Linear time delay systems

from characteristic roots...
...to stability charts

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Outline of talk

- Introduction to DDEs

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- Stability of steady states

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Research in collaboration with

R. Vermiglio - Università di Udine

S. Maset - Università di Trieste

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- mathematical formulation by retarded functional differential equations (RFDEs)

DDEs intro: remember ODEs

- let $X = \mathbb{C}^m$
- let $(t, y) \in D \subseteq \mathbb{R} \times X$ and $f : D \to \mathbb{C}^m$ continuous. An ordinary differential equation (ODE) is a relation

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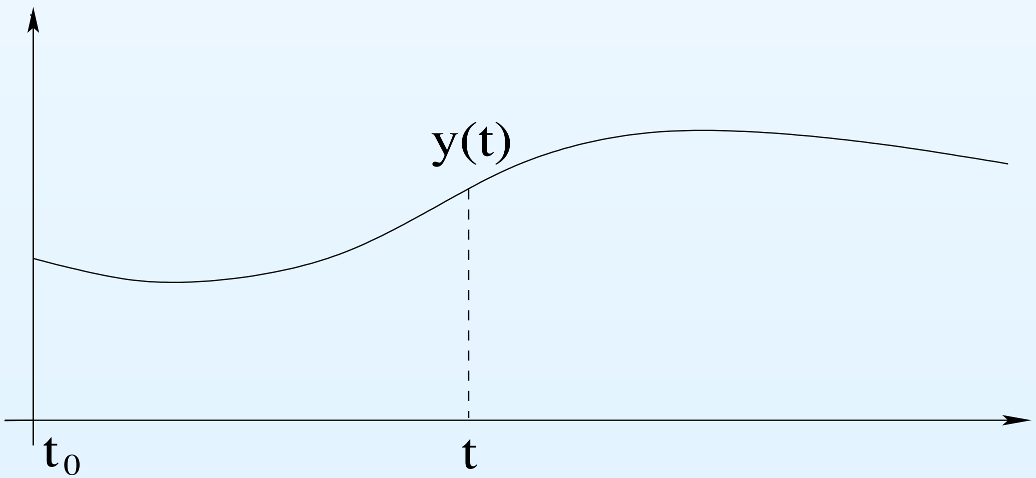
- state at $t \geq t_0$: $y(t) \in X$



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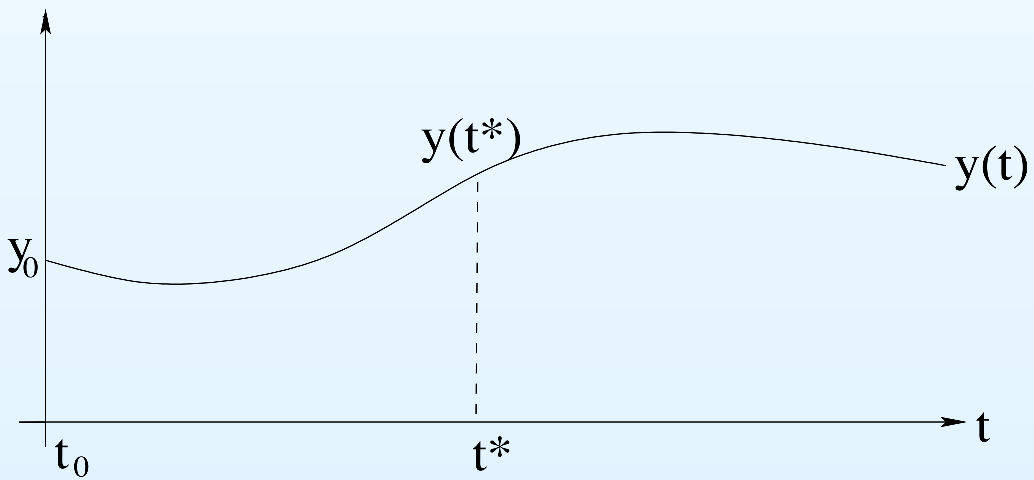

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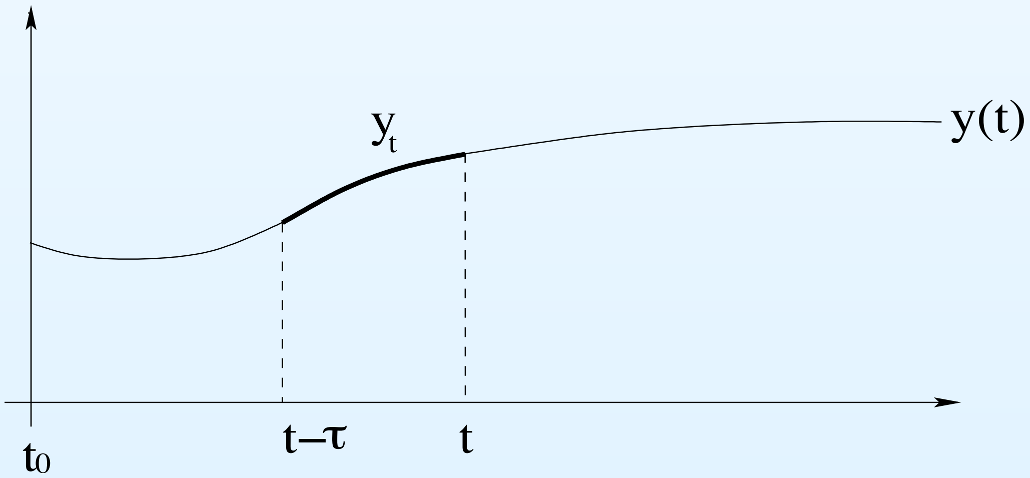
- state $y_t \in X$ at $t \geq t_0$: $y_t(\theta) = y(t + \theta)$, $\theta \in [-\tau, 0]$



DDEs intro: examples

- discrete delay: $f(t, \psi) = L_0\psi(0) + L_1\psi(-\tau)$, then for $\psi(\theta) = y_t(\theta)$ on $[-\tau, 0]$:

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- distributed delay: $f(t, \psi) = L_0\psi(0) + \int_{-\tau}^{0} M_1(\theta)\psi(\theta)d\theta$, then for $\psi(\theta) = y_t(\theta)$ on $[-\tau, 0]$:

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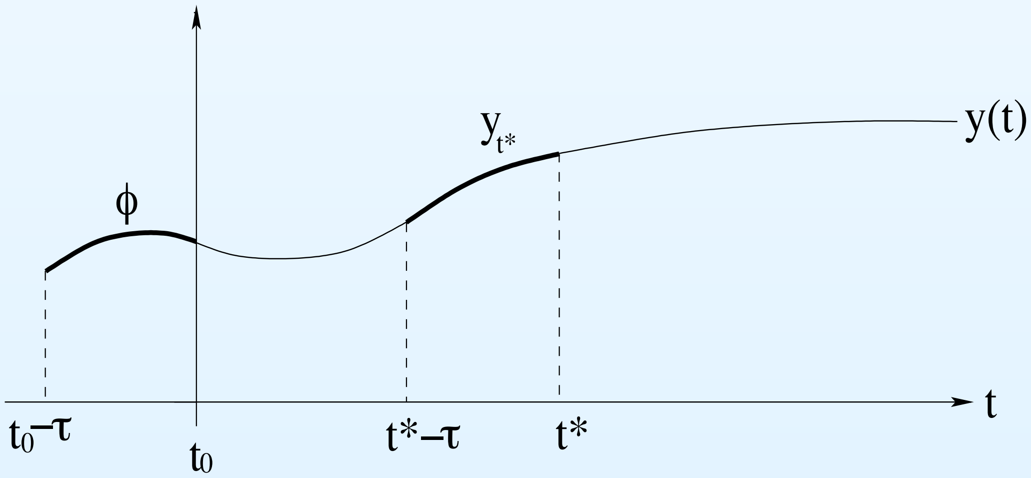

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Stability: question

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- answer: YES!

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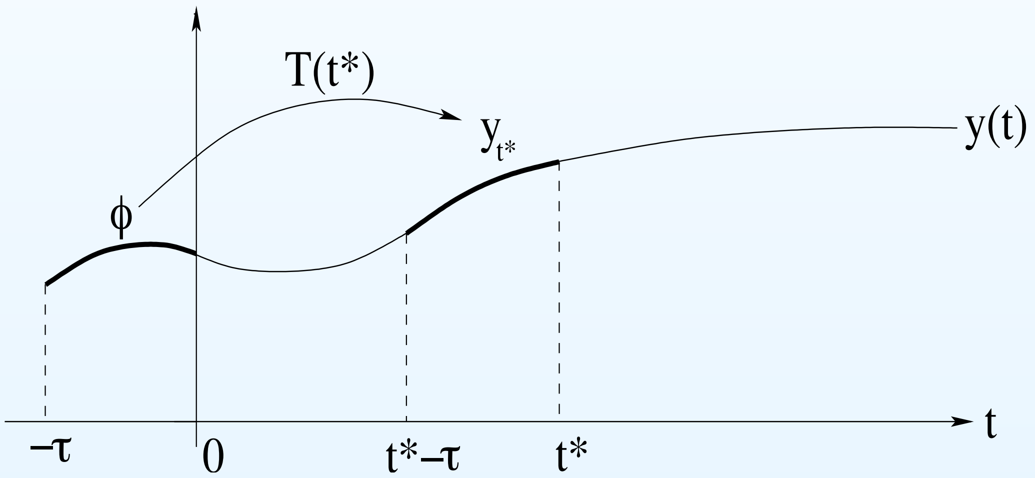
$$T(t)\phi = y_t$$



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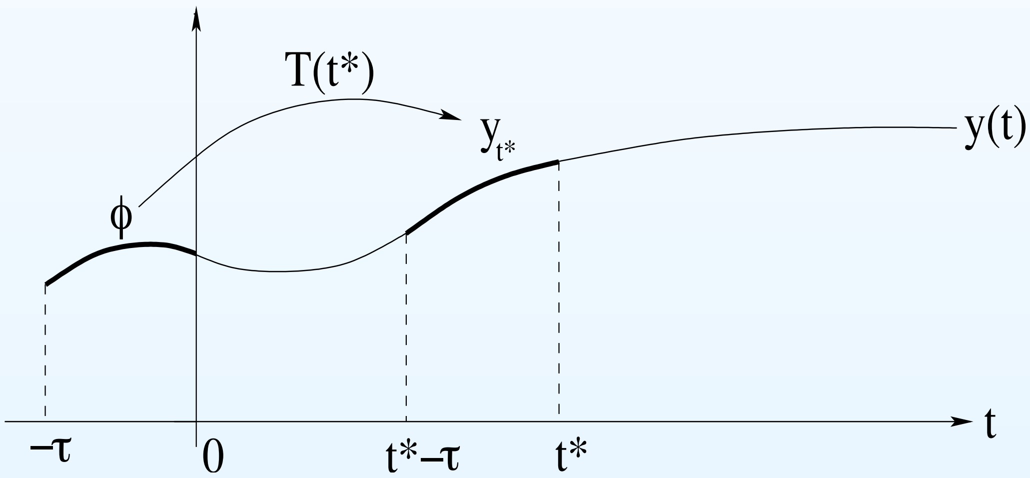

- $\{T(t)\}_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup

Stability: infinitesimal generator

- $\{T(t)\}_{t\geq 0}$ has infinitesimal generator (IG) $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$:

$$\begin{cases} \mathcal{A}\psi = \psi', \quad \psi \in D(\mathcal{A}) \\ D(\mathcal{A}) = \{\psi \in X \mid \psi' \in X \text{ and } \psi'(0) = L_0\psi(0) + L_1\psi(-\tau)\} \end{cases}$$

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- evolution of y_t depends on eigenvalues of $\mathcal{A}$

Stability: theorems

- $\lambda \in \mathbb{C}$ CR $\Leftrightarrow \det\left(\lambda I - L_0 - L_1 e^{-\lambda\tau}\right) = 0$
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∞ dimension!

Numerics: basics

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  - numerical differentiation $+$ splicing condition in $D(\mathcal{A})$
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- past: linear multistep (LMS) and Runge-Kutta (RK) methods
- present: pseudospectral differentiation (PSD)

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- Lagrange representation of ψ_N leads to...

Numerics: approximated IG

...

$$\mathcal{A}_N = \begin{pmatrix} L_0 & 0 & \cdots & 0 & L_1 \\ d_{10} & d_{11} & \cdots & d_{1N-1} & d_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{N0} & d_{N1} & \cdots & d_{NN-1} & d_{NN} \end{pmatrix} \in \mathbb{C}^{m(N+1)\times m(N+1)}$$

where $d_{ij} = l_j'(\theta_i) \otimes I$ with l_j's the Lagrange coefficients

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- analogous result for CMs

Numerics: example

$$x'(t) = L_0 x(t) + L_1 x(t-1) + \int_{-0.3}^{-0.1} M_1 x(t+\theta)\,d\theta + \int_{-1}^{-0.5} M_2 x(t+\theta)\,d\theta$$

$$L_0 = \begin{pmatrix} -3 & 1 \\ -24.646 & -35.430 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 1 & 0 \\ 2.35553 & 2.00365 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 2 & 2.5 \\ 0 & -0.5 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

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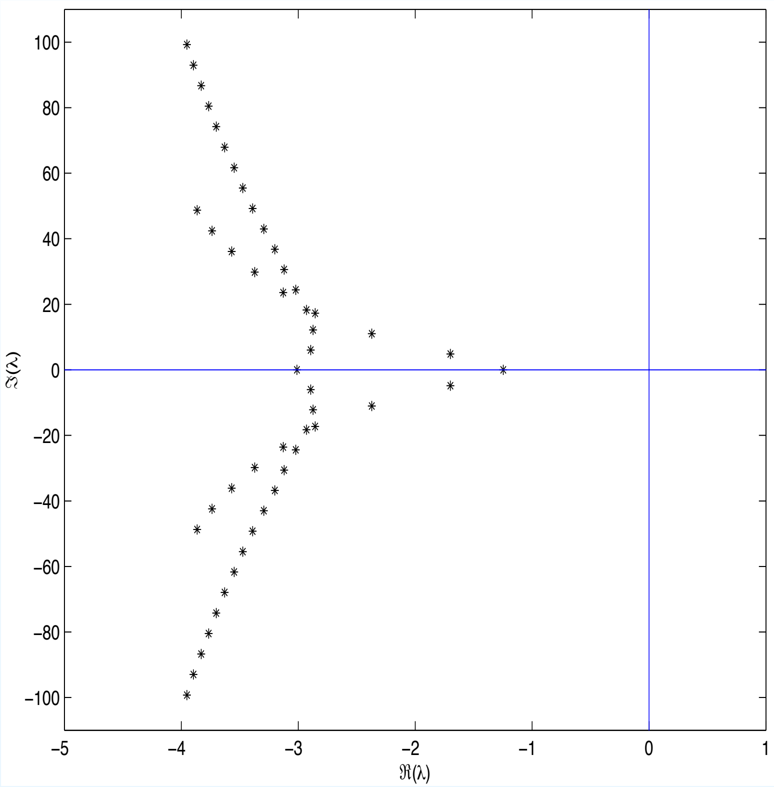

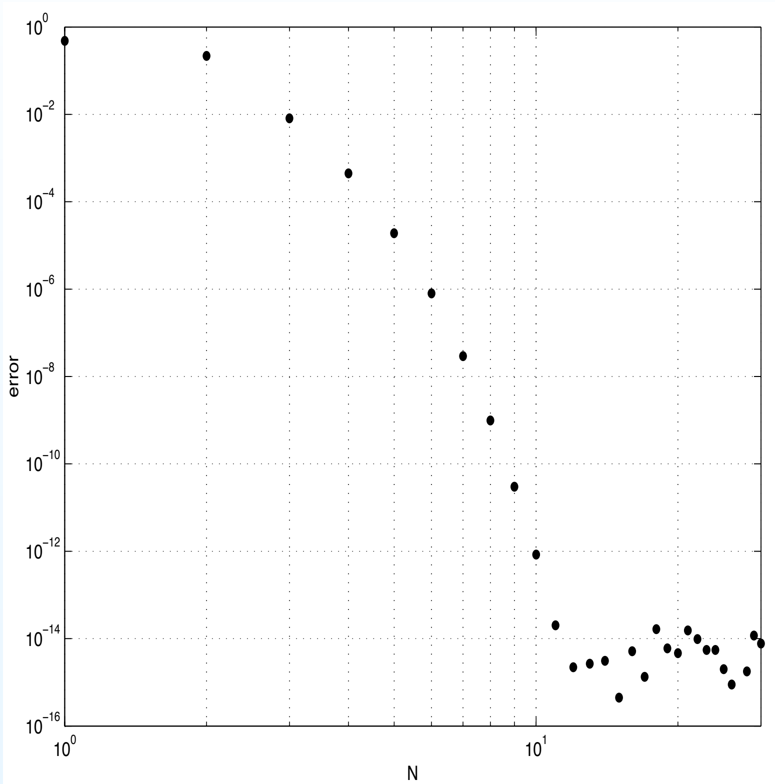

Numerics: generalization

- linear multiple discrete/distributed DDEs:

$$y'(t) = L_0 y(t) + \sum_{l=1}^{k} \left(L_l y(t-\tau_l) + \int_{-\tau_{l-1}}^{-\tau_l} M_l(\theta) y(t+\theta) d\theta \right)$$

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- but also extension to more general linear time delay systems (LTDS):
  - neutral DDEs
  - periodic coefficients DDEs
  - age-structured population dynamics
  - mixed-type FDEs
  - PDEs with delay

Stability charts: what?

- consider a LTDS depending on two parameters p_1 and p_2 (e.g. delays, but not only...) varying in given intervals
- determine where the system is stable or not in the (p_1, p_2)-plane
- how?

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- determine where the system is stable or not in the (p_1, p_2)-plane
- how?
  - point-by-point investigation of the (p_1, p_2)-plane determining the real part of the rightmost CR $r_{CR}(p_1, p_2)$ (or the absolute value of the dominant CM $d_{CM}(p_1, p_2)$)
  - also other approaches...

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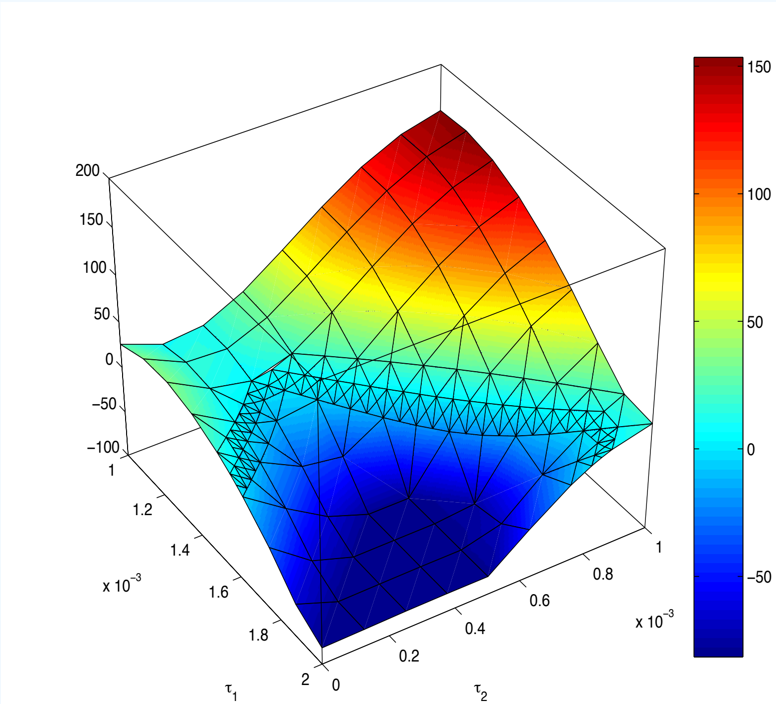
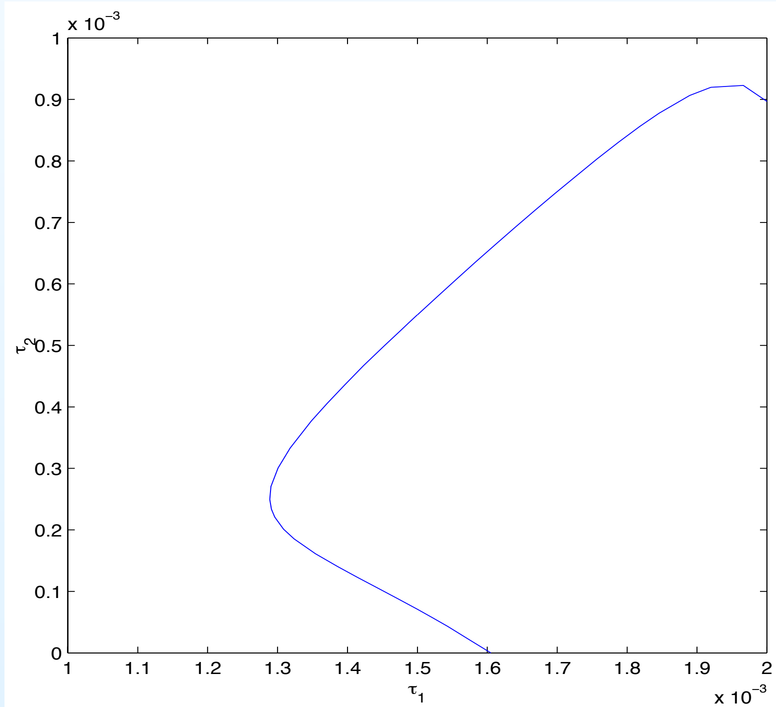
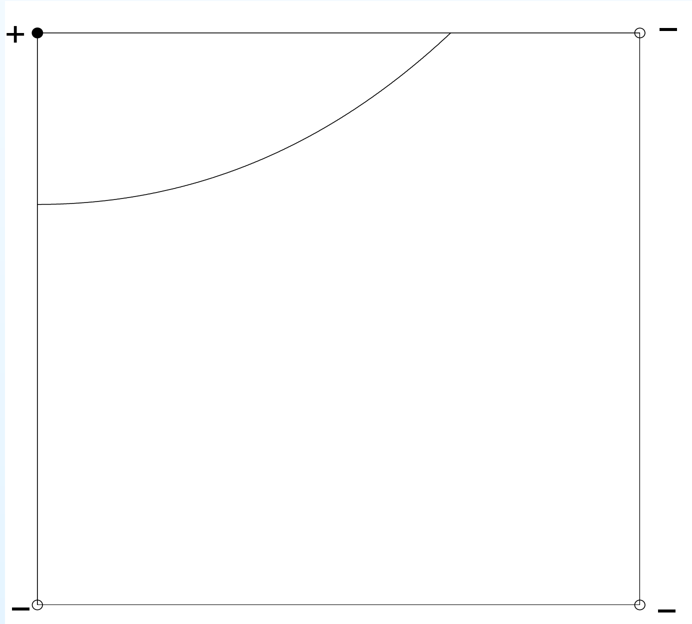
$$\begin{aligned} y'(t) &= L_0 y(t) + L_1 y(t-\tau_1) + L_1 y(t-\tau_2) + L_2 y(t-2\tau_1) + \\ &+ L_2 y(t-2\tau_2) + L_3 y(t-\tau_1-\tau_2), \quad L_i \in \mathbb{C}^{8\times 8} \end{aligned}$$





Stability charts: point-by-point investigation

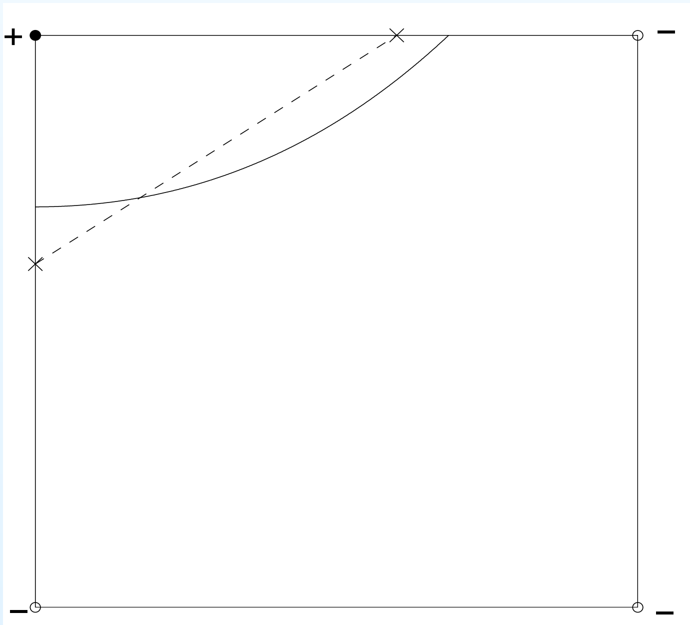
- consider a square cell in the (p_1, p_2)-plane
- evaluate stability of the four vertices by computing r_{CR}
- if a sign change occurs, a SB passes through the cell



- a sort of $2d$-bisection

Stability charts: location of SB

- consider edges with sign change in r_{CR} at the vertices
- for each edge, determine the point (p_1, p_2) such that $r_{CR}(p_1, p_2) = 0$ by linear interpolation of vertex values
- approximate SB by joining the zeros



Stability charts: uniform square grid

- set a uniform square grid on the (p_1, p_2)-plane
- determine SB on each cell
- accuracy of SB depends on the grid size

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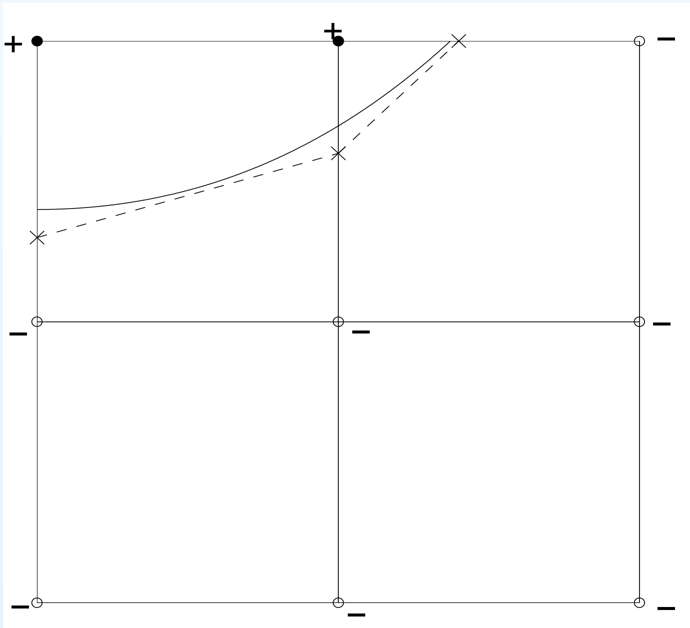
- set a uniform square grid on the (p_1, p_2)-plane
- determine SB on each cell
- accuracy of SB depends on the grid size
- this is behind *MATLAB's contour* for surface level-curves
- not efficient: each stability evaluation is expensive, hence uniform grid is not a good choice

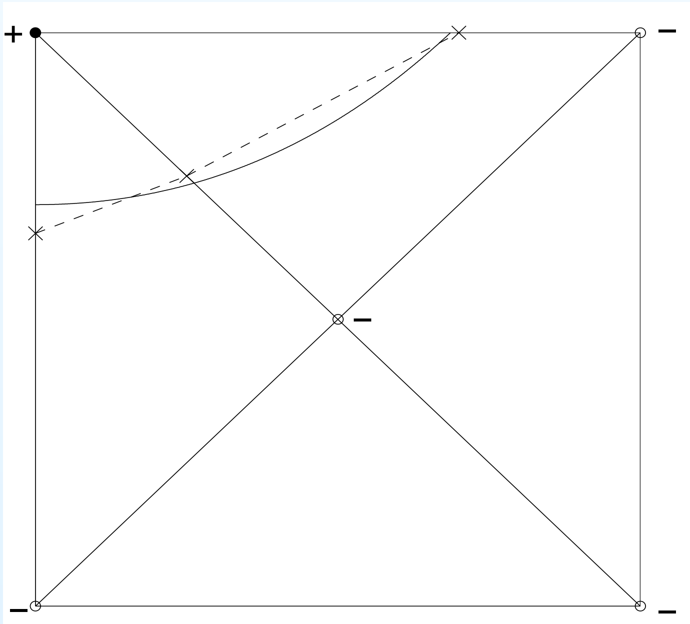
Stability charts: adaptive square refinement

- start from a coarse square grid
- refine only cells with sign change by dividing into four square cells by the center point
- five new stability evaluations required



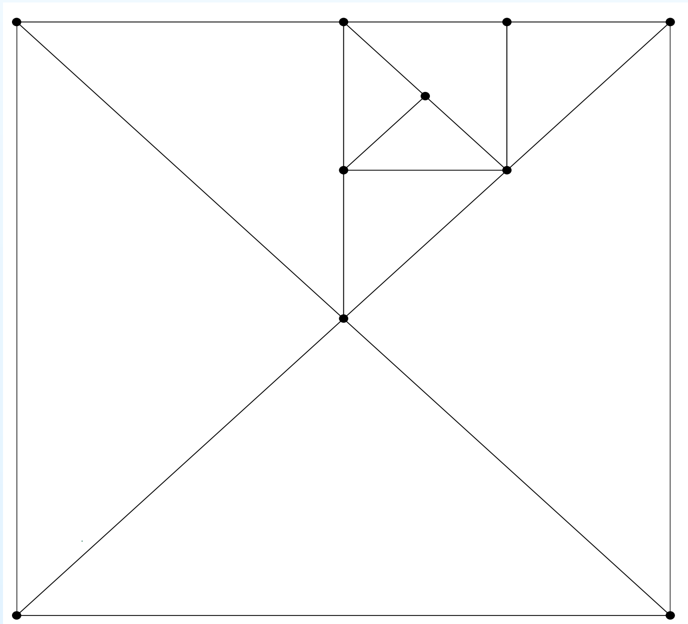
Stability charts: adaptive triangulation 1

- start from a coarse square grid
- refine square cells with sign change by dividing into four triangular cells by the center point
- only one new stability evaluation required



Stability charts: adaptive triangulation 2

- start from a triangular cell with sign change
- refine by dividing into two triangular cells by the mid point of the hypotenuse
- only one new stability evaluation required



Stability charts: squares vs triangles

- start from a square with area A
- consider the number of stability evaluations necessary to reduce the area to a

Stability charts: squares vs triangles

- start from a square with area A
- consider the number of stability evaluations necessary to reduce the area to a
- using square refinement

$$n = \frac{5}{\log 4} \log \frac{A}{a}$$

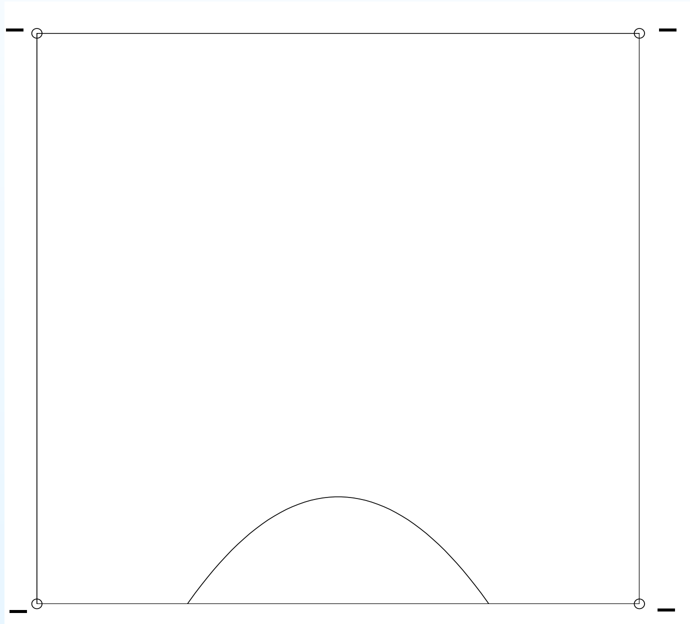
- using triangulation

$$n = \frac{2}{\log 4} \log \frac{A}{a} - \log 4$$

- less than half!

Stability charts: if no sign change?

- if all vertices of a cell have same sign there might be a SB crossing the cell (at only one edge)



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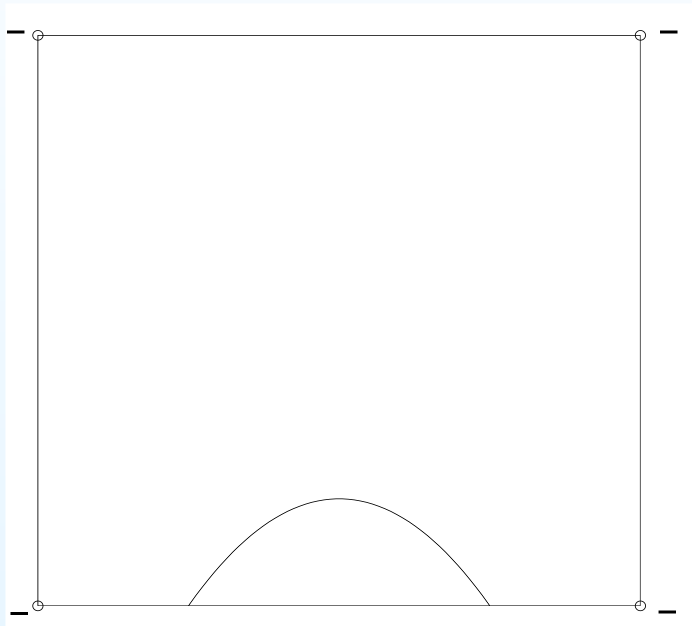
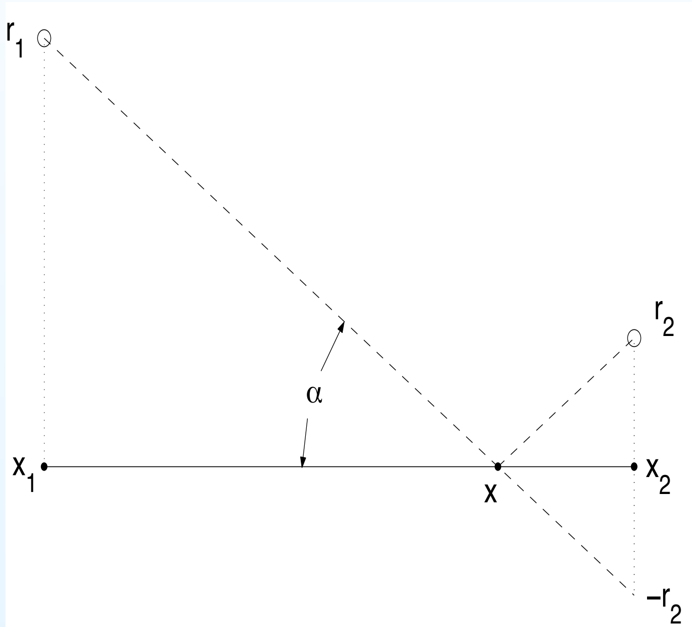


- how to recognize this possibility?

Stability charts: slope test

- measure the minimum slope $s = \tan\alpha = \frac{|r_1+r_2|}{|x_2-x_1|}$ at which $r_{CR} = 0$ is reached from both edge vertices



- if s is too large exclude refinement, else refine
- not a sufficient condition: heuristic test

Stability charts: multiple evaluations

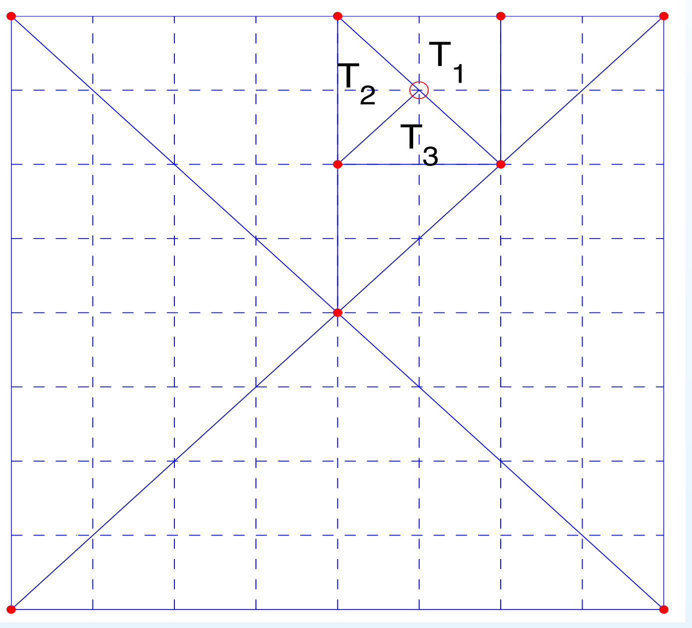
- possible multiple evaluations for neighboring cells
- considerable increase of computational time: not to underestimate

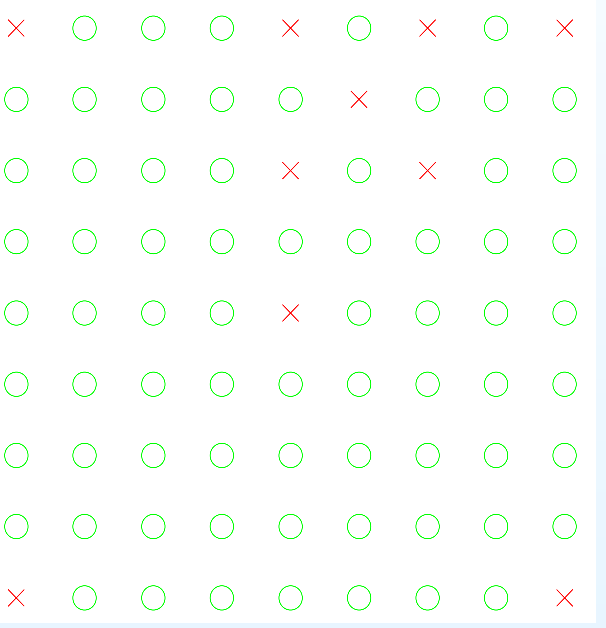
Stability charts: multiple evaluations

- possible multiple evaluations for neighboring cells
- considerable increase of computational time: not to underestimate
- “easy” to avoid for square grid by storing stability information in a rectangular matrix with entries corresponding to grid points
- before evaluating a grid point check the matrix if it already exists

Stability charts: information storage

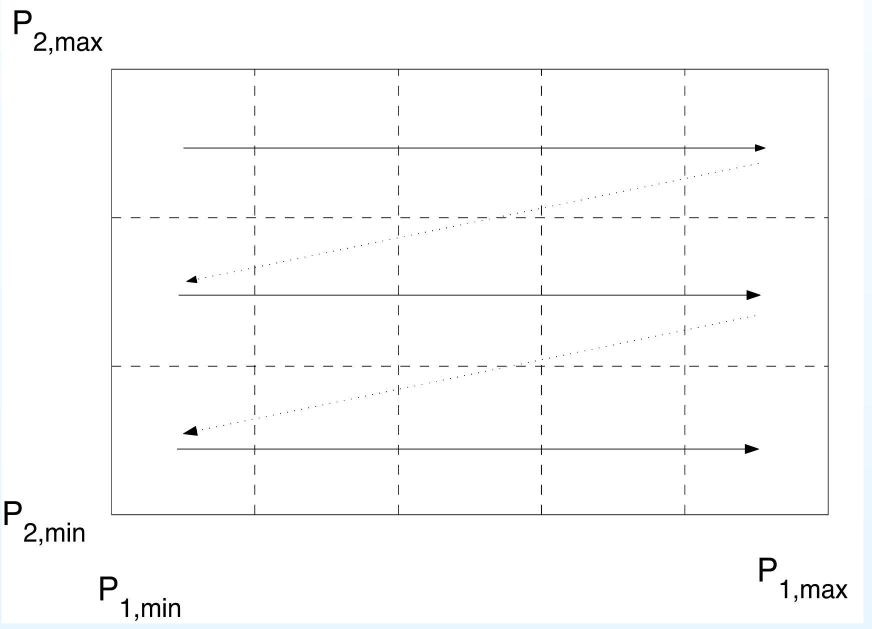
- more difficult with triangulation
- use a square matrix for each square cell





Stability charts: information passing

- update matrix from square to square scanning the whole grid in the usual reading/writing sense



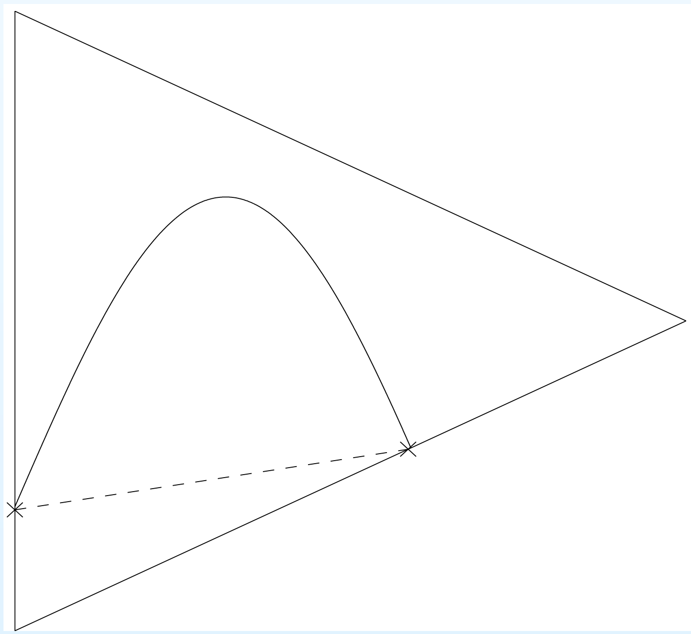
- avoid multiple evaluations in each step!

Stability charts: location of SB

- improve using secant method on each edge
- better control of accuracy along the edge

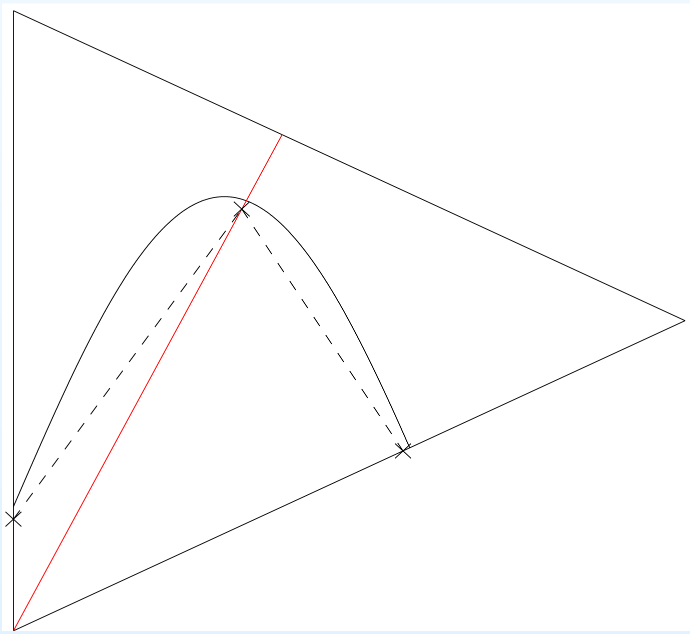
Stability charts: location of SB

- improve using secant method on each edge
- better control of accuracy along the edge
- but...no sense in find zeros with high accuracy and then join them with a line: curvature of SB is determining



Stability charts: adaptive curvature determination

- take an extra mid edge and find its zero
- measure the height of the triangle formed by the zeros of the three edges
- if this height is too large, refine by an extra mid edge and iterate

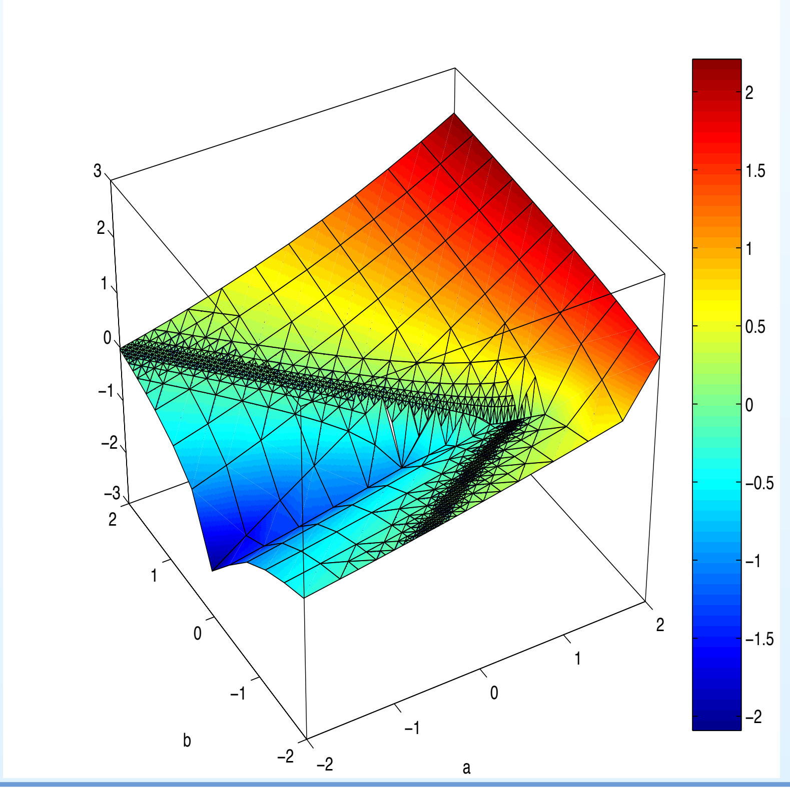


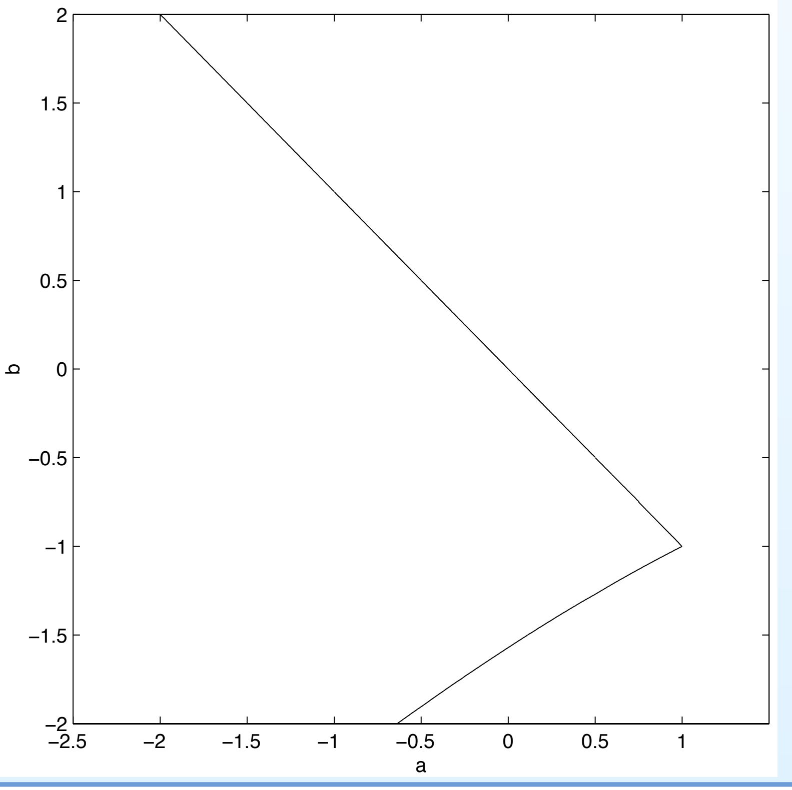
Examples: single delay

- consider the scalar single DDE

$$y'(t) = ay(t) + by(t-1)$$

- stability chart analytically well-known

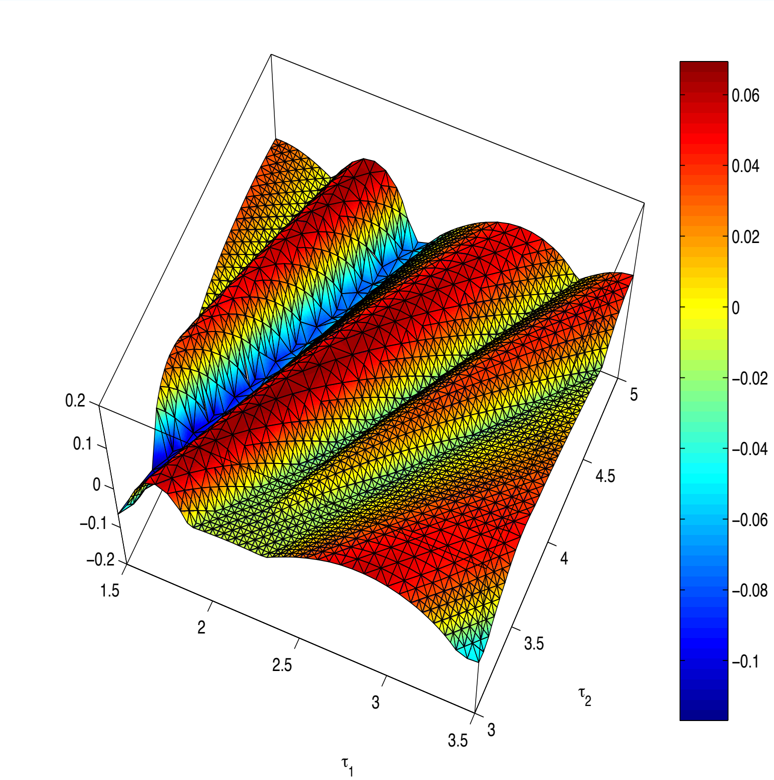


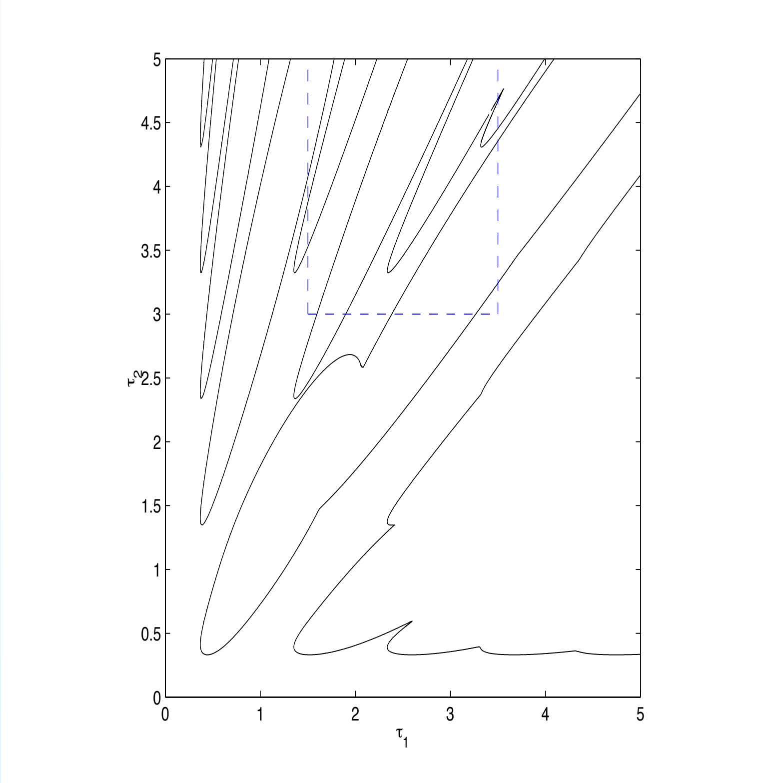


Examples: multiple delays

- consider the $2d$ DDE

$$y'(t) = \begin{pmatrix} -6.45 & -12.1 \\ 1.5 & -0.45 \end{pmatrix} y(t) + \begin{pmatrix} -6 & 0 \\ 1 & 0 \end{pmatrix} y(t-\tau_1) + \begin{pmatrix} 0 & 4 \\ 0 & -2 \end{pmatrix} y(t-\tau_2)$$

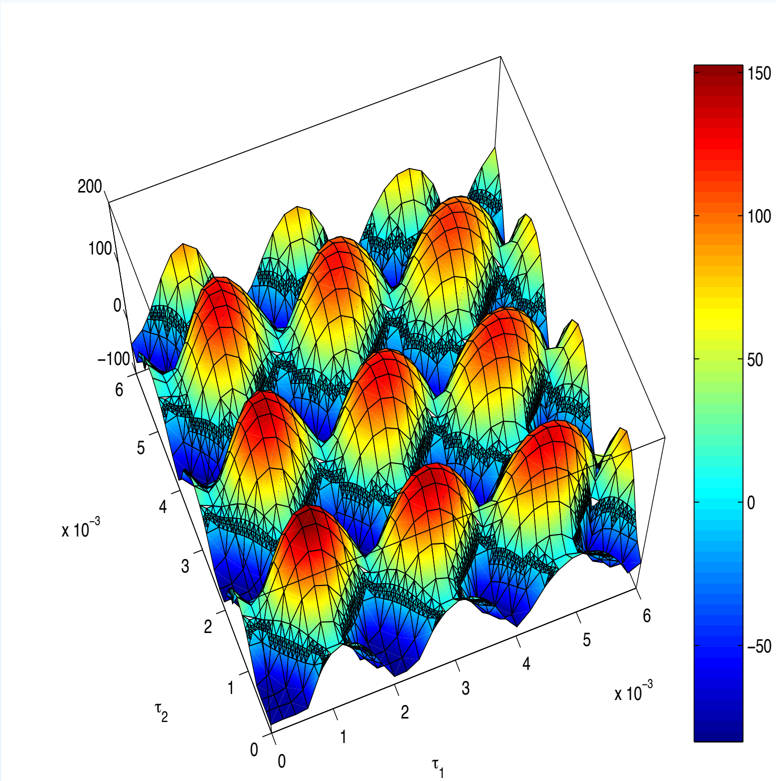


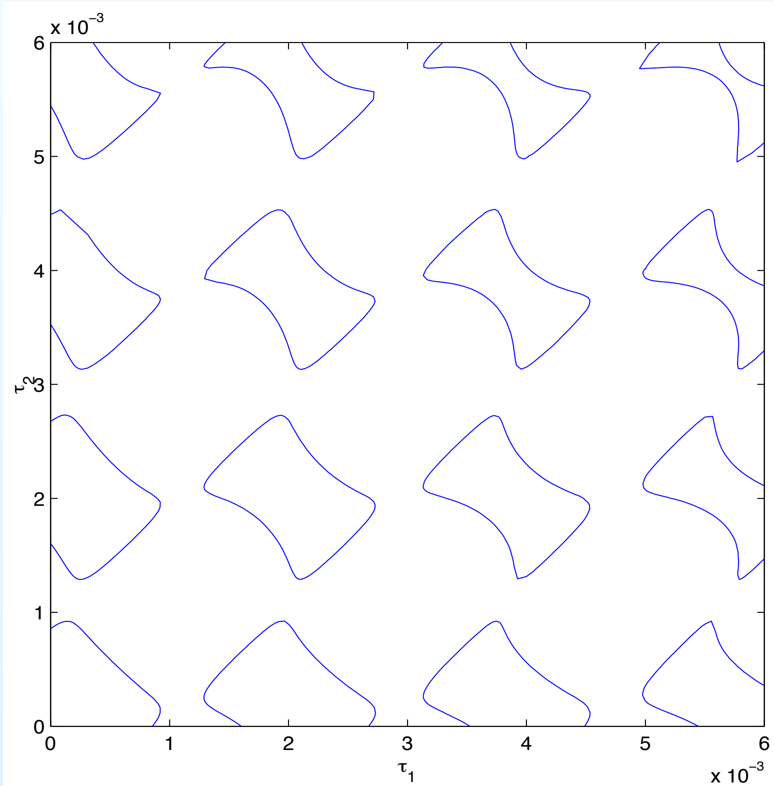


Examples: multiple delays

- consider the $8d$ DDE

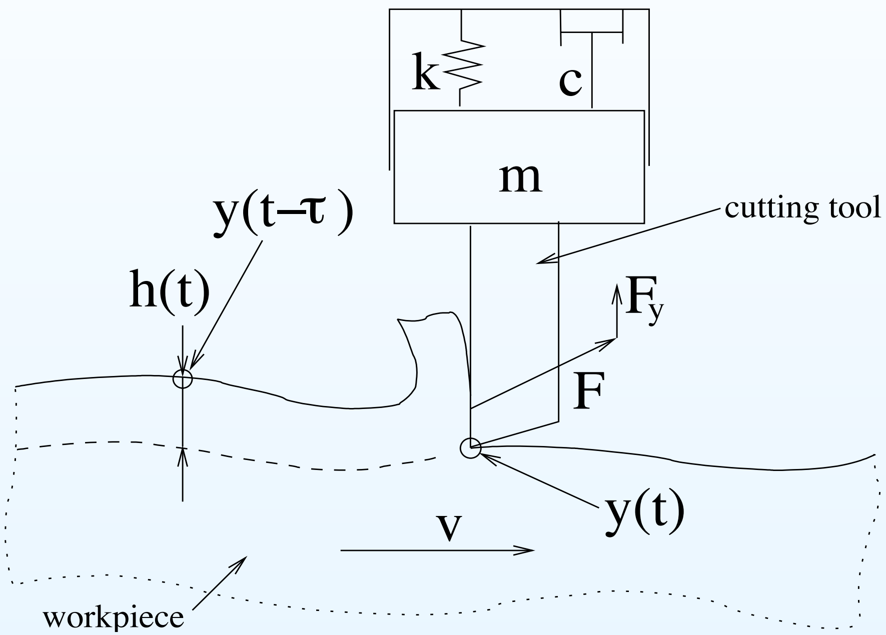
$$
\begin{aligned}
y'(t) &= L_0 y(t) + L_1 y(t-\tau_1) + L_1 y(t-\tau_2) + L_2 y(t-2\tau_1) + \\
&+ L_2 y(t-2\tau_2) + L_3 y(t-\tau_1-\tau_2), \quad L_i \in \mathbb{C}^{8\times 8}
\end{aligned}
$$





Applications: metal cutting

- consider 1dof model of orthogonal metal cutting



$$y''(t) + 2\zeta\omega_n y'(t) + \omega_n^2 y(t) = \frac{F_y}{m}$$

Applications: regenerative effect

- relative vibrations between tool and workpiece produces wavy surface

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- relative vibrations between tool and workpiece produces wavy surface
- after a round of the tool (or workpiece) chip thickness will vary
- cutting force depends on actual and delayed values of relative displacement between tool and workpiece
- this is called regenerative effect

Applications: delay model

- with regenerative effect the model becomes

$$y''(t) + 2\zeta\omega_n y'(t) + \omega_n^2 y(t) = -\frac{K(t)w}{m}(y(t) - y(t-\tau))$$

- $K(t)$ possibly time periodic (e.g. milling process)

Applications: delay model

- with regenerative effect the model becomes

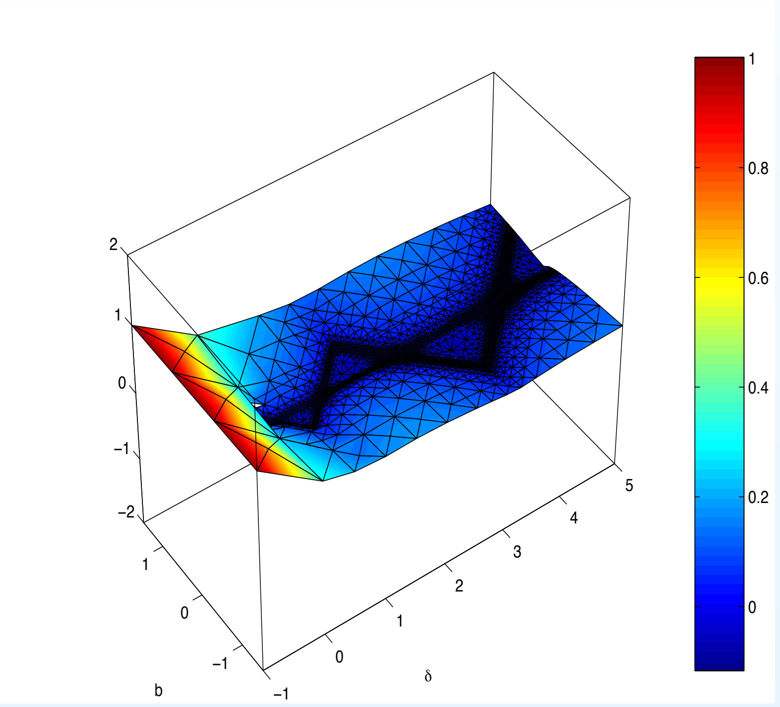
$$y''(t) + 2\zeta\omega_n y'(t) + \omega_n^2 y(t) = -\frac{K(t)w}{m}(y(t) - y(t-\tau))$$

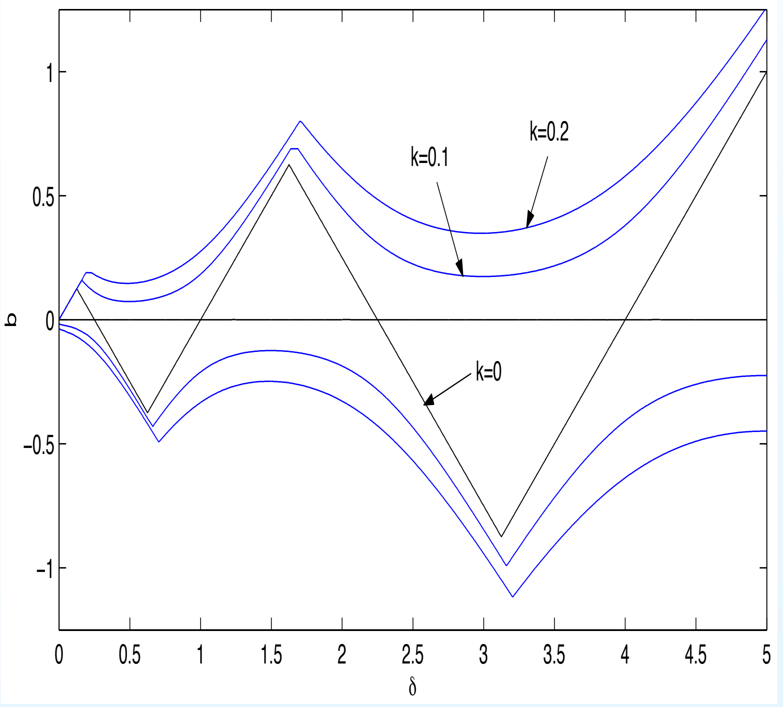
- $K(t)$ possibly time periodic (e.g. milling process)
- reduce to a model similar to the damped delayed Mathieu equation

$$y''(t) + ky'(t) + (\delta + \varepsilon\cos 2\pi t/T)y(t) = by(t - 2\pi)$$

Applications: stability chart

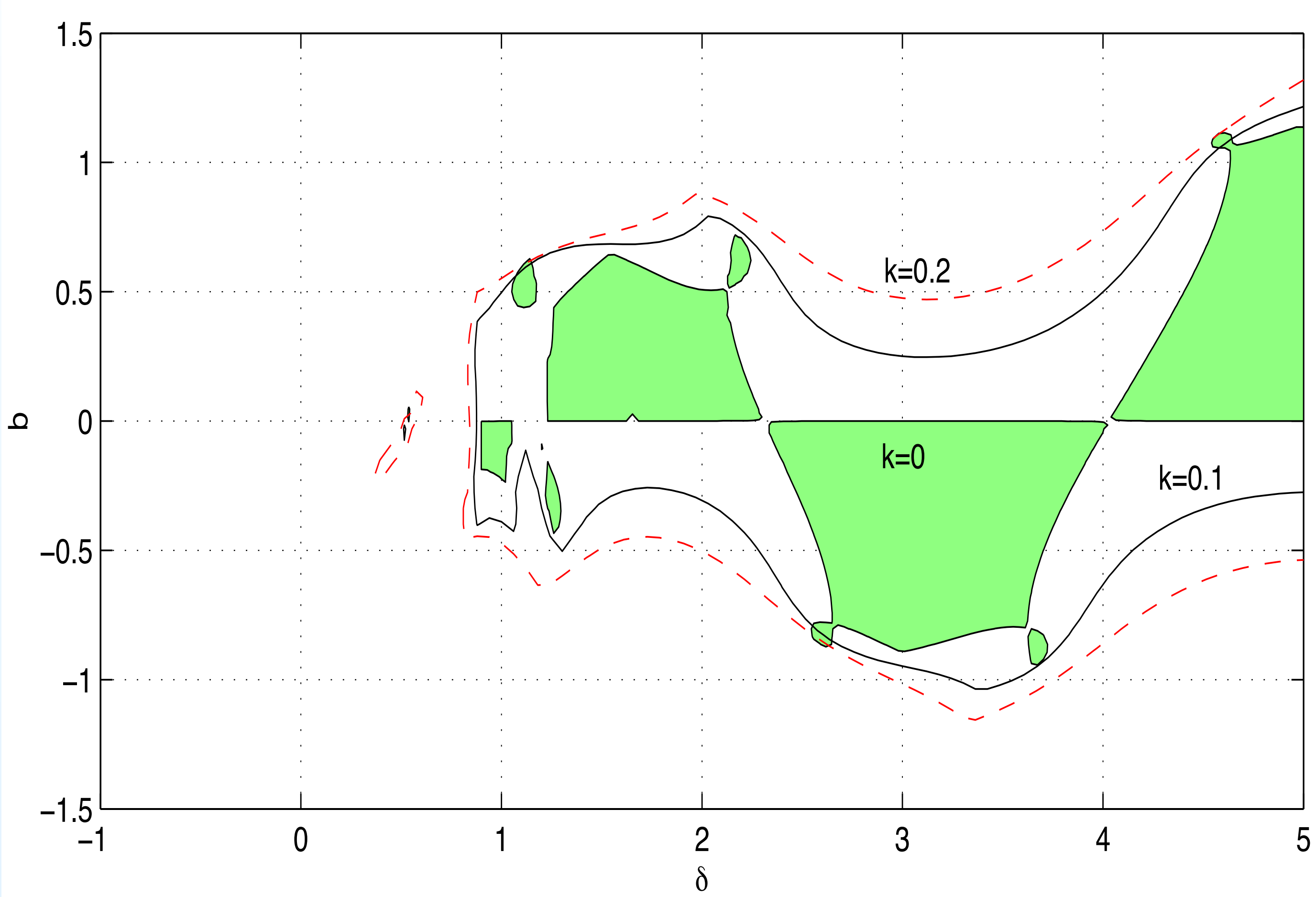
- consider $\varepsilon = 0$, δ and b as varying parameters
- Hsu-Bhatt-Vyshnegradskii stability chart





Applications: periodic case

- consider the periodic case $\varepsilon = 1$



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- increasing interest in time delay systems
- stability is an infinite dimensional problem

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- increasing interest in time delay systems
- stability is an infinite dimensional problem
- use numerical techniques to solve
- special attention to computational cost
- robust study of stability wrt varying parameters
- efficient computation of stability charts
- match best compromise among all tolerances

The end

...and thanks for your attention!