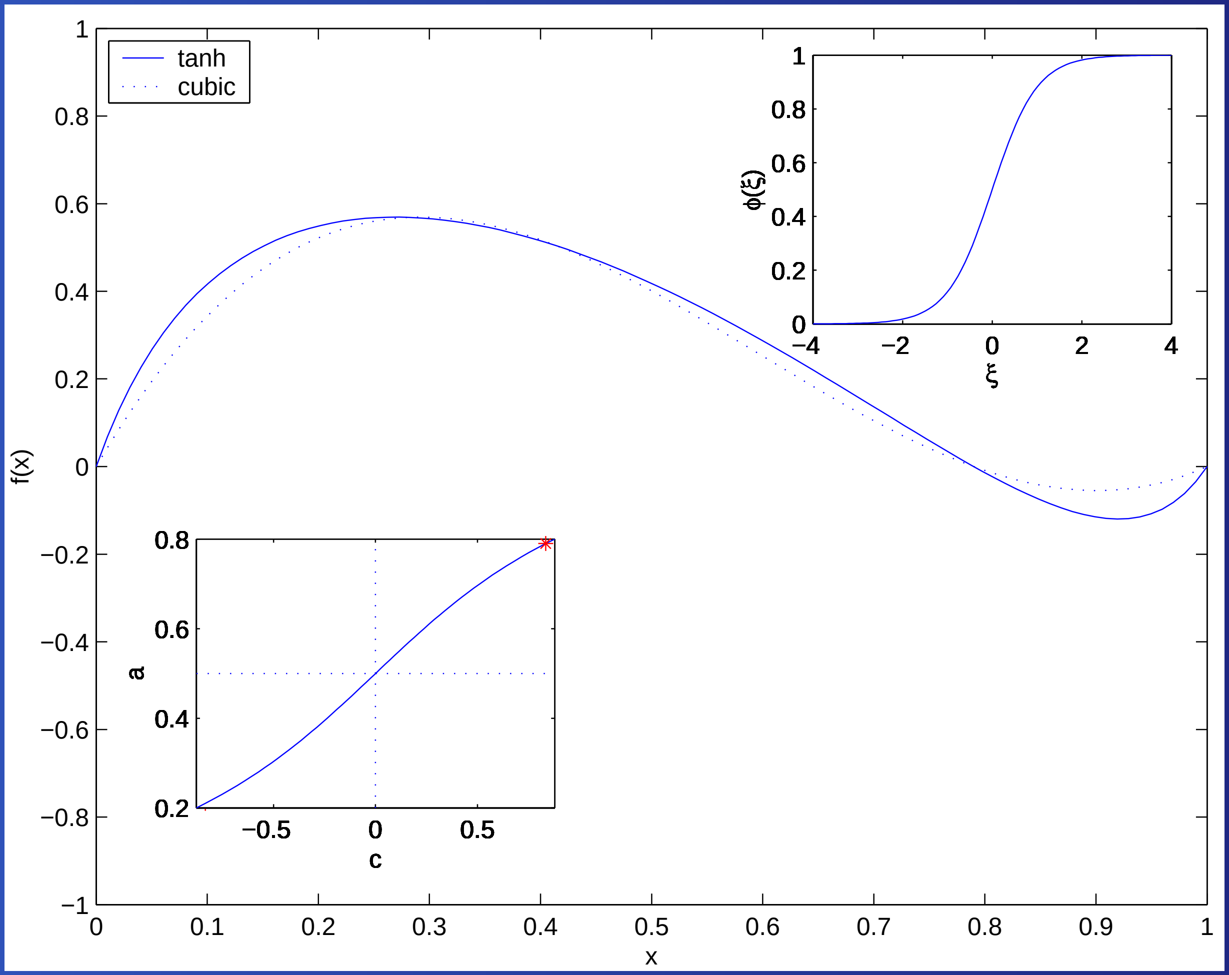
with c determined by $f(a) = 0$
($f(0) = f(1) = 0$ follows from choice of solution)

Test Problem 2
The tanh Solution



Test Problem 2

The tanh Solution



Problems with
The Tanh Test Problem

- No propagation failure
  Wave speed determined by

$$g(a,c) := 2ca(1-a) + \frac{a}{a+e^2(1-a)} - 2a + \frac{a}{a+e^{-2}(1-a)} = 0$$

Has solution with $a = 1/2$ and $c = 0$ and simple application of implicit function theorem shows $\frac{dc}{da}\Big|_{a=1/2} > 0$.

Problems with The Tanh Test Problem

- No propagation failure
- Symmetric Tails
  $\varphi(\xi) + \varphi(-\xi) = 1$ like McKean, not like cubic.

Problems with The Tanh Test Problem

- No propagation failure
- Symmetric Tails
- Non-generic connecting orbit?

$$\varphi(\xi > 0) = (1 + e^{-2\xi})^{-1} = \sum_{k=0}^{\infty} (-1)^k e^{-2k\xi} = 1 - e^{2\xi} + h.o.t.$$

$$\varphi(\xi < 0) = e^{2\xi}(1 + e^{2\xi})^{-1} = e^{2\xi} \sum_{k=0}^{\infty} (-1)^k e^{2k\xi} = e^{2\xi} + h.o.t.$$

Problems with The Tanh Test Problem

- No propagation failure
- Symmetric Tails
- Non-generic connecting orbit?
  But for $\xi < 0$ ($\xi > 0$ is similar) linearizing about $\varphi = 0$ gives

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - \beta f'(0)\varphi(\xi)$$

with solutions $\varphi(\xi) = \sum_{n=1}^{\infty} k_n e^{\lambda_n \xi}$ where each λ satisfies $\Re(\lambda) < 0$ and

$$0 = c\lambda + e^{\lambda} - 2 + e^{-\lambda} - \beta f'(0).$$

Problems with The Tanh Test Problem

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$$0 = c\lambda + e^{\lambda} - 2 + e^{-\lambda} - [2c + e^{2} - 2 + e^{-2}].$$

Which has real root $\lambda = 2$, and infinitely many complex roots with $Re(\lambda_n) > 2$, $Re(\lambda_n) \to \infty$ as $n \to \infty$.

Problems with The Tanh Test Problem

- No propagation failure
- Symmetric Tails
- Non-generic connecting orbit?

  Linear unstable manifold of 0 consists of functions $e^{2\xi}$ and infinitely many $e^{\lambda\xi}$ with $Re(\lambda) > 2$, $Im(\lambda) \neq 0$. Heteroclinic orbit will be tangent to this space, with $e^{2\xi}$ as dominant component, but also with other components

  But nonlinear solution has $e^{2\xi}$ as dominant component with other components $(e^{2\xi})^k$, $k = 2, 3, \ldots$ These are Chi,Bell, Hassard nonlinear corrections to linear boundary function

  This is a very special solution. Approaches fixed point exactly in direction of dominant linear component with no oscillatory components

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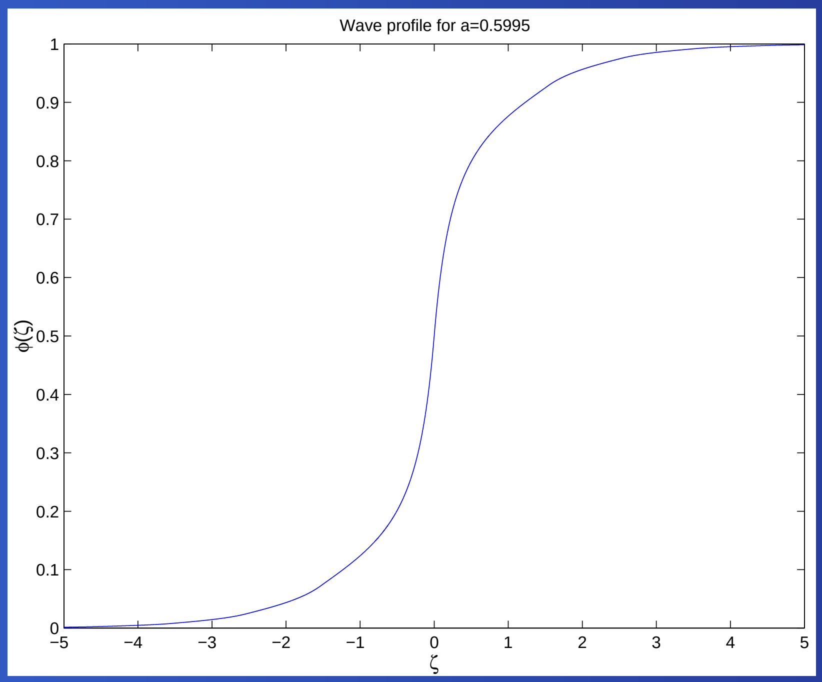
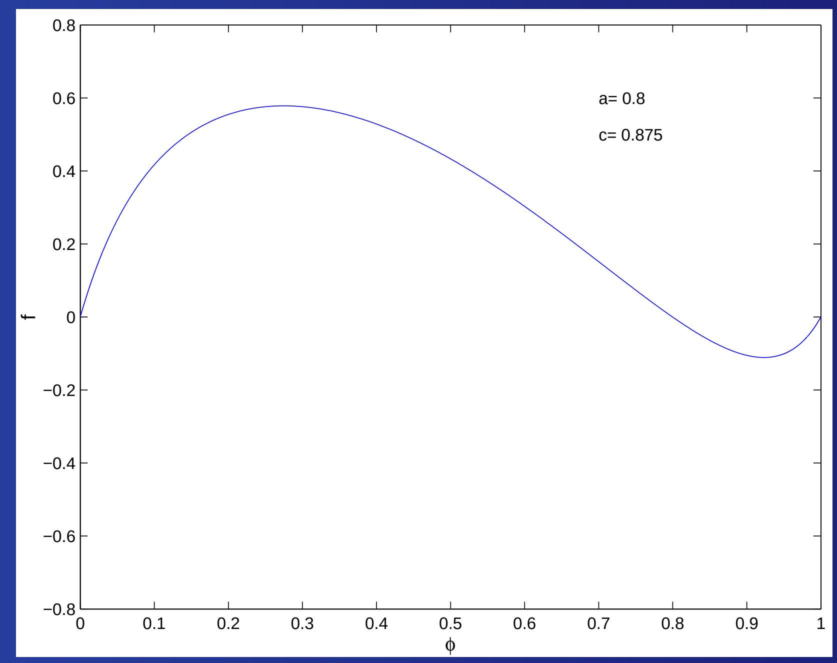
Problems with The Tanh Test Problem

- No propagation failure
- Symmetric Tails
- Non-generic connecting orbit
- This is not a good test problem to compare our truncated problems
  Does not have behaviour like cubic f
  Special solution matches one truncation method, but no reason why general solutions should be like this

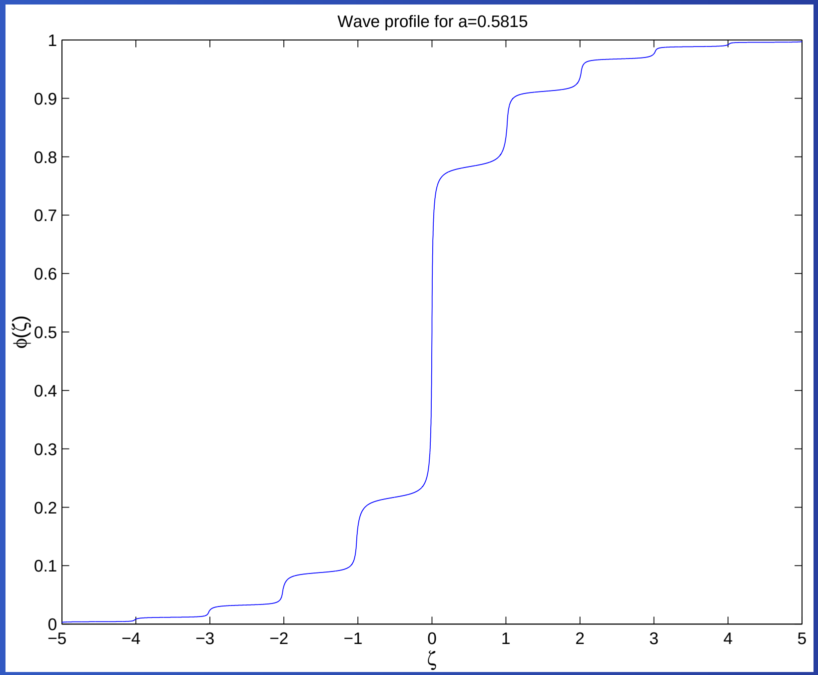
An Exact Solution with Propagation Failure [Elmer,Rodrigo,Muira]

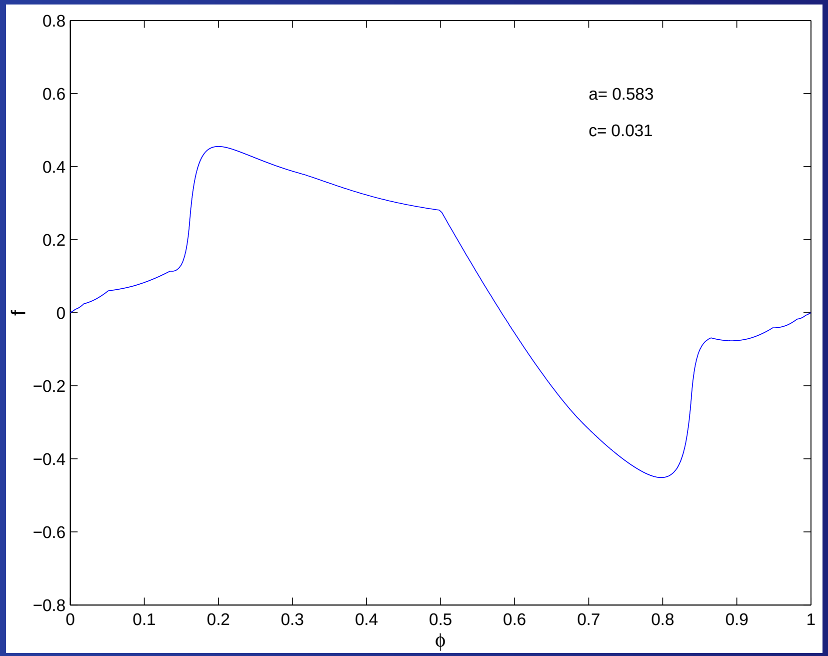




$$\varphi(\xi) = \frac{1}{2}(1 + \tanh(b\xi + g(\xi)))$$

An Exact Solution with Propagation Failure [Elmer,Rodrigo,Muira]





- Propagation failure
- But symmetric and $f'(\xi)$ has discontinuities at $\xi = \varphi(z)$, $z \in \mathbb{Z}$, with accumulation points at 0 and 1.

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An Exact Solution for Cubic-like f

$$\varphi(\xi) = \left(\frac{e^{b\xi}}{1+e^{b\xi}}\right)^r$$

$$\varphi(\xi<0) = e^{br\xi} + h.o.t. \qquad \varphi(\xi>0) = 1 - re^{b\xi} + h.o.t.$$

Characteristic equation

$$0 = c\lambda + e^{\lambda} - 2 + e^{-\lambda} - \beta f'(0 or 1).$$

implies b, c, r determined by $f(a) = 0$ and

$$0 = cbr + e^{br} - 2 + e^{-br} - \beta a$$

$$0 = cb + e^{b} - 2 + e^{-b} - \beta(1-a)$$

satisfies $f(0) = f(a) = f(1) = 0$, $f'(0) = a$, $f'(1) = 1-a$

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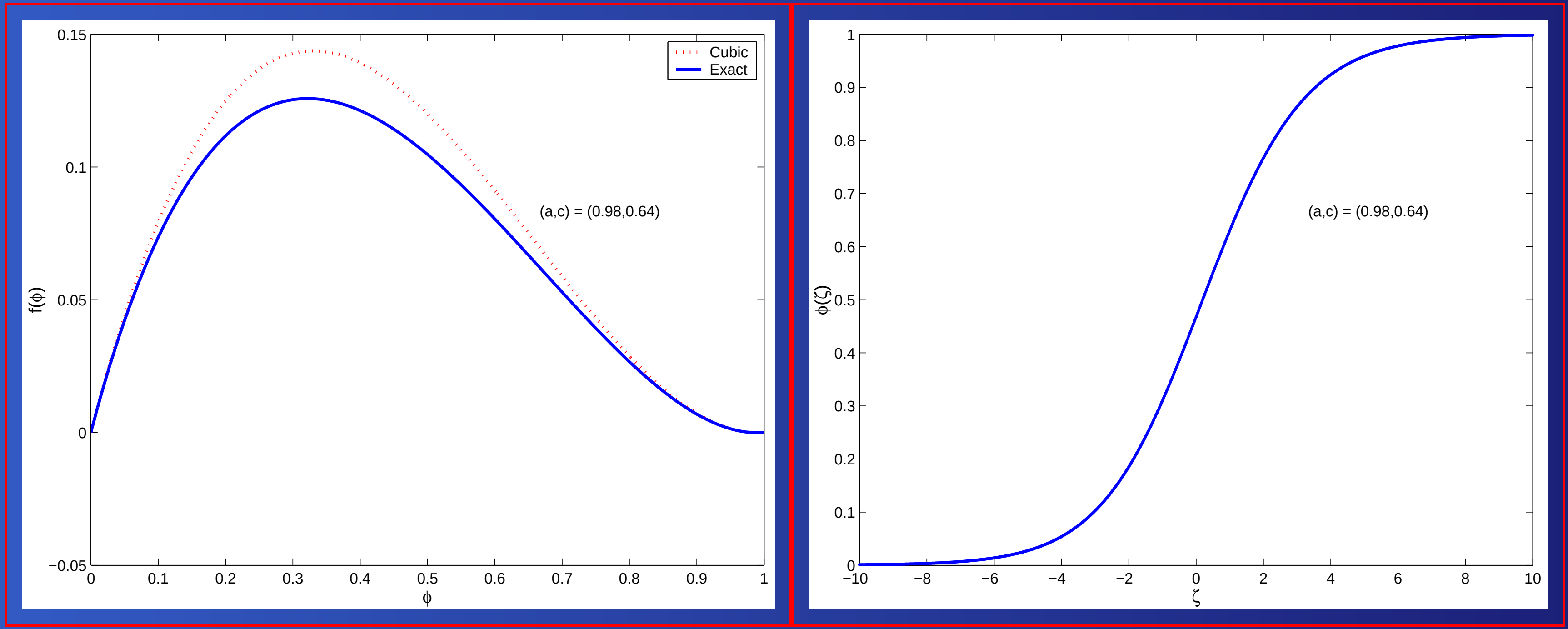
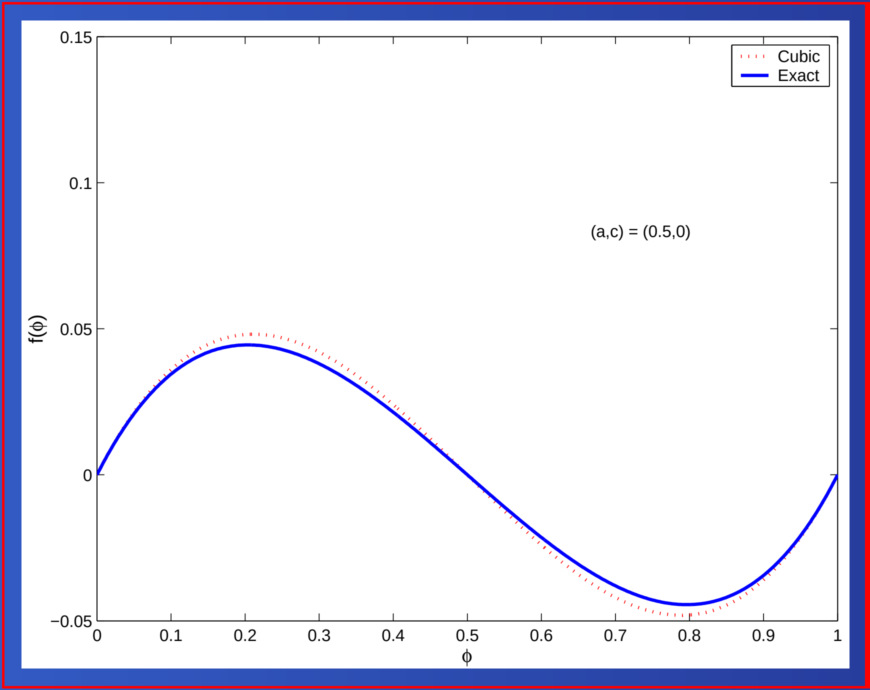
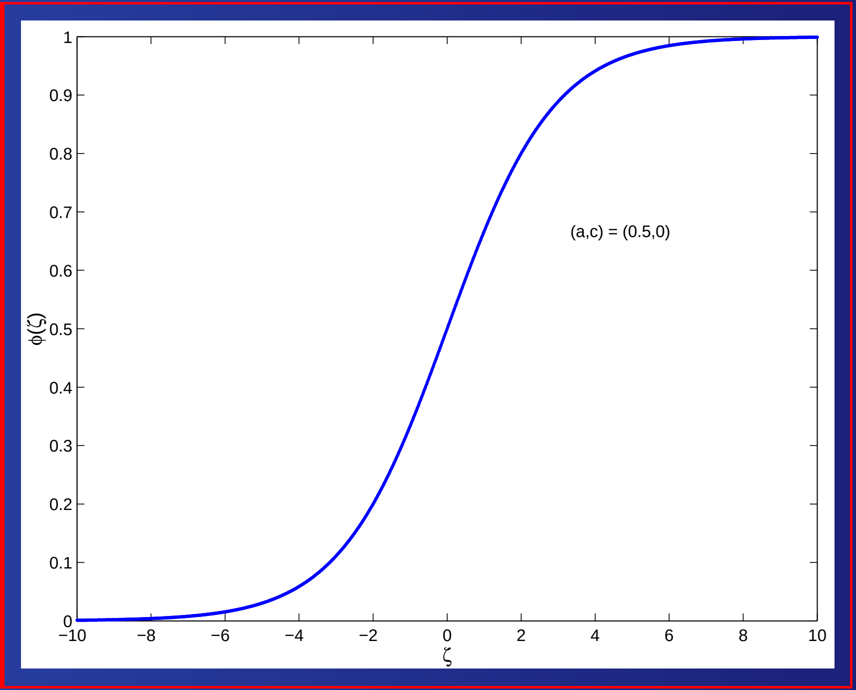
An Exact Solution for Cubic-like f



An Exact Solution for Cubic-like f





- Symmetry is broken, $f'(0) = a$, $f'(1) = 1 - a$
- Propagation Failure Missing, as yet......

Future Directions & Conclusions

- Mixed type FDE theory is incomplete
- Good numerics are needed to inform analysis
- Good analysis is needed to inform the numerics
- Development of suitable test problems, will allow benchmarking of numerical solution algorithms

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