Derivative operators with non-local boundary conditions:

Pseudospectral approximation of eigenvalues

Dimitri Breda

dbreda@dimi.uniud.it - http://www.dimi.uniud.it/dbreda

Dipartimento di Matematica e Informatica

Università degli Studi di Udine

Outline of talk - part I

The “prototype” problem:

- from a retarded functional differential equation...
- ...to a derivative operator $\mathcal{A}$ with boundary conditions
- asymptotic stability and eigenvalues of $\mathcal{A}$

Outline of talk - part II

Derivative operators with non-local boundary conditions:

- linear autonomous differential systems with multiple discrete and distributed delays
- linear autonomous neutral retarded functional differential equations
- linear abstract functional differential equations
- age-structured population models
- mixed-type functional differential equations

Outline of talk - part III

Numerical methods:

- general structure
- pseudospectral differencing approach
- convergence analysis
- numerical results
- conclusions

Collaboration

Research activity with

R. Vermiglio - Università di Udine

S. Maset - Università di Trieste

I - RFDEs

"A retarded functional differential equation (RFDE) consists of a **rule to extend** *the initial function beyond its original domain of definition"*

[Diekmann *et al*, 1994]

I - The protoype problem

- let $\mathcal{C} = C([-\tau, 0], \mathbb{C}^m)$ with supremum norm

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- consider the initial value problem for the linear autonomous RFDE

$$\begin{cases} x'(t) = L_0 x(t) + L_1 x(t-\tau), \quad L_0, L_1 \in \mathbb{C}^{m \times m}, \quad t \geq 0 \\ x(t) = \phi(t), \quad \phi \in \mathcal{C}, \quad t \in [-\tau, 0] \end{cases}$$

- constant delay $\tau > 0$

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From a dynamical system point of view:

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- to use same space $\mathcal{C}$, shift back to $[-\tau, 0]$:

$$x_t(\theta) := x(t+\theta), \quad \theta \in [-\tau, 0]$$

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- state space: $\mathcal{C}$
- state at time $t \geq 0$: $x_t \in \mathcal{C}$

I - Abstract Cauchy problem

- analyze behavior of solution following time evolution of initial state in $\mathcal{C}$

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- analyze behavior of solution following time evolution of initial state in $\mathcal{C}$
- description by the abstract Cauchy problem

$$\begin{cases} \frac{d}{dt}u(t) = \mathcal{A}u(t), & t > 0 \\ u(0) = \phi, & \end{cases}$$

where...

I - derivative operator with boundary conditions

- ...$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{C} \to \mathcal{C}$ linear unbounded operator

$\mathcal{A}\psi = \psi'$ **shift = derivative**

$$D\left(\mathcal{A}\right) = \left\{ \psi \in \mathcal{C} \mid \psi' \in \mathcal{C}, \quad \psi'(0) = L_0\psi(0) + L_1\psi(-\tau) \right\}$$

extension rule = boundary condition

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NOTE: $\mathcal{A}$ is the infinitesimal generator of the $\mathcal{C}_0$-semigroup $\{T(t)\}_{t\geq 0}$ of linear bounded operators on $\mathcal{C}$ where $T(t)$ is the solution operator associated to the problem and defined by $T(t)\varphi = x_t$, $t \geq 0$

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- λ is eigenvalue of $\mathcal{A}$ iff there exists eigenvector $\psi \in D(\mathcal{A}) \setminus \{0\}$ st (λ, ψ) satisfies the characteristic equation

$$\lambda \in \sigma(\mathcal{A}),\ \psi \in M_\lambda \Leftrightarrow \mathcal{D}(\lambda)\psi = 0$$

I - Prototype problem: characteristic equation

- $\lambda \in \mathbb{C}$ is eigenvalue of $\mathcal{A}$ iff there exists $\psi \in \mathcal{C} \setminus \{0\}$ st

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- i.e. there exists $\psi(0) \in \mathbb{C}^m \setminus \{0\}$ such that

$$\left(\lambda I - L_0 - L_1 e^{-\lambda\tau}\right)\psi(0) = 0$$

$$\Updownarrow$$

$$\det\left(\lambda I - L_0 - L_1 e^{-\lambda\tau}\right) = 0$$

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This theory represents an important mathematical tool to investigate the behavior of solutions of more general classes of linear autonomous RFDEs and other types of functional differential systems.

II - Delay differential equations (DDEs)

Let $\mathcal{C} = C([-\tau, 0], \mathbb{C}^m)$ and consider the linear autonomous system of DDEs

$$x'(t) = \mathcal{L}(x_t), \quad t \geq 0,$$

with $\mathcal{L} : \mathcal{C} \to \mathbb{C}^m$ given by

$$\mathcal{L}(\psi) = L_0\psi(0) + \sum_{l=1}^{k} \left(L_l\psi(-\tau_l) + \int_{-\tau_{l-1}}^{-\tau_l} M_l(\theta)\psi(\theta)d\theta \right)$$

with $0 = \tau_0 < \cdots < \tau_k = \tau$, $L_l \in \mathbb{C}^{m\times m}$ and
$M_l : [-\tau_{l-1}, -\tau_l] \to \mathbb{C}^{m\times m}$ suff. smooth for $l = 1, \ldots, k$

II - DDEs

- **State space:** $\mathcal{C}$ with supremum norm
- **State:** $x_t \in \mathcal{C}, \quad t \geq 0$
- **Operator:** $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{C} \to \mathcal{C}$ given by

$$\mathcal{A}\psi = \psi'$$

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NOTE: $\mathcal{A}$ is the infinitesimal generator of the $\mathcal{C}_0$-semigroup $\{T(t)\}_{t\geq 0}$ of solution operators on $\mathcal{C}$ given by $T(t)x_0 = x_t$, $t \geq 0$

II - Neutral RFDEs

Let $\mathcal{C} = C([-\tau, 0], \mathbb{C}^m)$ and consider the linear autonomous system of neutral RFDEs

$$\frac{d}{dt}[x(t) + \mathcal{N}(x_t)] = \mathcal{L}(x_t), \quad t \geq 0,$$

with $\mathcal{N}, \mathcal{L} : \mathcal{C} \to \mathbb{C}^m$ continuous, linear and $\mathcal{N}$ atomic at zero given by

$$\mathcal{N}(\psi) = \int_{-\tau}^{0} d\eta(\theta)\psi(\theta), \quad \mathcal{L}(\psi) = \int_{-\tau}^{0} d\mu(\theta)\psi(\theta)$$

where η, μ have not singular part

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II - Abstract FDEs

Let X and $\mathcal{C} = C([-\tau, 0], X)$ be Banach spaces and consider the system of linear abstract FDEs

$$\frac{dx(t)}{dt} = \mathcal{A}_T x(t) + \mathcal{L}(x_t), \quad t \geq 0,$$

where $\mathcal{A}_T : D(\mathcal{A}_T) \subset X \to X$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of linear bounded operators on X and $\mathcal{L} : \mathcal{C} \to X$ is a linear mapping

II - Abstract FDEs

- **State space:** $\mathcal{C}$ with supremum norm

$$\|\psi\| = \sup_{\theta\in[-\tau,0]} |\psi(\theta)|, \qquad |\cdot| \text{ norm of } X$$

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NOTE: $\mathcal{A}$ is the infinitesimal generator of...

II - Age-structured population models

Let $\mathcal{C} = C([0, a_\dagger], \mathbb{C}^m)$ and consider the age-structured population model

$$\begin{cases} \frac{\partial x}{\partial t}(a,t) + \frac{\partial x}{\partial a}(a,t) = 0, \quad a \in [0, a_\dagger], \quad t \geq 0, \\ x(0,t) = Kx(\cdot, t), \quad t \geq 0, \\ x(a,0) = \varphi(a), \quad a \in [0, a_\dagger], \end{cases}$$

where $x \in C\left([0, a_\dagger] \times [0, +\infty), \mathbb{C}^m\right)$, $\varphi \in \mathcal{C}$ and $K : \mathcal{C} \to \mathbb{C}^m$:

$$K\psi = \sum_{l=1}^{d} \int_{a_{l-1}}^{a_l} k^{(l)}(a)\psi(a)da, \quad 0 = a_0 < a_1 < \cdots < a_d = a_\dagger,$$

with sufficiently smooth matrix kernels $k^{(l)}$, $l = 1, \ldots, d$

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$T(t)x(\cdot, 0) = x(\cdot, t)$, $t \geq 0$

II - Mixed-type FDEs

Let $\mathcal{C} = C([-p,q],\mathbb{C}^m)$, $p,q>0$, and consider the system of linear autonomous FDEs of mixed-type (i.e. advanced and retarded)

$$\frac{dx(t)}{dt} = \int_{-p}^{q} dk(\theta)x(t+\theta),$$

where $x(t)\in\mathbb{C}^m$ and $dk(\theta)$ is an $m\times m$ matrix of Lebesgue-Stiltjes measures on $[-p,q]$.

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Although such equations do not generate semigroups, it is still useful to take...

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NOTE: under suitable hypothesis on $\sigma(\mathcal{A})$, $\mathcal{A}$ is exponential dichotomous, i.e. there exists a direct sum $\mathcal{C} = P \oplus Q$, such that $\mathcal{A}_P : D(\mathcal{A}) \cap P \to P$, $\mathcal{A}_Q : D(\mathcal{A}) \cap Q \to Q$ are infinitesimal generators of (exp.ly stable) semigroups.

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- asymptotic behavior of solutions depends on the position on $\mathbb{C}$ of the eigenvalues of a derivative operator $\mathcal{A}$
- $\mathcal{A}$ has non-local boundary conditions contained in its domain

III - General structure

- **State space:** $\mathcal{C} := C([\alpha, \beta], X)$ with supremum norm

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- **Derivative operator:** $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{C} \to \mathcal{C}$ given by

$$\mathcal{A}\psi = (\pm)\psi'$$

$$D(\mathcal{A}) = \Big\{ \psi \in \mathcal{C} \mid \psi' \in \mathcal{C},$$

$$\psi'(\bar{\theta}) + \mathcal{N}(\psi') = \mathcal{L}(\psi) \text{ or } \psi(\bar{\theta}) = \mathcal{L}(\psi) \Big\}$$

with $\bar{\theta} \in [\alpha, \beta]$ and $\mathcal{L}, \mathcal{N} : \mathcal{C} \to X$ linear

III - Infinite dimension

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- how to compute its infinitely many eigenvalues?

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- how to compute its infinitely many eigenvalues?
- need of numerical approximation

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  - approximate functions of $\mathcal{C}$ by interpolating polynomials on certain nodes
  - approximate exact derivative with that of interpolating polynomials
  - use boundary condition applied to the polynomials

III - Mesh

Given a positive integer N, discretize $[\alpha, \beta]$ with the mesh

$$\Omega_N = \{\theta_i \mid i = 0, \ldots, N\}$$

of $N + 1$ distinct points

III - State space

- replace the continuous space $\mathcal{C}$ by the space $\mathcal{C}_N = (\mathbb{C}^m)^{\Omega_N} \cong \mathbb{C}^{m(N+1)}$ of the discrete functions defined on the mesh Ω_N

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- replace the continuous space $\mathcal{C}$ by the space $\mathcal{C}_N = (\mathbb{C}^m)^{\Omega_N} \cong \mathbb{C}^{m(N+1)}$ of the discrete functions defined on the mesh Ω_N
- i.e. any $\psi \in \mathcal{C}$ is discretized into the block-vector $x \in \mathcal{C}_N$ of components

$$x_i = \psi(\theta_i) \in \mathbb{C}^m, \quad i = 0, \ldots, N$$

III - Operator: $\psi'(\bar{\theta}) + \mathcal{N}(\psi') = \mathcal{L}(\psi)$

- assume $\bar{\theta} \in \Omega_N$, i.e. $\theta_{\bar{i}} = \bar{\theta}$ for some $\bar{i} \in \{0, 1, \ldots, N\}$

III - Operator: $\psi'(\bar\theta) + \mathcal{N}(\psi') = \mathcal{L}(\psi)$

- assume $\bar\theta \in \Omega_N$, i.e. $\theta_{\bar i} = \bar\theta$ for some $\bar i \in \{0, 1, \ldots, N\}$
- let $\mathcal{P}_N x$, $x \in \mathcal{C}_N$, be the unique $\mathbb{C}^m$-valued interpolating polynomial of degree $\leq N$ such that

$$(\mathcal{P}_N x)(\theta_i) = x_i, \quad i = 0, 1, \ldots, N$$

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- approximate $\mathcal{A}$ by the matrix $\mathcal{A}_N : \mathcal{C}_N \to \mathcal{C}_N$ given by

$$\begin{cases} (\mathcal{A}_N x)_{\bar{i}} = \mathcal{L}_N(\mathcal{P}_N x) - \mathcal{N}_N((\mathcal{P}_N x)') \\ (\mathcal{A}_N x)_i = (\mathcal{P}_N x)'(\theta_i), \quad i = 0, 1, \ldots, N, \quad i \neq \bar{i} \end{cases}$$

with $\mathcal{L}_N$ and $\mathcal{N}_N$ possibly approximating $\mathcal{L}$ and $\mathcal{N}$

III - Operator: $\psi'(\bar{\theta}) + \mathcal{N}(\psi') = \mathcal{L}(\psi)$

- use Lagrange representation

$$(\mathcal{P}_N x)(\theta) = \sum_{j=0}^{N} l_j(\theta) x_j, \quad \theta \in [\alpha, \beta],$$

with l_j's the Lagrange polynomials on Ω_N

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- the matrix $\mathcal{A}_N$ is defined by the relations

$$\begin{cases} (\mathcal{P}_N x)'(\theta_{\bar{i}}) = \sum_{j=0}^{N} \left(\mathcal{L}_N\left((l_j(\cdot)I\right) - \mathcal{N}_N\left((l'_j(\cdot)I\right)\right) x_j \\ (\mathcal{P}_N x)'(\theta_i) = \sum_{j=0}^{N} l'_j(\theta_i) x_j, \quad i = 0, 1, \ldots, N, \quad i \neq \bar{i} \end{cases}$$

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III - Operator: $\psi(\bar{\theta}) = \mathcal{L}(\psi)$

- use Lagrange representation

$$(\mathcal{Q}_{N+1}x)(\theta) = \sum_{j=0}^{N} m_j(\theta)x_j + m_{N+1}(\theta)x_{N+1}, \quad \theta \in [\alpha, \beta],$$

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III - Convergence assumptions

Eigenvalues of $\mathcal{A}_N$ approximate a finite number of the eigenvalues of $\mathcal{A}$. How accurate are these approximations?

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- choose θ_i, $i = 0, 1, \ldots, N$, as the Chebyshev extremal nodes in $[\alpha, \beta]$
- possibly assume

$$\sup_{N \in \mathbb{N}} \|\mathcal{L}_N\| < +\infty$$

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$$\sup_{N \in \mathbb{N}} \|\mathcal{N}_N\| < +\infty$$

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- $\mathcal{A}_N$ has ν eigenvalues λ_i, $i = 1, \ldots, \nu$, such that

$$\max_{i=1,\ldots,\nu} |\overline{\lambda} - \lambda_i| \leq C_2^{1/\nu} \left(\varepsilon_N(\mathcal{L}) + \varepsilon_N(\mathcal{N}) + \frac{1}{\sqrt{N}} \left(\frac{C_1}{N} \right)^N \right)^{1/\nu}$$

$$\varepsilon_N(\mathcal{L}) = \sup_{\lambda \in B(\overline{\lambda}, r)} \frac{|\mathcal{L}\left(e^{-\lambda \cdot} u\right) - \mathcal{L}_N\left(e^{-\lambda \cdot} u\right)|}{|u|}, \quad u \in \mathbb{C}^m$$

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- $C_1 = C_1(\overline{\lambda})$, $C_2 = C_2(\overline{\lambda})$ constants independent of N

III - Ghost roots

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- sure that approximated roots converge to exact ones?
- let $\{\Omega_{N^{(i)}}\}_{i\geq 1}$ be a sequence of meshes on $[\alpha,\beta]$ such that $N^{(i)} \longrightarrow \infty$ as $i \longrightarrow \infty$
- if $\lambda^{(i)} \longrightarrow \overline{\lambda}$ as $i \longrightarrow \infty$ for $\lambda^{(i)}$ eigenvalue of $\mathcal{A}_N^{(i)}$, then $\overline{\lambda}$ is eigenvalue of $\mathcal{A}$

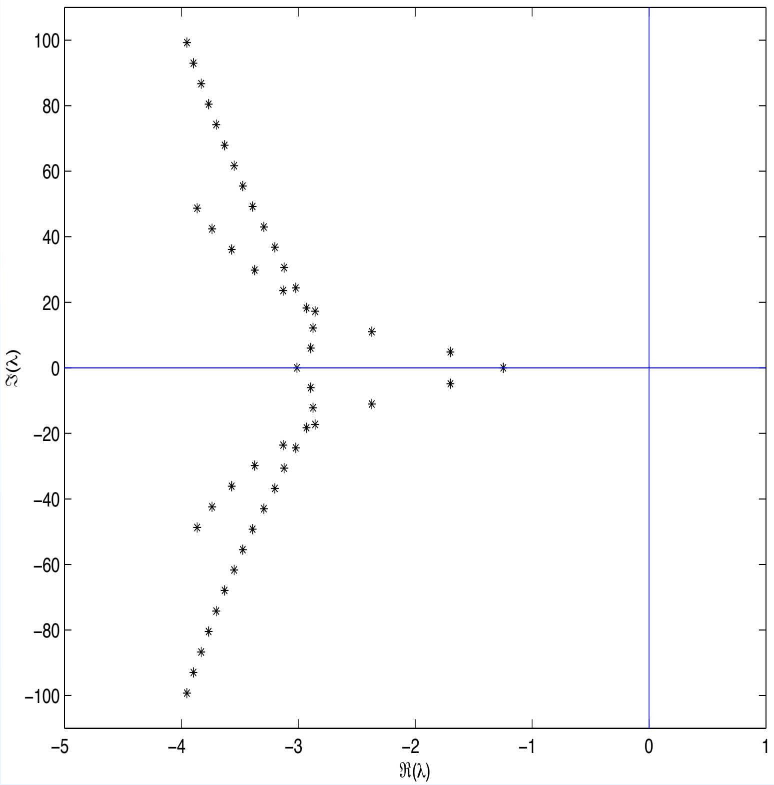
III - Results: DDEs

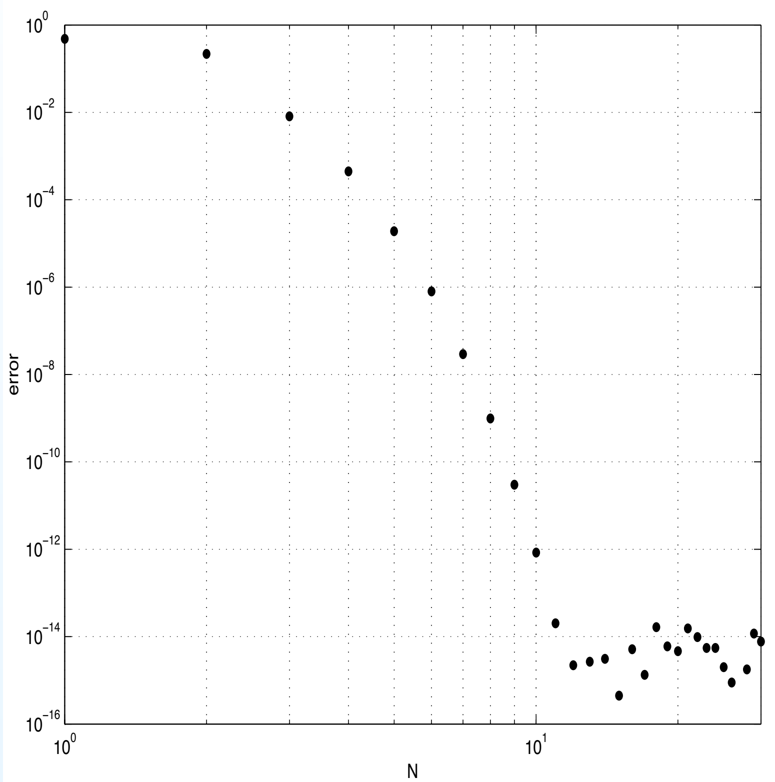
$$x'(t) = L_0 x(t) + L_1 x(t-1) + \int_{-0.3}^{-0.1} M_1 x(t+\theta)\,d\theta + \int_{-1}^{-0.5} M_2 x(t+\theta)\,d\theta$$

$$L_0 = \begin{pmatrix} -3 & 1 \\ -24.646 & -35.430 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 1 & 0 \\ 2.35553 & 2.00365 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 2 & 2.5 \\ 0 & -0.5 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

III - Results: DDEs, spectrum and convergence

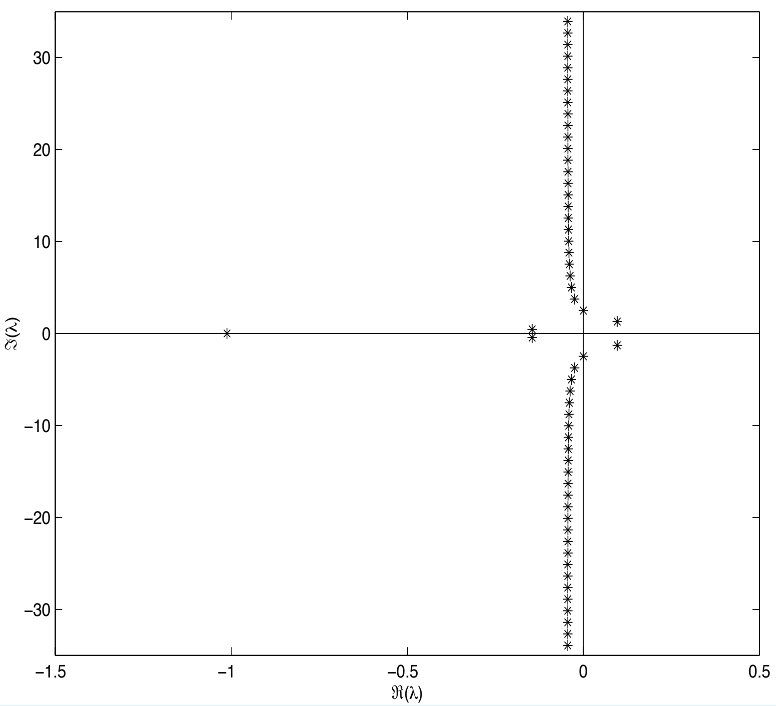


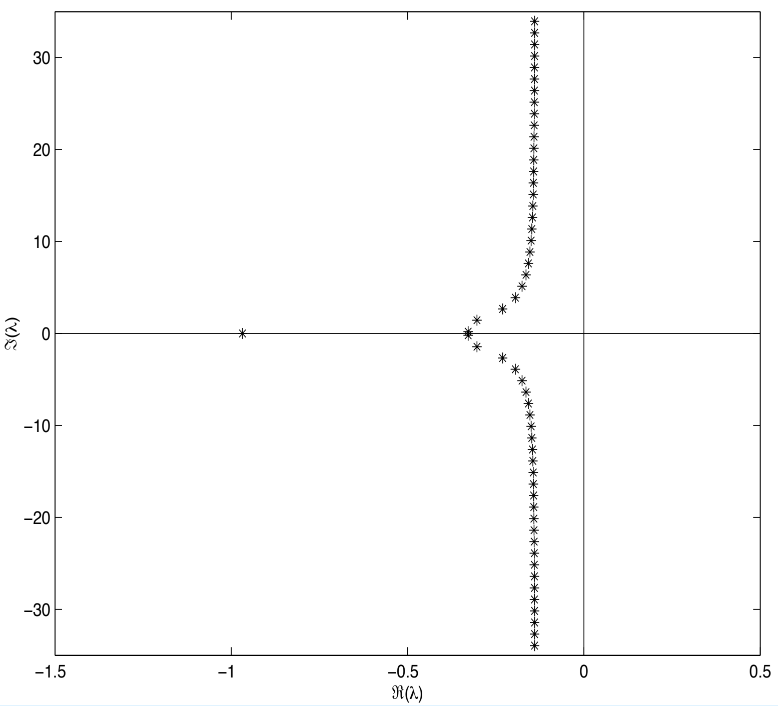


III - Results: NDDEs, spcetrum

$$x'(t) = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix} x'(t-\tau)$$

$a = 1,\ b = 0.5,\ h = 0.8,\ \tau = 5,$ $\quad a = 1,\ b = 4,\ h = 0.5,\ \tau = 5$





III - Results: abstract RFDEs

$$\begin{cases} \frac{\partial x}{\partial t}(a,t) = \frac{\partial^2 x}{\partial a^2}(a,t) - bx(a,t) - cx(a,t-\tau), \quad a \in [0,\pi], \quad t > 0 \\ x(0,t) = x(\pi,t) = 0, \quad t \geq 0 \\ x(a,t) = \varphi(t)(a), \quad a \in [0,\pi], \quad t \in [-\tau,0], \quad \varphi \in C([-\tau,0],X) \end{cases}$$

$$X := L^2([0,\pi],\mathbb{R})$$

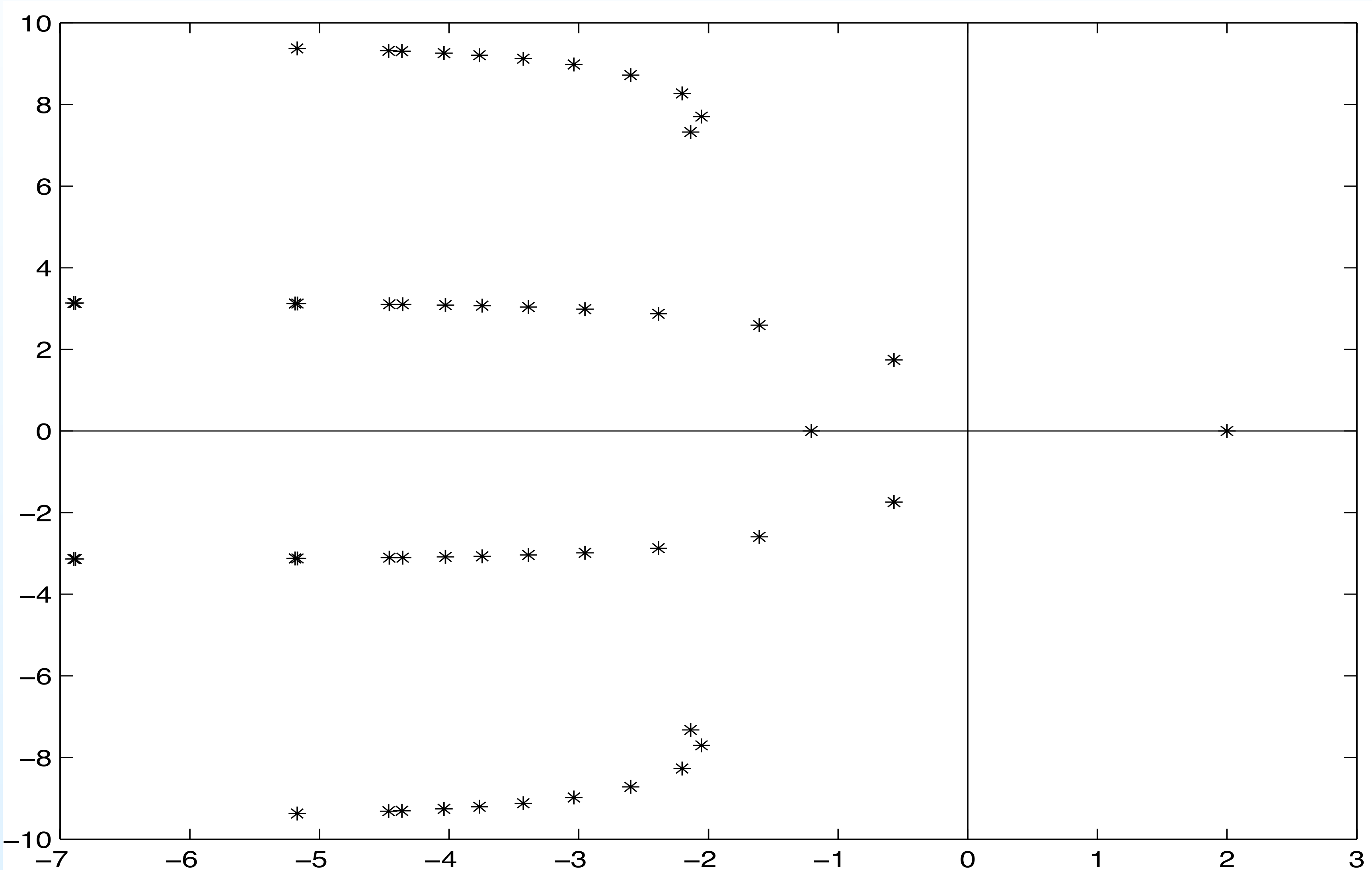
$$\mathcal{A}_T : D(\mathcal{A}_T) \subset X \to X$$

$$\mathcal{A}_T y = y''$$

$$D(\mathcal{A}_T) : \left\{ y \in C^2([0,\pi],\mathbb{R}) \mid y(0) = y(\pi) = 0 \right\}$$

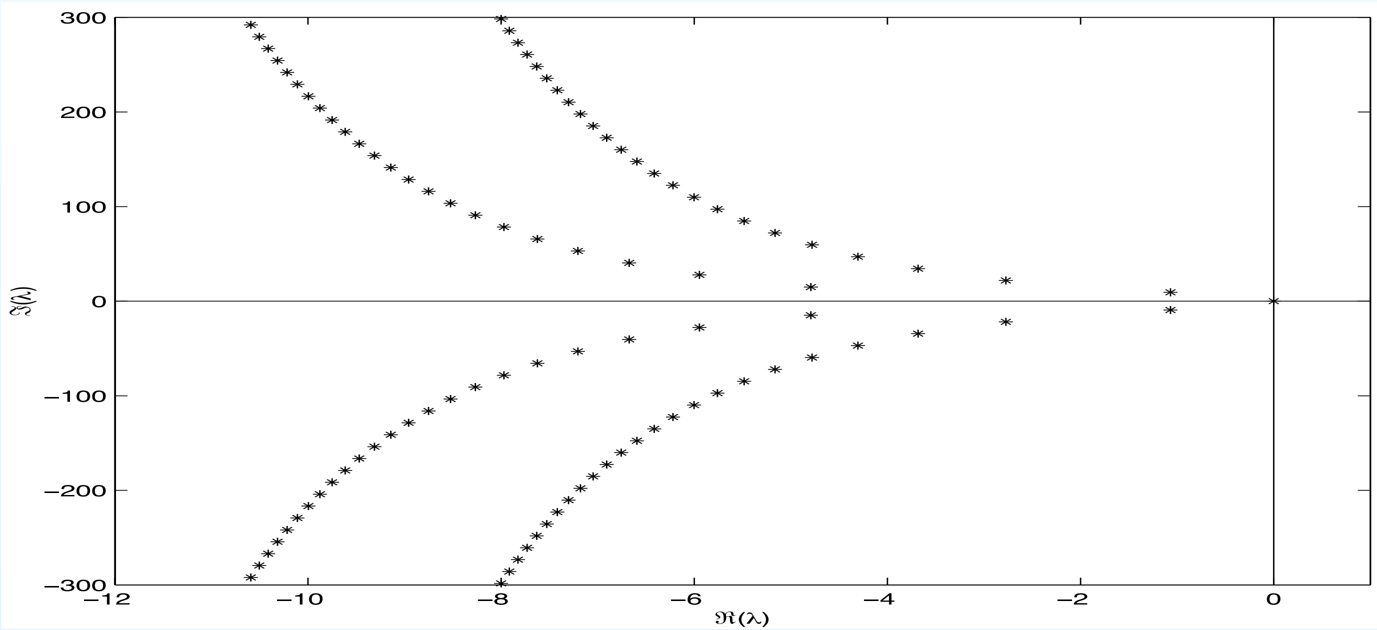
III - Results: abstract RFDEs, spectrum

$$b = -3 - e^{-2},\ c = 1,\ \tau = 1$$

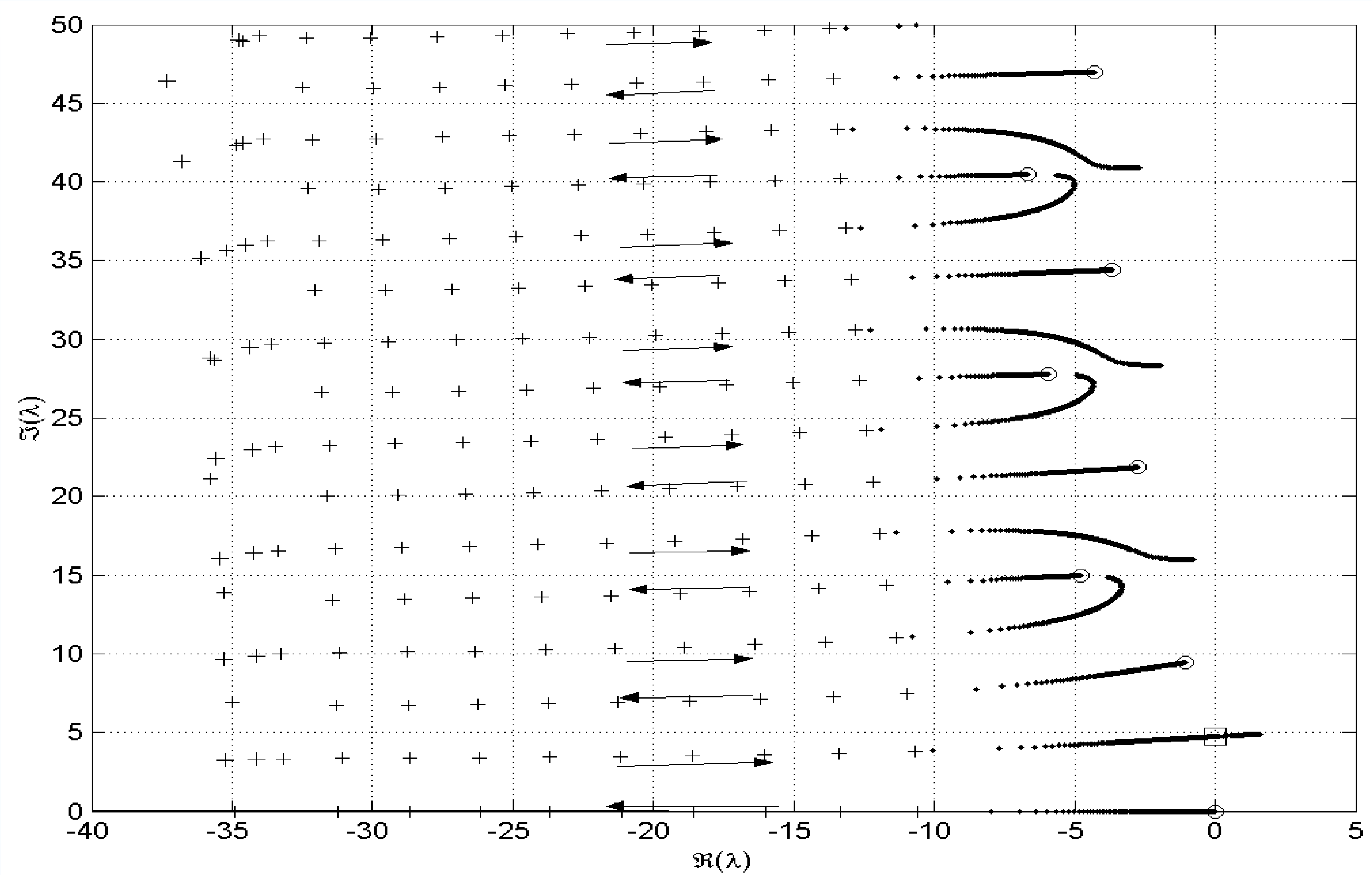


III - Results: population model, spectrum

$$\begin{cases} \dfrac{\partial x}{\partial t}(a,t) + \dfrac{\partial x}{\partial a}(a,t) = 0, \quad a \in [0, a_\dagger], \quad t \geq 0 \\ x(0,t) = \displaystyle\int_0^{a_\dagger} 8\left[1 - \ln R_0\right](1-a)\chi_{[\frac{1}{2},1]}(a)x(a,t)da, \quad t \geq 0 \\ x(a,0) = \varphi(a), \quad a \in [0, a_\dagger] \end{cases}$$



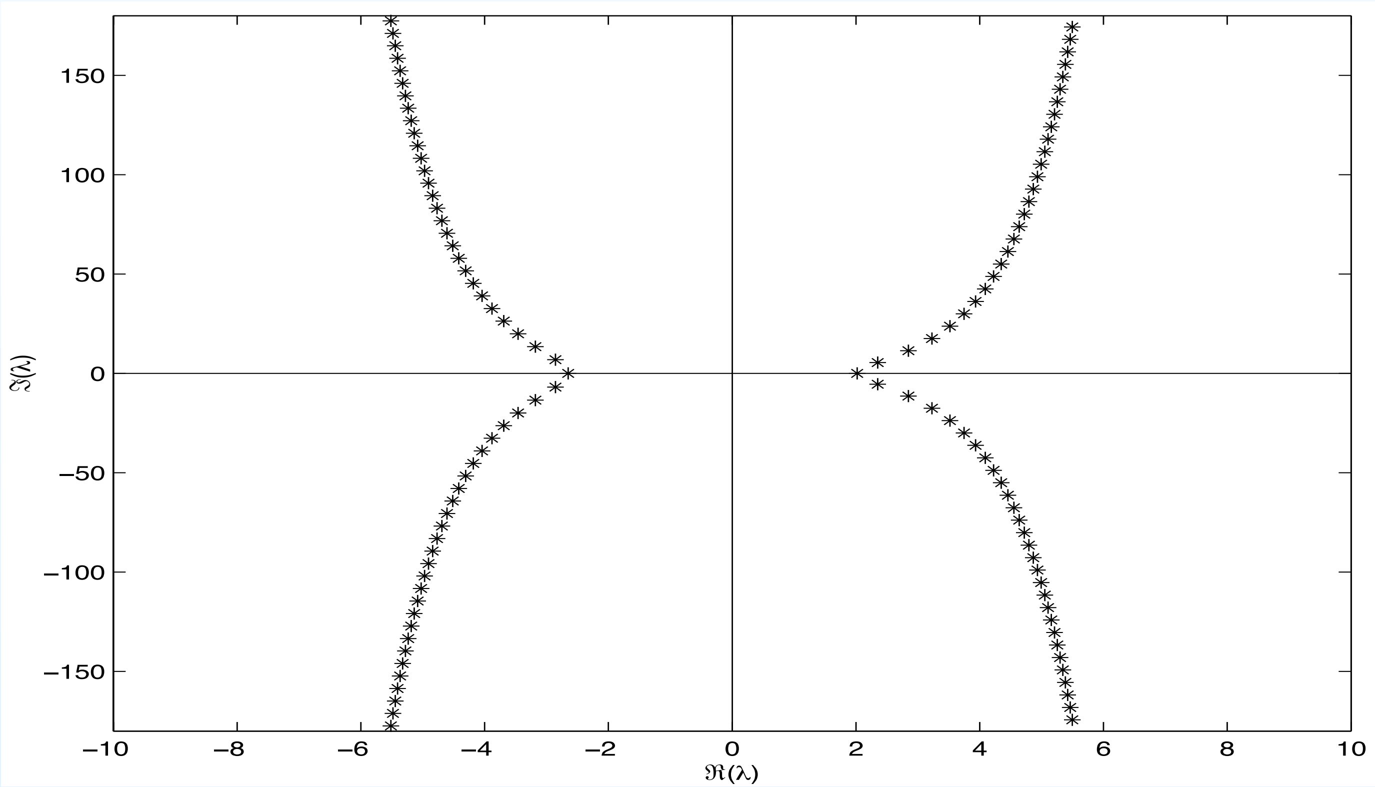
III - Results: population model, spectrum variation



III - Results: mixed-type RFDEs, spectrum

$$x'(t) = ax(t+1) + bx(t) + cx(t-1)$$

$$a = c = -0.714, \quad b = 7.5$$



III - Results: space discretized PDE, delay in diffusion

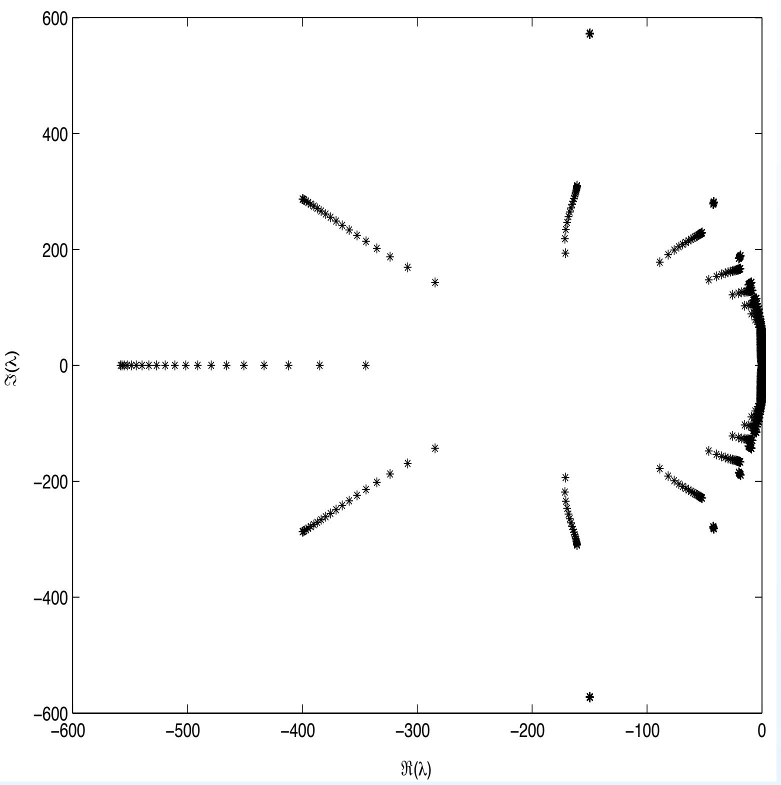
$$x'(t) = L_0 x(t) + L_1 x(t-\tau), \quad L_0, L_1 \in \mathbb{C}^{m\times m}$$

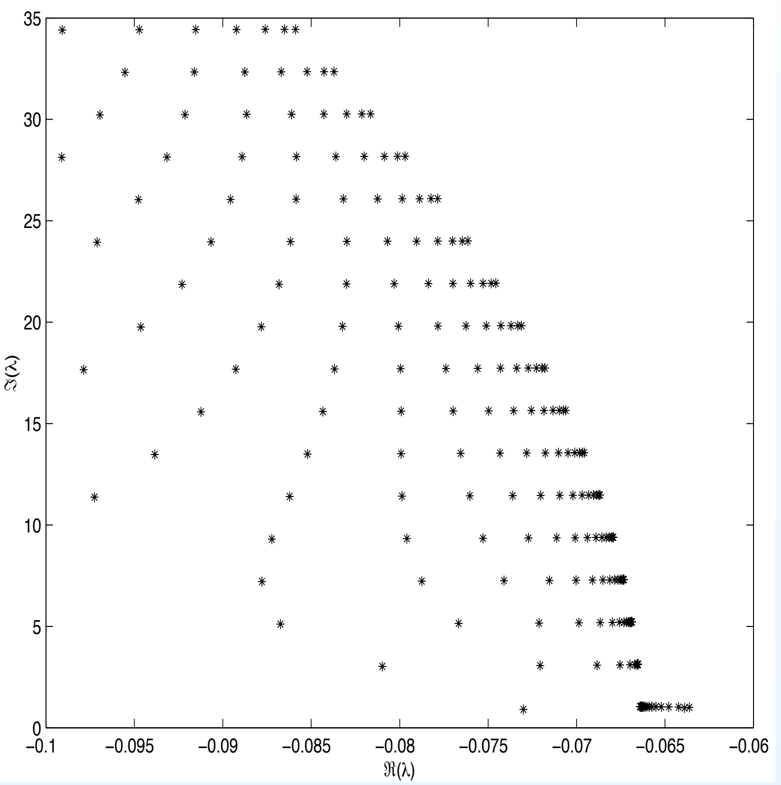
$$L_0 = \frac{1}{h^2}\left(\frac{1-\alpha}{\beta} + \alpha\right)\begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix} + RI_m$$

$$L_1 = \frac{1}{h^2}\frac{1-\alpha}{\beta}\begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}$$

III - Results: space discretized PDE, delay in diffusion

$$m = 20, \quad h = \frac{\pi}{m+1}, \quad \alpha = 0.1, \quad \beta = 2, \quad R = 0.08$$





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- behavior of solution as time-evolution of state
- asymtpotic stability depends on eigenvalues of derivative operator
- holds for more than DDEs
- ∞ eigenvalues: discretization via pseudospectral methods
- fast convergence

The end

...and thanks for your attention!