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Mosaic solutions and entropy for spatially discrete Cahn–Hilliard equations

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We consider arrays of scalar differential equations organized on a spatial lattice in a form analogous to the Cahn–Hilliard partial differential equation which on the integer lattice has couplings of nearest and next nearest neighbour type. The coupling strengths are not restricted in magnitude or in sign and need not be near a continuum limit. With the so-called double obstacle nonlinearity we prove existence and uniqueness results for the initial value problem and consider the existence and stability of a class of equilibrium solutions called mosaic solutions. These equilibrium solutions take only the values $+1$, -1 and 0 at each lattice point. Using the notion of a weakly forward invariant set we provide criteria for weak Lyapunov and weak asymptotic stability. Rigorous results are then obtained for the spatial entropy of these stable mosaic solutions and it is shown that the existence and stability results obtained on the integer lattice can be used to obtain similar results on an arbitrary lattice. Numerical results are presented that illustrate the importance of the analytical results.

Keywords: lattice differential equations; mosaic solutions; phase transitions; Cahn–Hilliard equation.

1. Introduction

When a high-temperature homogeneous mixture of two metals is quenched to a lower temperature, the mixture exhibits phase separation. Cahn & Hilliard (1958) proposed a fourth-order parabolic partial differential equation, which describes this process of phase separation and is given by

$$u_t = -\Delta(\varepsilon^2 \Delta u - f(u)) \quad \forall x \in \Omega, \tag{1.1}$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \forall x \in \partial\Omega,$$

where ν is the unit outward normal, ε is a small parameter, and f is the derivative of a double-well potential W, Δ denotes the standard Laplacian, $u_t = \partial u/\partial t$, and $\Omega \subset \mathbb{R}^d$ is a bounded domain with sufficiently smooth boundary given by $\partial\Omega$, where $d = 1, 2$ or 3.

Cahn & Hilliard (1958) first derived this equation through Fick's law of diffusion using the van der Waals free energy functional

$$E[u] := \int_{\Omega} \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) \mathrm{d}x.$$

In the Cahn–Hilliard equation (1.1), the variable u represents the concentration of one of the two metallic components, so that $\int_{\Omega} u \, \mathrm{d}x$ represents the total mass of that component. Mass is conserved in (1.1) and the concentration of the other component v can be determined by the equation $v = (1 - u)/2$. This means that a concentration of 0 or 1 in one of the components corresponds to u being -1 or 1, depending on which component is being considered. A derivation of this equation can also be found in Elliott (1989). It has been the focus of much mathematical study (cf. Novick & Segal, 1984; Elliott & French, 1987; Elliott, 1989; Blowey & Elliott, 1991).

Cahn & Hilliard (1958) suggested the following form for the free energy:

$$W(u) = \tfrac{1}{2} k T_c (1 - u^2) + \tfrac{1}{2} k T [(1 - u) \ln(1 - u) + (1 + u) \ln(1 + u) - \ln(2)], \tag{1.2}$$

where u varies between ± 1, k is Boltzmann's constant, T is temperature, and T_c is a critical temperature whose value depends on the particular materials. In the so-called deep quench limit when $T/T_c \to 0$, W'' tends to a constant and we consider the following potential function:

$$W(u) = \begin{cases} \frac{1}{2}(1 + \gamma u^2) & \text{if } |u| < 1, \\ +\infty & \text{if } |u| > 1, \end{cases} \tag{1.3}$$

where $\gamma \in \mathbb{R}$ is a parameter (for the potential to be double-welled $\gamma < 0$ is required, and $\gamma = -1$ is traditionally considered). This corresponds to $f = W'$ in (1.5) being the set-valued function known as the 'double obstacle nonlinearity' (see Oono & Puri, 1988; Blowey & Elliott, 1991; Elliott *et al.*, 1994; Chow *et al.*, 1996)

$$f(u) = \begin{cases} (-\infty, -\gamma] & \text{if } u = -1, \\ \gamma u & \text{if } |u| < 1, \\ [\gamma, \infty) & \text{if } u = 1, \\ \phi & \text{if } |u| > 1. \end{cases} \tag{1.4}$$

The argument of f is restricted to the interval $[-1, 1]$ with the values ± 1 acting as barriers.

In this paper we consider a one-dimensional spatially discrete version of the Cahn–Hilliard equation which has the form

$$\dot{u}_i = -\alpha \Delta [-\beta \Delta u_i - f(u_i)] \quad \forall i \in \mathbb{Z}, \tag{1.5}$$

where Δ denotes the discrete Laplacian operator defined by

$$\Delta u_i = u_{i+1} - 2u_i + u_{i-1}, \tag{1.6}$$

and f is given by (1.4). Note that we do not restrict attention to the double-well case, and so here γ can be positive, negative or zero.

It is assumed throughout that α is non-zero. Taking $\alpha > 0$ and $\beta < 0$ corresponds to a finite difference spatial discretization of the partial differential equation

$$u_t = -(\varepsilon u_{xx} - f(u))_{xx} \quad \forall x \in \mathbb{R}. \tag{1.7}$$

However, in what follows no restriction is placed on the sign of α and β and the resulting system need not necessarily be near a partial differential equation (PDE) continuum limit. Following the example in (Chow *et al.*, 1996), the system is written with negative coupling coefficients since the cases with $\alpha < 0$ and $\beta > 0$, which have no spatially continuous counterparts, are particularly interesting.

Our approach in this paper is to consider a spatially discrete model of Cahn–Hilliard type similar to that considered in (Cahn *et al.*, 1995) using techniques in the spirit of (Chow *et al.*, 1996) used there for spatially discrete Allen–Cahn type equations with nearest and next nearest neighbour interactions.

Our motivation for studying (1.5) is threefold.

(i) In many cases spatially discrete models allow for microscopic effects that cannot easily be modelled with continuum models. For example, anisotropy arises naturally in discrete models (Cahn *et al.*, 1999), also phenomena with fixed interaction length are easily modelled. (Other spatially discrete models for phase separation of binary solutions include that of Hillert (1961) and that of Cook *et al.* (1969).)

(ii) Often there is interesting dynamical behaviour in spatially discrete models that is not present in the analogous continuum models. For example propagation failure of travelling waves arises and can be studied in discrete models (Laplante & Erneux, 1992; Cahn *et al.*, 1999) (spatially continuous models fail to represent this phenomenon). Also discrete models are often applicable in parameter regions which are physically reasonable, but for which the PDE arising from the corresponding spatially continuous model is ill-posed.

(iii) In cases in which the spatially discrete model corresponds to the spatial discretization of a PDE model, careful study may lead to a better understanding of the effects of discretization. Note that (1.6) could be viewed as a finite difference discretization of the continuous Laplacian (where the mesh size h is factored into the parameter β).

With f given by (1.4) the differential equation is interpreted as a differential inclusion and the values of the variables are restricted to the range $|u_i| \leqslant 1$, so the phase space of the system is

$$[-1, 1]^{\mathbb{Z}} = \{u : \mathbb{Z} \to \mathbb{R}^{\mathbb{Z}} | u_i \in [-1, 1]\ \forall i \in \mathbb{Z}\}. \tag{1.8}$$

Using the notation

$$x = A + B$$

for addition of sets $A, B \subseteq \mathbb{R}$ to mean

$$x = a + b, \text{ where } a \in A, b \in B,$$

we write equation (1.5) as

$$\dot{u}_i = -\alpha[-\beta(u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}) - f(u_{i+1}) + 2f(u_i) - f(u_{i-1})] \tag{1.9}$$

in both the cases where f is uniquely valued so that (1.9) has a conventional meaning as a differential equation and in the case where f is set-valued so that (1.9) must be interpreted as a differential inclusion with the addition of sets on the right hand-side of (1.9) taking the meaning given above.

The time evolution of our system is described by an infinite system of ordinary differential equations that we call a *lattice differential equation* (see Chow *et al.*, 1996). With $i \in \mathbb{Z}$ denoting the space variable, the state of our dynamical system is an infinite vector $\{u_i\}_{i\in\mathbb{Z}}$. We are interested in bounded solutions and take $u \in l^\infty(\mathbb{Z})$, where $l^\infty(\mathbb{Z})$ is the Banach space with norm $\|\cdot\|_{l^\infty(\mathbb{Z})}$ given by

$$l^\infty(\mathbb{Z}) = \{u : \mathbb{Z} \to \mathbb{R} \mid \|u\|_{l^\infty(\mathbb{Z})} < \infty\}, \qquad \|u\|_{l^\infty(\mathbb{Z})} = \sup_{i\in\mathbb{Z}} |u_i|.$$

We write the general autonomous system $\dot{u} = g(u)$, where $g : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$, in coordinate form as

$$\dot{u}_i = g_i(\{u_j\}_{j\in\mathbb{Z}}) \text{ for } i \in \mathbb{Z}.$$

We can solve the initial value problem with $u(0) = u^0$ for any given $u^0 \in l^\infty(\mathbb{Z})$, if g is a locally Lipschitz function on $l^\infty(\mathbb{Z})$. The familiar existence and uniqueness proof for finite-dimensional ordinary differential equations carries over to this infinite-dimensional setting (see, for example Pazy, 1983). Our contributions in this paper include providing conditions for existence and uniqueness of solutions of (1.5) and (1.4). For $\alpha > 0$ these conditions are independent of γ and require $\beta > 0$, but for $\alpha < 0$ are restricted to a range of the parameters β and γ. In addition to existence and uniqueness of time dependent solutions, we study the existence and stability of a class of equilibrium solutions of (1.5) and (1.4) that we call mosaic solutions, following (Chow *et al.*, 1996). Mosaic solutions are equilibrium solutions that are restricted to take only the values $+1$, -1 and 0. We obtain explicit criteria for their existence and stability. We generalize the notion of weakly forward invariant to obtain criteria for weakly Lyapunov stable and weakly asymptotically stable mosaic solutions. We then extend these results to include a broader class of higher space dimension spatially discrete Cahn–Hilliard equations defined on arbitrary lattices. Using transition matrices we extend the analysis in Chow *et al.* (1996) for spatially discrete Allen–Cahn equations with nonlinearity (1.4) to determine the spatial entropy over large ranges of parameter values for (1.5). This involves transitions between four-tuples (which overlap to form five-tuples) as opposed to two-tuples for the Allen–Cahn equations. Finally, numerical simulations are presented in which the solution to the initial value problem is approximated. These numerical results confirm the importance of the analytical results by showing how the asymptotic state of the system depends upon the given parameter values.

We note here that one of the important differences between Allen–Cahn type equations and Cahn–Hilliard equations is that typically (that is, with the appropriate boundary conditions on a finite domain) the Cahn–Hilliard equations conserve mass. Our stability results do not assume that the perturbed problem has the same mass as the mosaic type equilibrium solution being perturbed. Thus our stability results can be viewed as sufficient conditions for stability of Cahn–Hilliard type systems. On the other hand, due to our use of the double obstacle nonlinearity, we do not expect conservation of mass for solutions that have values of $u_i = \pm 1$ due to the set-valued nature of (1.4) at ± 1.

2. Existence and uniqueness of solutions

We will establish existence and uniqueness theorems for the initial value problem (1.5), (1.4). Because f defined by (1.4) is set-valued, the theorems that we present are non-standard. In addition, our results concern only forward time, as existence and uniqueness do not hold in general for such systems in backward time. Our treatment will closely follow that in (Chow *et al.*, 1996) where similar theorems were established for a spatially discrete Allen–Cahn equation.

Before proceeding, we must first define what we mean by a solution of such a system.

DEFINITION 2.1 By a *solution* of (1.5), (1.4), we mean a continuous function

$$u : I \to [-1, 1]^{\mathbb{Z}} \subseteq l^{\infty}(\mathbb{Z})$$

on some interval I, such that the coordinate function $u_i(t)$ is absolutely continuous in I for each $i \in \mathbb{Z}$, and such that the inclusion

$$\dot{u}_i(t) \in -\alpha \Delta[-\beta \Delta u_i(t) - f(u_i(t))] \quad \forall i \in \mathbb{Z} \tag{2.10}$$

holds for almost every $t \in I$.

Note that this is a very weak concept of solution. Other authors (see, for example Blowey & Elliott, 1991) prefer to define

$$\left.\begin{array}{rcl} \dot{u}_i & = & -\alpha \Delta w_i, \\ w_i & \in & -\beta \Delta u_i - f(u_i). \end{array}\right\} \tag{2.11}$$

Note that any solution in the sense of (2.11) is a solution in the sense of (2.10) but that the converse is not true. Under suitable conditions conservation of mass and uniqueness of solutions can be established for (2.11). However for (2.10) the direct action of the discrete Laplacian operator on the set-valued f can destroy the property of mass conservation and also lead to non-uniqueness of solutions.

We prefer (2.10) to (2.11), however, because (2.10) is amenable to a local component by component analysis. With the stronger definition of solution (2.11) global assumptions must be made to ensure existence and uniqueness, and the analysis of the stability of solutions then becomes a global analysis.

We have the following existence result in forward time. Note that this result is very different in spirit from that of Blowey & Elliott (1991). In Theorem 2.2 we prove existence of a solution u with $\|u(t)\|_{l_\infty} \leqslant 1$ on a restricted parameter range. In (Blowey & Elliott, 1991) existence of bounded solutions is proved for an unrestricted parameter range, but solutions are not shown to lie in the interval $[-1, 1]$.

THEOREM 2.2 Consider (1.5), (1.4) with $u^0 \in [-1, 1]^{\mathbb{Z}}$ given. If
(i) $\alpha > 0$ and $\beta \geqslant 0$, or
(ii) $\alpha < 0$ and $\gamma \leqslant 4\beta \leqslant 0$,
then there exists a solution $u : [0, \infty) \to [-1, 1]^{\mathbb{Z}}$ in forward time to the initial value problem $u(0) = u^0$.

Proof. We will prove the result by constructing solutions to a series of approximating problems, to which a standard existence result applies, and then taking limits of the approximating solutions, after having obtained the appropriate *a priori* estimates.

The approximating problem is given by replacing the set-valued nonlinearity (1.4) with

$$f^\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}(u+1) - \gamma & \text{if } u \leqslant -1, \\ \gamma u & \text{if } |u| \leqslant 1, \\ \dfrac{1}{\varepsilon}(u-1) + \gamma & \text{if } u \geqslant 1, \end{cases} \tag{2.12}$$

which for any $\varepsilon \neq 0$ is a globally Lipschitz function $f^\varepsilon : \mathbb{R} \to \mathbb{R}$. With $\varepsilon > 0$ fixed, consider (1.5) with f^ε replacing f. We shall restrict ε to be sufficiently small, specifically

$$|\gamma|\varepsilon \leqslant 1 \quad \text{and} \quad 4\varepsilon|\beta| \leqslant 1. \tag{2.13}$$

We may write this system, with the nonlinearity f^ε, abstractly as an ordinary differential equation $\dot{u}^\varepsilon = F^\varepsilon(u^\varepsilon)$ in the Banach space $l^\infty(\mathbb{Z})$, where $F^\varepsilon : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ is a globally Lipschitz function. As noted in Section 1, in a standard fashion we obtain a unique solution $u^\varepsilon : \mathbb{R} \to l^\infty(\mathbb{Z})$ to the initial value problem $u^\varepsilon(0) = u^0$ (where we always take u^0 as in the statement of the theorem, with $\|u^0\|_{l^\infty(\mathbb{Z})} \leqslant 1$). Observe that this solution satisfies

$$\dot{u}_i^\varepsilon = -\alpha\Delta[-\beta\Delta u_i^\varepsilon - \gamma u_i^\varepsilon - h_i^\varepsilon(t)] \quad \forall i \in \mathbb{Z}, \tag{2.14}$$

where the continuous functions $h_i^\varepsilon : \mathbb{R} \to \mathbb{R}$ are given by $h_i^\varepsilon(t) = f^\varepsilon(u_i^\varepsilon(t)) - \gamma u_i^\varepsilon(t)$, and satisfy

$$h_i^\varepsilon(t) \begin{cases} \leqslant 0 & \text{if } u_i^\varepsilon(t) \leqslant -1, \\ = 0 & \text{if } |u_i^\varepsilon(t)| \leqslant 1, \\ \geqslant 0 & \text{if } u_i^\varepsilon(t) \geqslant 1; \end{cases} \tag{2.15}$$

here we have used the first bound of (2.13) on ε.

We now establish the uniform bounds

$$|u_i^\varepsilon(t)| \leqslant 1 + K_1\varepsilon \quad \text{and} \quad |h_i^\varepsilon(t)| \leqslant K_2 \quad \forall i \in \mathbb{Z}, \quad \forall t \geqslant 0, \tag{2.16}$$

with constants K_1 and K_2 independent of ε, as well as of i and t. (The constants K_1 and K_2 generally depend on β and γ, however.)

First observe that the bound on $h_i^\varepsilon(t)$ follows from the bound on $u_i^\varepsilon(t)$. Indeed, if $1 \leqslant u_i^\varepsilon(t) \leqslant 1 + K_1\varepsilon$, then from (2.14) and the formula (2.12) for f^ε, we have from the definition of $h_i^\varepsilon(t)$ that

$$|h_i^\varepsilon(t)| = \left|\left(\frac{1}{\varepsilon} - \gamma\right)(u_i^\varepsilon(t) - 1)\right| \leqslant K_1|1 - \gamma\varepsilon| \leqslant 2K_1,$$

by (2.13). The same bound holds for $-1 - K_1\varepsilon \leqslant u_i^\varepsilon(t) \leqslant -1$, and of course $h_i^\varepsilon(t) = 0$ if $|u_i^\varepsilon(t)| < 1$. Thus the bound (2.16) on $h_i^\varepsilon(t)$ holds with $K_2 = 2K_1$.

To bound $u_i^\varepsilon(t)$, we show that the ball $\{u \in l^\infty(\mathbb{Z}) : \|u\|_{l^\infty(\mathbb{Z})} \leqslant 1 + K_1\varepsilon\}$ is positively invariant. To do this it is sufficient to prove that

$$\dot{u}_i^\varepsilon(t) \leqslant 0 \text{ whenever } u_i^\varepsilon(t) = 1 + K_1\varepsilon \text{ and } \|u^\varepsilon(t)\|_{l^\infty(\mathbb{Z})} = 1 + K_1\varepsilon, \tag{2.17}$$

along with the corresponding inequality $\dot{u}_i^\varepsilon(t) \geqslant 0$ whenever $u_i^\varepsilon(t) = -1 - K_1\varepsilon$ and $\|u^\varepsilon(t)\|_{l^\infty(\mathbb{Z})} = 1 + K_1\varepsilon$.

Let

$$g^\varepsilon(u) = f^\varepsilon(u) - 4\beta u. \tag{2.18}$$

Then from (1.5) and (2.12) we have

$$\dot{u}_i^\varepsilon = \alpha[\beta(6u_i^\varepsilon + u_{i+2}^\varepsilon + u_{i-2}^\varepsilon) + g^\varepsilon(u_{i+1}^\varepsilon) - 2f^\varepsilon(u_i^\varepsilon) + g^\varepsilon(u_{i-1}^\varepsilon)]. \tag{2.19}$$

The following inequalities will be useful. If $\gamma \geqslant 4\beta$ or $4\beta \geqslant \gamma$ and

$$K_1 \geqslant \frac{2(4\beta - \gamma)}{1 - 4\varepsilon\beta}, \tag{2.20}$$

then if $\|u^\varepsilon(t)\|_{l^\infty(\mathbb{Z})} \leqslant 1 + K_1\varepsilon$ it follows that

$$g^\varepsilon(u_j) \leqslant \gamma - 4\beta + K_1(1 - 4\varepsilon\beta), \tag{2.21}$$

with the right hand-side non-negative (using (2.13)). It also follows from (2.12), (2.18) that if $4\beta \geqslant \gamma$ and

$$K_1 \leqslant \frac{2(4\beta - \gamma)}{1 - 4\varepsilon\beta}, \tag{2.22}$$

then if $\|u^\varepsilon(t)\|_{l^\infty(\mathbb{Z})} \leqslant 1 + K_1\varepsilon$

$$\gamma - 4\beta \leqslant g^\varepsilon(u_j) \leqslant 4\beta - \gamma\cdot \tag{2.23}$$

We use (2.18) to prove (2.17) in the two different cases.

(i) $\alpha > 0$ *and* $\beta \geqslant 0$
Let $K_1 > 0$ satisfy (2.20). Thus, if $u_i^\varepsilon = 1 + K_1\varepsilon$ and $\alpha > 0$, then it follows from (2.19) that

$$\begin{aligned}\dot{u}_i^\varepsilon &\leqslant 2\alpha[3\beta(1 + K_1\varepsilon) + |\beta|(1 + K_1\varepsilon) + \gamma - 4\beta + K_1(1 - 4\varepsilon\beta) - (K_1 + \gamma)]\\ &= 2\alpha[(|\beta| - \beta)(1 + K_1\varepsilon)]\\ &= 0,\end{aligned}$$

provided $\beta \geqslant 0$. This establishes (2.17).

(ii) $\alpha < 0$ *and* $\gamma \leqslant 4\beta \leqslant 0$
Let $K_1 = 0$. Thus (2.22) is satisfied and hence (2.23) holds. Thus, if $u_i^\varepsilon = 1 + K_1\varepsilon$ and $\alpha < 0$, then it follows from (2.19) that

$$\begin{aligned}\dot{u}_i^\varepsilon &\leqslant 2\alpha[3\beta(1 + K_1\varepsilon) - |\beta|(1 + K_1\varepsilon) + \gamma - 4\beta - (K_1 + \gamma)]\\ &= 2\alpha[-(\beta + |\beta|)]\\ &\leqslant 0,\end{aligned}$$

provided $\beta \leqslant 0$. This establishes (2.17).

The proof of the corresponding inequality at $-1 - K_1\varepsilon$ is similar in both cases.

Thus the ball $\{u \in l^\infty(\mathbb{Z}) : \|u\|_{l^\infty(\mathbb{Z})} \leqslant 1 + K_1\varepsilon\}$ is positively invariant, which establishes the remaining bound on $|u_i^\varepsilon(t)|$ in (2.16).

From (2.16), we obtain from the differential equation (2.14) the additional bound on the derivative of $u_i^\varepsilon(t)$ that $|\dot{u}_i^\varepsilon(t)| \leqslant K_3$, for some K_3 valid for all i, non-negative t, and $\varepsilon \leqslant 1$. Upon taking a sequence $\varepsilon_n \to 0$, and possibly passing to a subsequence, we have with a standard application of Ascoli's theorem, the limits

$$u_i^{\varepsilon_n}(t) \to u_i(t) \text{ uniformly for } 0 \leqslant t \leqslant T,$$

and also

$$h_i^{\varepsilon_n}(t) \to h_i(t) \text{ weak* in } L^\infty(0, T),$$

for each $i \in \mathbb{Z}$ and $T > 0$. The limiting functions $u_i(t)$ are absolutely continuous, enjoy the bound $|u_i(t)| \leqslant 1$ for all $t \geqslant 0$, and satisfy the initial condition $u(0) = u^0$. They also satisfy the equation

$$\dot{u}_i = -\alpha\Delta[-\beta\Delta u_i - \gamma u_i - h_i(t)] \quad \forall i \in \mathbb{Z}, \tag{2.24}$$

for almost every $t \geqslant 0$, as one sees by integrating the equation (2.14) from 0 to any $t > 0$, and taking the limit $\varepsilon_n \to 0$. Finally, one sees from (2.15) that the functions $h_i(t)$ satisfy

$$h_i(t) \begin{cases} \leqslant 0 & \text{if } u_i(t) = -1, \\ = 0 & \text{if } |u_i(t)| < 1, \\ \geqslant 0 & \text{if } u_i(t) = 1, \end{cases} \tag{2.25}$$

for almost every $t \geqslant 0$. With this, it is now clear that $u : [0, \infty) \to [-1, 1]^{\mathbb{Z}}$ is a solution to (1.5), (1.4) with the initial condition $u(0) = u^0$, as desired. We note in particular, that the uniform bound $|\dot{u}_i(t)| \leqslant K_3$ ensures the continuity of $u(t)$ in t, as an element of $l^\infty(\mathbb{Z})$. □

We will not have backward existence of solutions in general. In both parts of the proof we show that the ball $\{u \in l^\infty(\mathbb{Z}) : \|u\|_{l^\infty(\mathbb{Z})} \leqslant 1\}$ is forward invariant, However this ball is not backward invariant and so solutions can escape from $[-1, 1]^{\mathbb{Z}}$ in backwards time.

We will consider existence and stability of mosaic solutions for all parameter values, not just those that satisfy Theorem 2.2. In the case of parameter values which do not satisfy Theorem 2.2 we may not have infinite time existence of solutions for arbitrary initial conditions in $[-1, 1]^{\mathbb{Z}}$. However, since a mosaic solution is an equilibrium solution, we will always have infinite time existence for these solutions. Our first step to establishing stability of mosaic solutions will be to show that a neighbourhood of the mosaic solution is forward invariant. In this case, existence of solutions for all initial conditions within the neighbourhood will follow using the techniques above, and indeed for any initial conditions within the basin of attraction of the forward invariant neighbourhood.

We will now prove that the solution constructed above is unique provided that $\|u(t)\|_{l^\infty(\mathbb{Z})} < 1$. The Laplacian operator acting on the set-valued function f precludes uniqueness, in general, when $|u_i(t)| = 1$ for some i.

THEOREM 2.3 Let $u^1, u^2 : [0, \infty) \to [-1, 1]^{\mathbb{Z}}$ be two solutions of (1.5), (1.4) with the same initial condition $u^1(0) = u^2(0) = u^0$, and with $u^1(t), u^2(t) \in (-1, 1)^{\mathbb{Z}}$ for all $t \in [0, \tau]$ for some $\tau > 0$. Then $u^1(t) = u^2(t)$ for all $t \in [0, \tau]$.

Proof. Consider (1.5) with f replaced by f^γ, defined by

$$f^\gamma(u) = \gamma u \quad \forall u \in \mathbb{R}.$$

Writing this system abstractly as an ordinary differential equation $\dot{u}^\gamma = F^\gamma(u^\gamma)$ in the Banach space $l^\infty(\mathbb{Z})$, where $F^\gamma : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ is a globally Lipschitz function, as noted in Section 1, in a standard fashion we obtain a unique solution $u^\varepsilon : \mathbb{R} \to l^\infty(\mathbb{Z})$ to the initial value problem $u^\varepsilon(0) = u^0$, and the result follows. □

It is straightforward to show that $|u_i| = 1$ can cause non-uniqueness in the solution for the neighbouring points on the lattice, as we now illustrate. Consider (1.5), (1.4) with $\alpha > 0$ and $(3\beta - \gamma) > 0$ and initial condition $u_0(0) = \widehat{u}$ and $u_j(0) = 0$ for all $j \neq 0$. Then if $\widehat{u} \in (1 - \delta, 1)$ we have $\dot{u}_0(0) = 2\alpha(3\beta - \gamma)\widehat{u} > 0$. But now since $\dot{u}_i$ satisfies (2.24) there exists some interval $[0, \varepsilon)$ such that $\dot{u}_0(t) > 0$ for all $t \in [0, \varepsilon)$ provided $\dot{u}_0(t) < 1$. It follows that if we set $\widehat{u} = 1$, then any solution of (1.5), (1.4) must satisfy $u_0(t) = 1$ for all $t \in [0, \varepsilon)$, and moreover Theorem 2.2 guarantees that at least one such solution exists. Now consider $u_1(t)$. We have $u_1(0) = 0$ and $\dot{u}_1(0) = \alpha(-4\beta + [\gamma, \infty))$, and hence is interval valued. Moreover since $u_0(t) = 1$ for all $t \in [0, \varepsilon)$ we have $\dot{u}_1(t)$ set-valued for all $t \in [0, \varepsilon)$ and non-uniqueness of the solution follows.

3. Equilibrium solutions

As in (Chow *et al.*, 1996), we use the following non-standard definition.

DEFINITION 3.1 A function $u \in [-1, 1]^{\mathbb{Z}}$ is said to be an *equilibrium solution* of (1.5), (1.4) if $0 \in -\alpha\Delta[-\beta\Delta u_i - f(u_i)]$ for all $i \in \mathbb{Z}$.

Note that due to the possible non-uniqueness of solutions, this is a generalization of the usual definition of an equilibrium solution. However, in Section 4 we will identify equilibrium solutions which are Lyapunov or asymptotically stable, and such solutions will be equilibrium solutions in the usual sense.

Before we turn our attention to mosaic solutions, we briefly consider equilibrium solutions which are constant in space. By (1.4) if $u_i \in (-1, 1)$, then $f(u_i) = \gamma u_i$. Hence, if $u_i \in (-1, 1)$ for all $i \in \mathbb{Z}$, then (1.5), (1.4) reduces to

$$\dot{u}_i = \alpha[\beta(u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}) + \gamma(u_{i+1} - 2u_i + u_{i-1})]. \tag{3.1}$$

Thus, if

$$u_i = \mu \quad \forall i \in \mathbb{Z}, \quad \mu \in (-1, 1), \tag{3.2}$$

then $\dot{u}_i = 0$ for all $i \in \mathbb{Z}$ and hence this defines an equilibrium solution.

DEFINITION 3.2 An equilibrium solution of (1.5), (1.4) is called a *mosaic solution* if $u_\alpha \in \{-1, 0, 1\}$ for all $\alpha \in \mathbb{Z}$. In general, any function $u : \mathbb{Z} \to \{-1, 0, 1\}^{\mathbb{Z}}$ is a *one-dimensional mosaic*, and the set of all such mosaics is denoted by $\mathcal{M}_1 = \{-1, 0, 1\}^{\mathbb{Z}}$.

For any $u \in \mathcal{M}_1$ we set

$$\sigma_i^j = u_{i+j} + u_{i-j} \quad \forall\, i \in \mathbb{Z},\, j \in \mathbb{N}. \tag{3.3}$$

Clearly σ_i^j is an integer such that $-2 \leqslant \sigma_i^j \leqslant 2$. With this notation we can classify the mosaic solutions.

THEOREM 3.3 A mosaic $u \in \mathcal{M}_1$ is an equilibrium solution of (1.5), (1.4), that is, u is a mosaic solution, if and only if

(i) $u_{i-1} = u_i = u_{i+1} = 0$ and $\beta\sigma_i^2 = 0$,
(ii) $u_i = 0$ and $u_{i-1}u_{i+1} = -1$,
(iii) $u_i u_{i\pm 1} = 1$, or
(iv) $2u_i \neq \sigma_i^1$ and $\beta\left(4 - \dfrac{2u_i - \sigma_i^2}{2u_i - \sigma_i^1}\right) \geqslant \gamma$,

for each $i \in \mathbb{Z}$.

Proof. Expanding (1.5), it is clear that u is a mosaic solution if and only if

$$0 \in -\alpha[-\beta(u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}) - f(u_{i+1}) + 2f(u_i) - f(u_{i-1})], \tag{3.4}$$

so that u is a mosaic solution of (1.5), (1.4) if and only if

$$-\beta(\sigma_i^2 - 4\sigma_i^1 + 6u_i) \in f(u_{i+1}) - 2f(u_i) + f(u_{i-1}). \tag{3.5}$$

(a) Assume first that $u_i = 0$, which implies that $f(u_i) = 0$. Thus for u to be a mosaic solution, the inclusion

$$-\beta(\sigma_i^2 - 4\sigma_i^1) \in f(u_{i+1}) + f(u_{i-1}) \tag{3.6}$$

must be satisfied. We deal with the cases $\sigma_i^1 = 0$ and $\sigma_i^1 \neq 0$ separately.

(i) For $\sigma_i^1 = 0$ there are two sub-cases.
(1) $u_{i+1} = u_{i-1} = 0$ implies $f(u_{i+1}) = f(u_{i-1}) = 0$. Hence, we have a mosaic solution from (3.6) if and only if $\beta\sigma_i^2 = 0$. This is case (i) in the statement of the theorem.
(2) $u_{i+1} = -u_{i-1} = \pm 1$ implies one of $f(u_{i+1})$, $f(u_{i-1})$ is equal to $(-\infty, -\gamma]$ whilst the other is equal to $[\gamma, \infty)$. Thus $f(u_{i+1}) + f(u_{i-1}) = (-\infty, \infty)$, and (3.6) is trivially satisfied. This is included in case (ii) in the statement of the theorem.

(ii) Now consider $\sigma_i^1 \neq 0$. If $\sigma_i^1 = -2$, then $u_{i+1} = u_{i-1} = -1$ and hence $f(u_{i+1}) + f(u_{i-1}) = (-\infty, -2\gamma]$. If $\sigma_i^1 = -1$, then one of u_{i+1}, u_{i-1} is equal to -1 and the other is equal to 0, thus $f(u_{i+1}) + f(u_{i-1}) = (-\infty, -\gamma]$. The cases of $\sigma_i^1 > 0$ are similar, and in all four cases we have

$$f(u_{i+1}) + f(u_{i-1}) = (\operatorname{sgn}(\sigma_i^1)\infty, \sigma_i^1\gamma].$$

Thus by (3.6) for a mosaic solution we require

$$-\beta(\sigma_i^2 - 4\sigma_i^1) \in (\operatorname{sgn}(\sigma_i^1)\infty, \sigma_i^1\gamma]$$

which is equivalent to

$$\frac{\beta(4\sigma_i^1 - \sigma_i^2)}{\sigma_i^1} \geqslant \gamma.$$

But for $u_i = 0$ (and recalling that $\sigma_i^1 \neq 0$) we have

$$\frac{\beta(4\sigma_i^1 - \sigma_i^2)}{\sigma_i^1} = \beta\left(4 - \frac{\sigma_i^2}{\sigma_i^1}\right) = \beta\left(4 - \frac{2u_i - \sigma_i^2}{2u_i - \sigma_i^1}\right),$$

and so this corresponds to case (iv) in the statement of the theorem.

(b) Now assume $u_i = 1$, which implies that $f(u_i) = [\gamma, \infty)$. Thus (3.5) becomes

$$-\beta(\sigma_i^2 - 4\sigma_i^1 + 6u_i) \in f(u_{i+1}) - 2[\gamma, \infty) + f(u_{i-1}). \tag{3.7}$$

There are two separate cases to consider.

(i) If $\sigma_i^1 < 0$ or $\sigma_i^1 = u_{i+1} = u_{i-1} = 0$, then in each of these cases $f(u_{i+1})$ and $f(u_{i-1})$ are either equal to 0 or $(-\infty, -\gamma]$ and

$$f(u_{i+1}) - 2[\gamma, \infty) + f(u_{i-1}) = (-\infty, (\sigma_i^1 - 2)\gamma].$$

Thus, by (3.7), for a mosaic solution we require

$$-\beta(\sigma_i^2 - 4\sigma_i^1 + 6u_i) \in (-\infty, (\sigma_i^1 - 2)\gamma]$$

or, equivalently,

$$-\beta(\sigma_i^2 - 4\sigma_i^1 + 6u_i) \leqslant (\sigma_i^1 - 2)\gamma,$$

and since we are considering $\sigma_i^1 \leqslant 0$, we have $(\sigma_i^1 - 2) < 0$ and hence

$$\beta\left(\frac{\sigma_i^2 - 4\sigma_i^1 + 6u_i}{2 - \sigma_i^1}\right) \geqslant \gamma.$$

But for $u_i = 1$ we have

$$\beta\left(\frac{\sigma_i^2 - 4\sigma_i^1 + 6u_i}{2 - \sigma_i^1}\right) = \beta\left(\frac{8u_i - 4\sigma_i^1 + \sigma_i^2 - 2u_i}{2u_i - \sigma_i^1}\right) = \beta\left(4 - \frac{2u_i - \sigma_i^2}{2u_i - \sigma_i^1}\right),$$

and so this corresponds to case (iv) in the statement of the theorem.

(ii) If $\sigma_i^1 > 0$ or $\sigma_i^1 = 0$ with $u_{i+1} = -u_{i-1} = \pm 1$, then at least one of u_{i+1}, u_{i-1} is equal to 1, and so at least one of $f(u_{i+1})$, $f(u_{i-1})$ is equal to $[\gamma, \infty)$. It then follows that $f(u_{i+1}) - 2[\gamma, \infty) + f(u_{i-1}) = (-\infty, \infty)$ and so (3.7) is trivially satisfied. This corresponds to case (iii) in the statement of the theorem.

(c) The case $u_i = -1$ is similar to the case $u_i = 1$. This completes the proof. □

Finally in this section we note that not all equilibrium solutions are mosaic solutions or constant in space. In particular, the following type of equilibrium solution, which we refer to as a semi-mosaic solution may arise. Suppose

$$u_{2i} = 1, \qquad u_{2i+1} = \mu \qquad \forall i \in \mathbb{Z}, \quad 0 < |\mu| < 1. \tag{3.8}$$

Then it follows from (1.5), (1.4) that

$$\dot{u}_{2i} = 2\alpha[4\beta + (\gamma - 4\beta)\mu - [\gamma, \infty)]$$

and

$$\dot{u}_{2i+1} = -2\alpha[4\beta + (\gamma - 4\beta)\mu - [\gamma, \infty)].$$

Now provided $4\beta \geqslant \gamma$ it follows that

$$0 \in 4\beta + (\gamma - 4\beta)\mu - [\gamma, \infty)$$

and hence $0 \in \dot{u}_{2i}$ and $0 \in \dot{u}_{2i+1}$ so that this defines an equilibrium solution.

4. Stability of mosaic solutions

To study the stability of mosaic equilibrium solutions, $u \in \mathcal{M}_1$, we will need to consider the behaviour of solutions $v \in [-1, 1]^{\mathbb{Z}}$ which are perturbations of mosaic equilibrium solutions.

Analogously to (3.3), for any $v \in [-1, 1]^{\mathbb{Z}}$ we define

$$\tau_i^j = v_{i+j} + v_{i-j} \quad \forall\, i \in \mathbb{Z}, j \in \mathbb{N}. \tag{4.1}$$

Note that, as for σ_i^j, we have $-2 \leqslant \tau_i^j \leqslant 2$; however, τ_i^j will not in general be an integer.

For $u \in \mathcal{M}_1$ and $\delta, \theta > 0$, define the set

$$\mathcal{N}(u, \theta, \delta) = \left\{ v : \mathbb{Z} \to [-1, 1]^{\mathbb{Z}} : \begin{array}{l} |v_\alpha - u_\alpha| \leqslant \theta \text{ if } u_\alpha = 0, \\ |v_\alpha - u_\alpha| \leqslant \delta \text{ if } u_\alpha = \pm 1 \end{array} \right\}. \tag{4.2}$$

Thus $\mathcal{N}(u, \theta, \delta)$ defines a neighbourhood of u in the phase space $[-1, 1]^{\mathbb{Z}}$. Note that for any $v \in \mathcal{N}(u, \theta, \delta)$ the following inequalities hold:

$$\left. \begin{array}{rcrcr} u_i & = & 1 & \Rightarrow & 1 - \delta \leqslant v_i \leqslant 1, \\ u_i & = & 0 & \Rightarrow & -\theta \leqslant v_i \leqslant \theta, \\ u_i & = & -1 & \Rightarrow & -1 \leqslant v_i \leqslant -1 + \delta, \end{array} \right\} \tag{4.3}$$

and hence defining

$$M := \max\,\{\theta, \delta\} \tag{4.4}$$

we have

$$|v_i - u_i| \leqslant M \tag{4.5}$$

and

$$|\tau_i^j - \sigma_i^j| \leqslant 2M. \tag{4.6}$$

These inequalities will be useful in Section 4.3.

4.1 *Weak stability*

The concepts of asymptotic and Lyapunov stability are well established and widely known for dynamical systems. However, as we found in Section 2, we do not have uniqueness of solutions for (1.5), (1.4) due to the set-valued nonlinearity (1.4), hence we do not have a dynamical system and so the standard definitions do not apply. We will use the following standard definitions for differential inclusions (see, for example Kloeden, 1978).

DEFINITION 4.1 Let $u : [0, \infty) \to [-1, 1]^{\mathbb{Z}}$ be a solution of (1.5), (1.4) in the sense of Definition 2.1, with $u(0) = u^0$. Then

$$\Gamma(u^0) = \{u(t) : t \geqslant 0\} \tag{4.7}$$

is said to be a *forward orbit* of u^0.

In general for a differential inclusion, unlike a dynamical system, the forward orbit need not be unique. This leads to new definitions of forward invariance, asymptotic and Lyapunov stability.

DEFINITION 4.2 The set $B \subset [-1, 1]^{\mathbb{Z}}$ is *weakly forward invariant* for (1.5), (1.4) if every $u^0 \in B$ has a forward orbit $\Gamma(u^0)$ such that $\Gamma(u^0) \subset B$. The set B is *strongly forward invariant* if every forward orbit $\Gamma(u^0)$ of every point $u^0 \in B$ satisfies $\Gamma(u^0) \subset B$.

Note that the concepts of weakly and strongly forward invariant would be equivalent to each other and to the standard definition of forward invariance if we had uniqueness of solutions.

DEFINITION 4.3 Let $u \in [-1, 1]^{\mathbb{Z}}$ be an equilibrium solution of (1.5), (1.4) in the sense of Definition 3.1. Then u is *weakly Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every $v \in \mathcal{N}(u, \delta, \delta)$ has a forward orbit $\Gamma(v) \subset \mathcal{N}(u, \varepsilon, \varepsilon)$. Moreover, u is *strongly Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every forward orbit $\Gamma(v)$ of every point $v \in \mathcal{N}(u, \delta, \delta)$ satisfies $\Gamma(v) \subset \mathcal{N}(u, \varepsilon, \varepsilon)$.

DEFINITION 4.4 Let $u \in [-1, 1]^{\mathbb{Z}}$ be an equilibrium solution of (1.5), (1.4) in the sense of Definition 3.1. Then u is *weakly asymptotically stable* if it is weakly Lyapunov stable and for any $\varepsilon > 0$ there exists $\delta > 0$ such that every $v \in \mathcal{N}(u, \delta, \delta)$ has a forward orbit $\Gamma(v) \subset \mathcal{N}(u, \varepsilon, \varepsilon)$ which satisfies $v(t) \to u$ as $t \to \infty$. Moreover, u is *strongly asymptotically stable* if it is strongly Lyapunov stable and for any $\varepsilon > 0$ there exists $\delta > 0$ such that every forward orbit $\Gamma(v)$ of every point $v \in \mathcal{N}(u, \delta, \delta)$ satisfies $\Gamma(v) \subset \mathcal{N}(u, \varepsilon, \varepsilon)$ and $v(t) \to u$ as $t \to \infty$.

Thus, if $\mathcal{N}(u, \theta, \delta)$ is weakly forward invariant for (1.5), (1.4) with $\delta, \theta > 0$ arbitrarily small, then u is weakly Lyapunov stable. If, in addition, every $v^0 \in \mathcal{N}(u, \theta, \delta)$ has a forward orbit which satisfies $v(t) \to u$ as $t \to \infty$, then u is weakly asymptotically stable.

We will establish weak Lyapunov and weak asymptotic stability of mosaic solutions in Section 4.3. Although we are not able to show strong stability of the mosaic solutions, we note that only certain points v have forward orbits which can escape $\mathcal{N}(u, \delta, \delta)$. This follows from the fact that strong and weak stability are equivalent if solutions are unique, and uniqueness of solutions only breaks down when $|v_j(t)| = 1$. Specifically, a necessary condition for a point $v_0 \in \mathcal{N}(u, \delta, \delta)$ to have a forward orbit whose jth component v_j escapes $\mathcal{N}(u, \delta, \delta)$ when u is weakly Lyapunov stable is that $|v_{j\pm1}(t)| = 1$ for a set of times t of non-zero measure.

4.2 *Spectral theory for the zero solution*

Since the function $f(u)$ is linear in a neighbourhood of the origin, it is possible to study the stability of the zero solution, $u_i = 0$ for all $i \in \mathbb{Z}$, of (1.5), (1.4) using spectral theory. A similar approach is followed to that given for the two dimensional spatially discretized Allen–Cahn equation in (Chow *et al.*, 1996).

Define $(Su)_i = u_{i+1}$, so that S is the bounded shift operator on $l^\infty(\mathbb{Z})$, and write the j-fold composition of S as $(S^j u)_i = u_{i+j}$. Then for $u_i \in (-1, 1)$ for all i, equation (1.5), (1.4) can be written as a linear equation of the form

$$\dot{u} = Au,$$

where $\|u\|_{l^\infty(\mathbb{Z})} < 1$ and A is the bounded linear operator

$$A = -\alpha[-\beta(S^2 - 4S - 4S^{-1} + S^{-2} + 6I) - \gamma S + 2\gamma I - \gamma S^{-1}]. \tag{4.8}$$

We would like to show that the spectrum of A is wholly contained in the left half-plane, that is, that $\operatorname{spec}(A) \in (-\infty, 0)$, which would give asymptotic stability of the origin (see, for example Coppel, 1965, Theorem 2, p. 56). However, we find that it is only possible to prove the following result.

THEOREM 4.5 Let A be the bounded linear operator defined in (4.8). Then $\operatorname{spec}(A) \in (-\infty, 0]$ if and only if $\alpha\gamma \geqslant \max\{4\alpha\beta, 0\}$.

Proof. We have $\operatorname{spec}(S) = \{e^{i\theta} | \theta \in \mathbb{R}\} = S^1$, the unit circle in the complex plane, since if $\lambda \in S^1$, then taking $u_j = \lambda^j$ for $j \in \mathbb{Z}$ implies that $\lambda u_j = e^{i(j+1)\theta} = u_{j+1} = (Su)_j$. Hence λ is an eigenvalue of S, and since $\|S\| = \|S^{-1}\| = 1$, every $\lambda \in \operatorname{spec}(S_H)$ belongs to S^1.

By the spectral mapping theorem (see, for example Kreyszig, 1978, Theorem 7.4-2, p. 381), $\operatorname{spec}(S^j) = \{e^{ij\theta} | \theta \in \mathbb{R}\}$.

Now

$$\begin{aligned}
\operatorname{spec}(A) &= \{-\alpha[-\beta(e^{2i\theta} - 4e^{i\theta} - 4e^{-i\theta} + e^{-2i\theta} + 6) - \gamma e^{i\theta} - \gamma e^{-i\theta} + 2\gamma]|\theta \in \mathbb{R}\} \\
&= \{-\alpha[-\beta(2\cos 2\theta - 8\cos\theta + 6) - \gamma(2\cos\theta - 2)]|\theta \in \mathbb{R}\} \\
&= \{(2\cos\theta - 2)[\alpha\beta(2\cos\theta - 2) + \alpha\gamma] \mid \theta \in \mathbb{R}\}.
\end{aligned}$$

Clearly $(2\cos\theta - 2) \in [-4, 0]$ for all $\theta \in \mathbb{R}$, and taking θ sufficiently close to zero implies that $(2\cos\theta - 2)[\alpha\beta(2\cos\theta - 2) + \alpha\gamma] > 0$ unless $\alpha\gamma \geqslant 0$. Now, assuming $\alpha\gamma \geqslant 0$ it is necessary and sufficient that $\alpha\gamma - 4\alpha\beta \geqslant 0$ to ensure that $\operatorname{spec}(A) \in (-\infty, 0]$. □

Note that this result gives a necessary condition for Lyapunov stability of the zero solution, but it does not imply Lyapunov stability of this solution, since $\operatorname{spec}(A) \in (-\infty, 0]$ is a necessary but not sufficient condition (see, Coppel, 1965, Theorem 2, p. 56) for Lyapunov stability. Theorem 4.8 will show that in fact we require the stronger condition that $0 \leqslant 4\alpha\beta \leqslant \alpha\gamma$ for Lyapunov stability of the solution $u_i = 0$ for all $i \in \mathbb{Z}$.

4.3 *Stability of mosaic solutions*

We begin with two lemmas which will be central to establishing stability.

LEMMA 4.6 Let $u \in \mathcal{M}_1$ be a mosaic solution of (1.5), (1.4) which satisfies $u_i = \pm 1$ and

$$u_i\alpha[\beta(6u_i - 4\sigma_i^1 + \sigma_i^2) - \gamma(2u_i - \sigma_i^1)] > 0 \tag{4.9}$$

for all $i \in \mathbb{N}$. Then $\mathcal{N}(u, \theta, \delta)$ is weakly forward invariant for all sufficiently small $\delta > 0$.

Proof. Let u be a mosaic solution satisfying the conditions of Theorem 3.3, let $v(t)$ be a solution of (1.5), (1.4) with $v(0) \in \mathcal{N}(u, \theta, \delta)$, and for $|v_i| \neq 1$ write (3.4) as

$$\begin{aligned}\dot{v}_i &= \alpha[\beta(6u_i - 4\sigma_i^1 + \sigma_i^2) - \gamma(2u_i - \sigma_i^1)] \\ &\quad + \alpha[(v_i - u_i)(6\beta - 2\gamma) - (\tau_i^1 - \sigma_i^1)(4\beta - \gamma) + (\tau_i^2 - \sigma_i^2)\beta].\end{aligned} \tag{4.10}$$

Note here that we have taken

$$f(v_{i-1}) + f(v_{i+1}) = \gamma\tau_i^1 = \gamma(v_{i-1} + v_{i+1}),$$

which corresponds to the uniquely defined value of f given by (1.4) when $|v_{i\pm1}| < 1$ but which corresponds to a specific choice of the value of f when $|v_{i\pm1}| = 1$. (Note that for $|v_i| \neq 1$ such a choice leads to a local solution of (4.10) and hence is valid.) Thus the solution defined by this equation will define one forward orbit of $v(0)$, and showing that this forward orbit remains in $\mathcal{N}(u, \theta, \delta)$ is sufficient to show weak forward invariance.

Now, applying (4.4)–(4.6) to (4.10), we have

$$|\dot{v}_i - \alpha[\beta(6u_i - 4\sigma_i^1 + \sigma_i^2) - \gamma(2u_i - \sigma_i^1)]| \leqslant M|\alpha|[|6\beta - 2\gamma| + 2|4\beta - \gamma| + 2|\beta|]. \tag{4.11}$$

Thus, noting that (4.9) implies

$$\alpha[\beta(6u_i - 4\sigma_i^1 + \sigma_i^2) - \gamma(2u_i - \sigma_i^1)] \neq 0,$$

by taking M sufficiently small we can ensure that

$$M|\alpha|[|6\beta - 2\gamma| + 2|4\beta - \gamma| + 2|\beta|] < |\alpha[\beta(6u_i - 4\sigma_i^1 + \sigma_i^2) - \gamma(2u_i - \sigma_i^1)]|. \tag{4.12}$$

If $v(t) \in \mathcal{N}(u, \theta, \delta)$, then $\dot{v}_i$ has the same sign as $\alpha[\beta(6u_i - 4\sigma_i^1 + \sigma_i^2) - \gamma(2u_i - \sigma_i^1)]$. Thus assuming that $u_i = \pm 1$, equation (4.9) ensures that $\dot{v}_i \geqslant 0$ when $u_i = 1$ and $v \in \mathcal{N}(u, \theta, \delta)$, and that $\dot{v}_i \leqslant 0$ when $u_i = -1$ and $v \in \mathcal{N}(u, \theta, \delta)$.

We can repeat this argument for all $i \in \mathbb{Z}$, noting that given any values of α, β, $\gamma \in \mathbb{R}$ there are only finitely many different non-zero values of $\alpha[\beta(6u_i - 4\sigma_i^1 + \sigma_i^2) - \gamma(2u_i - \sigma_i^1)]$, and hence we can choose $M > 0$. This establishes the forward invariance of $\mathcal{N}(u, \theta, \delta)$ and completes the proof. □

LEMMA 4.7 Let $u \in \mathcal{M}_1$ be a mosaic solution of (1.5), (1.4) which satisfies $0 \leqslant 4\alpha\beta \leqslant \alpha\gamma$ and

$$\beta(6u_i - 4\sigma_i^1 + \sigma_i^2) - \gamma(2u_i - \sigma_i^1) = 0 \tag{4.13}$$

for all $i \in \mathbb{N}$. Then $\mathcal{N}(u, \delta, \delta)$ is weakly forward invariant for all sufficiently small $\delta > 0$.

Proof. Let u be a mosaic solution satisfying the conditions of Theorem 3.3, and let $v(t)$ be a solution of (1.5), (1.4) with $v(0) \in \mathcal{N}(u, \theta, \delta)$, and recall that $v(0)$ has a forward orbit which satisfies (4.10). Thus by (4.13) we have

$$\dot{v}_i = \alpha[(v_i - u_i)(6\beta - 2\gamma) - (\tau_i^1 - \sigma_i^1)(4\beta - \gamma) + (\tau_i^2 - \sigma_i^2)\beta]. \tag{4.14}$$

Now let $0 < \delta = \theta = M < 1$ and $v(t) \in \mathcal{N}(u, \delta, \delta)$.

Suppose $v_i - u_i \leqslant -M$, but that $|u_j - v_j| \leqslant M$ for all $j \neq i$. Then noting that $\alpha(3\beta - \gamma) \leqslant \alpha(4\beta - \gamma) \leqslant 0$, by (4.14) we have

$$\begin{aligned}\dot{v}_i &\geqslant -2M\alpha(3\beta - \gamma) + 2M\alpha(4\beta - \gamma) - 2M|\alpha\beta| \\ &= 2M[|\alpha\beta| - \alpha\beta] = 0,\end{aligned}$$

since $\alpha\beta \geqslant 0$.

Similarly, if $v_i - u_i \geqslant M$ we find that $\dot{v}_i \leqslant 0$.

Now, if $u_i = 1$, then $v_i \leqslant 1 - \delta$ implies that $v_i - u_i \leqslant -M$ and hence $\dot{v}_i \geqslant 0$. Similarly, if $u_i = -1$, then $v_i \geqslant -1 + \delta$ implies that $v_i - u_i \geqslant M$ and hence $\dot{v}_i \leqslant 0$. Finally, if $u_i = 0$, then $v_i \leqslant -\delta$ implies that $v_i - u_i \leqslant -M$ and hence $\dot{v}_i \geqslant 0$ and $v_i \geqslant \delta$ implies that $v_i - u_i \geqslant M$ and hence $\dot{v}_i \leqslant 0$.

These implications show that if any component v_i of v is on the boundary of $\mathcal{N}(u, \delta, \delta)$, then $\dot{v}_i$ points into $\mathcal{N}(u, \delta, \delta)$ and so $\mathcal{N}(u, \delta, \delta)$ is forward invariant, completing the proof. □

Lemmas 4.6 and 4.7 could be combined, but are stated separately for clarity. Lemma 4.7 enables us to determine the stability of the constant mosaic solutions.

THEOREM 4.8 The constant mosaic solutions $u_i = -1$ and $u_i = 1$ for all $i \in \mathbb{Z}$ are weakly Lyapunov stable but not weakly asymptotically stable equilibrium solutions to (1.5), (1.4) provided $0 \leqslant 4\alpha\beta \leqslant \alpha\gamma$. The constant mosaic solution $u_i = 0$ for all $i \in \mathbb{Z}$ is a strongly Lyapunov stable but not strongly asymptotically stable equilibrium solution to (1.5), (1.4) provided $0 \leqslant 4\alpha\beta \leqslant \alpha\gamma$.

Proof. These solutions cannot be asymptotically stable. To see this, recall from Section 3 that there exists a class of equilibrium solutions $\widehat{u}_i = \mu$ for all $i \in \mathbb{Z}$, where $\mu \in (-1, 1)$. Thus clearly there are equilibrium solutions arbitrarily close to each constant mosaic solution which contradicts asymptotic stability.

However, weak Lyapunov stability follows directly from Lemma 4.7 since each of these solutions satisfies (4.13) and thus Lemma 4.7 shows that $\mathcal{N}(u, \delta, \delta)$ is forward invariant for $\delta > 0$ arbitrarily small, which implies weak Lyapunov stability.

Finally note that since f has unique values in a neighbourhood of 0, forward orbits are uniquely defined in a neighbourhood of the constant solution $u_i = 0$ for all $i \in \mathbb{Z}$, and so strong stability follows for this solution. □

REMARK Note that in the case of the constant zero mosaic solution $u_i = 0$ for all $i \in \mathbb{Z}$, the condition for weak Lyapunov stability from Theorem 4.8 is stronger than from Theorem 4.5.

Although the constant mosaic solutions are not asymptotically stable, there do exist asymptotically stable mosaic solutions, and we now identify a class of such solutions.

DEFINITION 4.9 Let $\mathcal{S}_A(\alpha, \beta, \gamma) \subset \mathcal{M}_1$ be the set of $u \in \mathcal{M}_1$ such that

(i) $u_i u_{i\pm 1} = 1$ and $\alpha[2(3\beta - \gamma) - u_i\sigma_i^1(4\beta - \gamma) + \beta u_i \sigma_i^2] > 0$, or

(ii) $u_i = \pm 1$, $2u_i \neq \sigma_i^1$, $\beta\left(4 - \dfrac{2u_i - \sigma_i^2}{2u_i - \sigma_i^1}\right) > \gamma$, and $\alpha > 0$,

for each $i \in \mathbb{N}$.

THEOREM 4.10 Let $u \in \mathcal{S}_A(\alpha, \beta, \gamma)$; then u is a weakly asymptotically stable mosaic solution of (1.5), (1.4).

Proof. Assume the conditions of Lemma 4.6 are satisfied which implies that $\mathcal{N}(u, \theta, \delta)$ is weakly forward invariant for all sufficiently small $\delta > 0$ and hence u is Lyapunov stable.

To establish weak asymptotic stability take M sufficiently small in the proof of Lemma 4.6 so that (4.12) holds. Now, assuming that $u_i = \pm 1$, condition (4.9) ensures that $\dot{v}_i > 0$ when $u_i = 1$ and $v \in \mathcal{N}(u, \theta, \delta)$, and also that $\dot{v}_i < 0$ when $u_i = -1$ and $v \in \mathcal{N}(u, \theta, \delta)$. Moreover, (4.11) and (4.12) bound $\dot{v}_i$ away from 0, and asymptotic stability follows.

It just remains to show that under (i) and (ii) of Definition 4.9 we have a mosaic solution (which satisfies the conditions of Theorem 3.3), and that the conditions of Lemma 4.6 are satisfied.

First consider case (i). Clearly $u_i u_{i\pm 1} = 1$ implies that Theorem 3.3(iii) is satisfied, so this does allow a mosaic solution. Moreover, this ensures that $u_i = \pm 1$, and hence the inequality in (i) is equivalent to (4.9). Thus the conditions of Lemma 4.6 are satisfied.

Finally consider case (ii). This clearly implies that Theorem 3.3(iv) is satisfied, so this also allows a mosaic solution. Moreover note that for $u_i = \pm 1$ it follows that $2u_i - \sigma_i^1$ and u_i have the same sign and so from (ii) we have

$$\alpha\left[\beta\left(4 - \frac{2u_i - \sigma_i^2}{2u_i - \sigma_i^1}\right) - \gamma\right] > 0 \Rightarrow u_i\alpha(2u_i - \sigma_i^1)\left[\beta\left(4 - \frac{2u_i - \sigma_i^2}{2u_i - \sigma_i^1}\right) - \gamma\right] > 0$$

and so (4.9) and the conditions of Lemma 4.6 are satisfied. □

We now identify mosaic solutions which are Lyapunov stable.

DEFINITION 4.11 Let $\mathcal{S}_L(\alpha, \beta, \gamma) \subset \mathcal{M}_1$ be the set of $u \in \mathcal{M}_1$ such that

(i) $u_i = \sigma_i^1 = \beta\sigma_i^2 = 0$, and $0 \leqslant 4\alpha\beta \leqslant \alpha\gamma$,

(ii) $u_i u_{i\pm 1} = 1$, and (4.9) holds,

(iii) $u_i u_{i\pm 1} = 1$, (4.13) holds and $0 \leqslant 4\alpha\beta \leqslant \alpha\gamma$,

(iv) $u_i = \pm 1$, $2u_i \neq \sigma_i^1$, $\beta\left(4 - \dfrac{2u_i - \sigma_i^2}{2u_i - \sigma_i^1}\right) > \gamma$, and $\alpha > 0$, or

(v) $2u_i \neq \sigma_i^1$, $\beta\left(4 - \dfrac{2u_i - \sigma_i^2}{2u_i - \sigma_i^1}\right) = \gamma$, and $0 \leqslant 4\alpha\beta \leqslant \alpha\gamma$,

for each $i \in \mathbb{N}$.

THEOREM 4.12 Let $u \in \mathcal{S}_L(\alpha, \beta, \gamma)$; then u is a weakly Lyapunov stable mosaic solution of (1.5), (1.4).

Proof. Consider the cases from Definition 4.11 separately for each i. In case (i) either (i) or (ii) of Theorem 3.3 is satisfied as is (4.13) and consequently the conditions of Lemma 4.7 hold. In case (ii) Theorem 3.3(iii) is satisfied, together with the conditions of Lemma 4.6. In case (iii) Theorem 3.3(iii) is satisfied, together with the conditions of Lemma 4.7. In case (iv) Theorem 3.3(iv) and Theorem 4.10(ii) are satisfied. Finally in case (v) for $2u_i \neq \sigma_i^1$ note that $\beta(4 - (2u_i - \sigma_i^2)/(2u_i - \sigma_i^1)) = \gamma$ is equivalent to (4.13) and hence Theorem 3.3(iv) is satisfied, together with the conditions of Lemma 4.7.

It follows that u is a mosaic solution and that $\mathcal{N}(u, \delta, \delta)$ is weakly forward invariant for all sufficiently small $\delta > 0$, which implies weak Lyapunov stability, as required. □

5. Existence and stability of mosaic solutions on arbitrary lattices

In this section we show how existence and stability results for mosaic solutions on the integer lattice can be used to obtain results on an arbitrary lattice. In addition, we show in a simple way that the 'diffusion' coefficients α and β can be allowed to vary in space.

We define an arbitrary n-dimensional lattice $\mathcal{L}$ (see, Senechal, 1990) in terms of the linearly independent vectors $\{b_j\}_{j=1}^n$ as the set of points defined by all integer combinations of the form

$$\sum_{j=1}^{n} a_j b_j, \quad a_j \in \mathbb{Z} \quad \text{for} \quad j = 1, \ldots, n. \tag{5.15}$$

In addition, define the vectors $\{b_{n+1}, \ldots, b_N\}$ to be arbitrary linear combinations of the vectors $\{b_j\}_{j=1}^n$. For example, in the plane we could set $b_1 = e_1$, $b_2 = e_2$, where e_i denotes the ith unit vector, and then set $b_3 = b_1 + b_2$ and $b_4 = b_1 - b_2$. In this case b_1 and b_2 are used to obtain the nearest neighbours, while b_3 and b_4 are used to obtain the next nearest neighbours.

Given the lattice $\mathcal{L}$ we define an equation that is discrete in space and of Cahn–Hilliard type for $\eta \in \mathcal{L}$ as

$$\begin{aligned}\dot{u}(\eta) = \sum_{j=1}^{N} \alpha_j(\eta)[\beta_j(\eta)\{u(\eta + 2b_j) - 4u(\eta + b_j) + 6u(\eta) - 4u(\eta - b_j) + u(\eta - 2b_j)\}\\ + f(u(\eta + b_j)) - 2f(u(\eta)) + f(u(\eta - b_j))].\end{aligned} \tag{5.16}$$

Notice that the right-hand side of (5.16) is simply the sum of N problems each effectively defined on an integer lattice. Thus, the existence and stability results for mosaic solutions on $\mathbb{Z}$ can be trivially applied to obtain results for (5.16). For instance, suppose $u(\eta) = 0$; then in each direction (that is, corresponding to each b_j) we can apply the criteria for existence from Theorem 3.3 (case (i), (ii), and (iv) with $u_i = 0$) and the criteria for stability from Theorems 4.5, 4.8, or 4.12 (case (i) and (v) with $u_i = 0$) to obtain criteria for existence and stability of mosaic solutions of (5.16). For $u(\eta) = \pm 1$ we apply Theorem 3.3 (case (iii), and (iv) with $u_i = \pm 1$) and Theorems 4.8, 4.10, or 4.12 (case (ii), (iii), (iv), and (v) with $u_i = \pm 1$).

6. Spatial entropy

To differentiate between sets of stable equilibria which exhibit regular patterns and those with a spatially disordered structure, we use the concept of spatial entropy. The spatial entropy, $h(\mathcal{U}) \geqslant 0$, of a given non-empty set of mosaics measures the degree of spatial disorder of the set, essentially by measuring the number of different patterns observable among the elements of $\mathcal{U}$ in finite subsets of the lattice $\mathbb{Z}$ and the rate at which this number grows with the size of the subset. Sets with large spatial entropy are more complex and it is therefore harder to predict the global appearance of a mosaic $u \in \mathcal{U}$ from its form on a finite subset of the lattice.

In this section we will calculate the spatial entropy of Lyapunov stable equilibrium solutions. We are particularly interested in the dependence of the spatial entropy on the parameters α, β and γ, and this leads us to consider the following class of Lyapunov stable equilibrium solutions.

DEFINITION 6.1 Let $\mathcal{S}_{L^*}(\alpha, \beta, \gamma) \subset \mathcal{M}_1$ be the set of $u \in \mathcal{M}_1$ such that

(i) $2u_i = \sigma_i^1 = \sigma_i^2$, and $0 \leqslant 4\alpha\beta \leqslant \alpha\gamma$,
(ii) $u_i u_{i\pm1} = 1$, and (4.9) holds, or
(iii) $u_i = \pm1$, $2u_i \neq \sigma_i^1$, $\beta\left(4 - \dfrac{2u_i - \sigma_i^2}{2u_i - \sigma_i^1}\right) > \gamma$, and $\alpha > 0$

for each $i \in \mathbb{N}$.

THEOREM 6.2 Let $u \in \mathcal{S}_{L^*}(\alpha, \beta, \gamma)$; then u is a weakly Lyapunov stable mosaic solution of (1.5), (1.4).

Proof. If Definition 6.1(i) holds, then either Definition 4.11(i) or (iii) holds, whilst Definition 6.1(ii) and (iii) correspond to Definition 4.11(ii) and (iv) respectively. Hence $\mathcal{S}_{L^*}(\alpha, \beta, \gamma) \subset \mathcal{S}_L(\alpha, \beta, \gamma)$ and the result now follows from Theorem 6.2. □

We consider Lyapunov stable mosaic solutions in the class $\mathcal{S}_{L^*}(\alpha, \beta, \gamma)$ rather than the larger class $\mathcal{S}_L(\alpha, \beta, \gamma)$ due to structural stability considerations. The Lyapunov stable mosaic solutions in $\mathcal{S}_{L^*}(\alpha, \beta, \gamma)$ will all persist for some range of values of α, β and γ. However, due to the equality in Definition 4.11(v) this condition can lead to mosaic solutions which are only stable for one fixed value of the ratio β/γ. Such solutions are not of practical interest, since any perturbation of the ratio β/γ will destroy the stability of the solutions. Similarly, Definition 4.11(i) and (iii) allow certain solutions which only occur at $\beta/\gamma = 0$ or $\beta/\gamma = 1/4$, and Definition 6.1(i) is a special case of these conditions which allows the same (constant) mosaic solutions over the whole range $0 \leqslant 4\alpha\beta \leqslant \alpha\gamma$.

6.1 *General entropy calculations*

We first define the concept of spatial entropy in a general one-dimensional setting. Let $\mathcal{A}$ be a finite set of d elements and define $\mathcal{A}^{\mathbb{Z}}$ to be the set of all functions $u : \mathbb{Z} \to \mathcal{A}$. Consider any non-empty subset $\mathcal{U} \subseteq \mathcal{A}^{\mathbb{Z}}$, and assume that $\mathcal{U}$ is translation invariant, so that $S(\mathcal{U}) = \mathcal{U}$, where S is the bounded shift operator $S : \mathcal{A} \to \mathcal{A}$

$$(Su)_i = u_{i+1} \quad \forall i \in \mathbb{Z}. \tag{6.17}$$

Given any positive integer N, define the set

$$E_N := \{i \in \mathbb{Z} \mid 0 \leqslant i \leqslant N-1\},$$

and consider the natural projection

$$\pi_N : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{E_N}$$

which is given by restricting any $u \in \mathcal{A}^{\mathbb{Z}}$ to the finite subset of the lattice $E_N \subseteq \mathbb{Z}$. Let

$$\Gamma_N(\mathcal{U}) = \operatorname{card}(\pi_N(\mathcal{U})),$$

so that $\Gamma_N(\mathcal{U})$ counts the number of patterns which can be observed among the elements of $\mathcal{U}$, restricting observation to the subset $E_N \subseteq \mathbb{Z}$. Clearly

$$1 \leqslant \Gamma_N(\mathcal{U}) \leqslant \operatorname{card}(\mathcal{A}^{E_N}) = d^N.$$

Note that since $\mathcal{U}$ is assumed to be translation invariant, there is no loss of generality in restricting to the coordinates $0 \leqslant i \leqslant N-1$ rather than to $c \leqslant i \leqslant N+c-1$ for some $c \neq 0$.

DEFINITION 6.3 The *spatial entropy* of the set $\mathcal{U}$ is defined to be the limit

$$h(\mathcal{U}) := \lim_{N\to\infty} \frac{1}{N} \log \Gamma_N(\mathcal{U}).$$

Existence of this limit together with the formula

$$h(\mathcal{U}) = \inf_{N\geqslant 1} \frac{1}{N} \log \Gamma_N(\mathcal{U}),$$

is established in (Chow *et al.*, 1996); see also (Abell, 2000). Note also that in general, since $\Gamma_N(\mathcal{A}^{\mathbb{Z}}) = d^N$ for any alphabet of d elements $\mathcal{A}$, it follows that $h(\mathcal{A}^{\mathbb{Z}}) = \ln d$.

In the one-dimensional case, it is possible to calculate $h(\mathcal{U})$ explicitly when $\mathcal{U}$ belongs to a certain class of translation invariant subsets known as *Markov shifts*, or *subshifts of finite type*. These are defined as follows (see, Robinson, 1995, p. 73).

Let M be a $d \times d$ matrix, all of whose entries are either 1 or 0, known as a *transition matrix*, and denote the (i, j)-th entry of M by $M_{i,j}$. Then define the set

$$\mathcal{U}(M) = \{u \in \mathcal{A}^{\mathbb{Z}} | M_{u_i,u_{i+1}} = 1 \quad \forall i \in \mathbb{Z}\}, \tag{6.18}$$

so that $\mathcal{U}(M)$ consists of the sequences from $\mathcal{A}^{\mathbb{Z}}$ allowed by the transition matrix M. Now note that by (6.17) and (6.18) we have $S\mathcal{U}(M) = \mathcal{U}(M)$, so that $\mathcal{U}(M)$ is translation invariant under the shift map S. Then the Markov shift for the matrix M is defined to be the map $S_d : \mathcal{U}(M) \to \mathcal{U}(M)$ defined by $S_d \equiv S|_{\mathcal{U}(M)}$.

In the case of a one-dimensional lattice, Markov shifts have been extensively studied and are well understood. In particular (see, for example Robinson, 1995) we have

$$h(\mathcal{U}(M)) = \ln(\lambda_1), \tag{6.19}$$

where λ_1 is the largest real positive eigenvalue of M.

It is also possible to use M to construct words $\pi_N(u) \in \mathcal{A}^{E_N}$ of any Markov chain u for $N \geqslant 2$, and we can calculate $\Gamma_N(\mathcal{U})$ using the following theorem.

THEOREM 6.4 There are $(M^k)_{i,j}$ allowable words of length $k+1$ starting at i and ending at j, that is, $(M^k)_{i,j}$ words of the form $is_1 \dots s_{k-1}j$.

Proof. See (Robinson, 1995, Lemma 2.2, p. 24). □

Clearly, the total number of words of length $k+1$ is given by summing the elements of M^k and hence

$$\Gamma_N(\mathcal{U}) = \sum_{i,j=1}^{d} (M^k)_{i,j}.$$

Now, the system (1.5), (1.4) is said to exhibit *spatial chaos* at a point (α, β, γ) in parameter space if the spatial entropy of a set of stable mosaic solutions of the system, $h(\mathcal{S}(\alpha, \beta, \gamma))$, is positive. The system is said to exhibit *pattern formation* at this point if the spatial entropy is zero.

We will see below that in parameter ranges where the system exhibits pattern formation there is a fixed finite number of patterns occurring in $\mathcal{S}(\alpha, \beta, \gamma)$. Where spatial chaos occurs, $h(\mathcal{S}(\alpha, \beta, \gamma))$ gives a measure of how fast $\Gamma_N(\mathcal{S}(\alpha, \beta, \gamma))$ grows with N.

6.2 *Entropy calculations for the Cahn–Hilliard system*

The techniques above can now be used to calculate the spatial entropy of sets of stable mosaic solutions of (1.5), (1.4). Let $\mathcal{A} = \{-1, 0, 1\}$ so that $d = 3$. Define $\mathcal{F} = \mathcal{A}^{E_5}$ to be the set of all five-tuples $\widetilde{u} = (\widetilde{u}_0, \dots, \widetilde{u}_4)$, where $\widetilde{u}_i \in \mathcal{A}$ for $i \in [0, 4]$. Clearly $\mathcal{F}$ has $3^5 = 243$ elements.

Now fix a non-empty subset $\mathcal{B} \subseteq \mathcal{F}$ and define a set $\widetilde{\mathcal{U}}_{\mathcal{B}} \subseteq \mathcal{M}_1$ such that

$$\widetilde{\mathcal{U}}_{\mathcal{B}} = \{u \in \mathcal{M}_1 \mid (u_{i+2}, u_{i+1}, u_i, u_{i-1}, u_{i-2}) \in \mathcal{B} \text{ for all } i \in \mathbb{Z}\}.$$

We can take $\widetilde{\mathcal{U}}_{\mathcal{B}}$ to be $\mathcal{S}_A(\alpha, \beta, \gamma)$, $\mathcal{S}_L(\alpha, \beta, \gamma)$, $\mathcal{S}_{L^*}(\alpha, \beta, \gamma)$ by taking $\mathcal{B}$ to be the set of five-tuples which satisfy the conditions in Definition 4.9, 4.11 or 6.1 respectively. Thus we refer to a five-tuple $\widetilde{u} \in \mathcal{B}$ as an *admissible five-tuple*, and to $\mathcal{B}$ as the *set of admissible five-tuples*. Then $\widetilde{\mathcal{U}}_{\mathcal{B}}$ is the set of mosaic solutions generated by the set of admissible five-tuples $\mathcal{B}$.

The conditions which define the stable mosaic solutions $S(\alpha, \beta, \gamma)$ in Definition 4.9, 4.11 and 6.1 clearly involve not only the values of u_i and u_{i+1}, but also u_{i-1} and $u_{i\pm 2}$, and so although $\widetilde{\mathcal{U}}_{\mathcal{B}}$ is translation invariant, it will not in general be a Markov shift. It must be shown that $\widetilde{\mathcal{U}}_{\mathcal{B}}$ is equivalent to a Markov shift in order to apply (6.19), This is done by defining an injective map which reinterprets a mosaic $u \in \mathcal{M}_1$ as an infinite array of four-tuples.

Let $\widehat{\mathcal{F}} = \mathcal{A}^{E_4}$ be the set of all possible four-tuples $\widehat{u} = (\widetilde{u}_0, \dots, \widetilde{u}_3)$ such that again $\widetilde{u}_i \in \mathcal{A}$ for $i \in [0, 3]$. Clearly, $\widehat{\mathcal{F}}$ has $3^4 = 81$ elements. Given a set of admissible five-tuples $\mathcal{B}$, the set of *available four-tuple pairs*, $\widetilde{\mathcal{B}} \subseteq \widehat{\mathcal{F}}$, is then defined to be the set of all pairs of elements of $\widehat{\mathcal{F}}$ which overlap to give an element of $\mathcal{B}$, so

$$\widetilde{\mathcal{B}} = \left\{ \widehat{u}, \widehat{v} \in \widehat{\mathcal{F}} \mid \exists w = (w_0, w_1, w_2, w_3, w_4) \in \mathcal{B} : \begin{array}{l} \widehat{u} = (w_0, w_1, w_2, w_3) \text{ and} \\ \widehat{v} = (w_1, w_2, w_3, w_4) \end{array} \right\}.$$

Define the set of *available four-tuples* $\widehat{\mathcal{B}}$ to be those elements of $\widehat{\mathcal{F}}$ which occur as adjacent points in elements of $\mathcal{B}$, so

$$\widehat{\mathcal{B}} = \left\{ \widehat{u} \in \widehat{\mathcal{F}} | \exists w = (w_0, w_1, w_2, w_3, w_4) \in \mathcal{B} : \begin{array}{l} \widehat{u} = (w_0, w_1, w_2, w_3) \text{ or} \\ \widehat{u} = (w_1, w_2, w_3, w_4) \end{array} \right\}.$$

Now, define a map $\psi : \mathcal{A}^{\mathbb{Z}} \to \widehat{\mathcal{F}}^{\mathbb{Z}}$ by

$$\psi(u)_i = \pi_4(S^i u) \quad \text{for } i \in \mathbb{Z},$$

so that ψ reinterprets a mosaic $u \in \mathcal{M}_1$ as an infinite array of four-tuples. The map ψ is clearly injective and now $\psi(\widetilde{\mathcal{U}}_{\mathcal{B}})$ is a Markov chain on the set $\widehat{\mathcal{B}}$, with transition matrix M given by setting $M_{\widehat{u},\widehat{v}} = 1$ if $\widehat{u},\widehat{v} \in \widetilde{\mathcal{B}}$, that is if $\widehat{u}$, $\widehat{v}$ are an available four-tuple pair, and setting $M_{\widehat{u},\widehat{v}} = 0$ otherwise.

Now, $\Gamma_N(\widetilde{\mathcal{U}}_{\mathcal{B}})$ is the number of different N-tuples $(\widetilde{u}_0, \ldots, \widetilde{u}_{N-1})$ which occur for the elements $u \in \widetilde{\mathcal{U}}_{\mathcal{B}}$, since $\Gamma_N(\widetilde{\mathcal{U}}_{\mathcal{B}}) = \text{card}(\pi_N(\widetilde{\mathcal{U}}_{\mathcal{B}}))$. Similarly, $\Gamma_{N-1}(\psi(\widetilde{\mathcal{U}}_{\mathcal{B}}))$ is the number of different $(N-1)$-tuples $(\psi(u)_0, \ldots, \psi(u)_{N-1})$ of elements of $\psi(\widetilde{\mathcal{U}}_{\mathcal{B}})$. From the definition of ψ we see that there is a one-to-one correspondence between $\pi_N(\widetilde{\mathcal{U}}_{\mathcal{B}})$ and $\pi_{N-1}(\psi(\widetilde{\mathcal{U}}_{\mathcal{B}}))$ and so for $N \geqslant 2$,

$$\Gamma_N(\widetilde{\mathcal{U}}_{\mathcal{B}}) = \Gamma_{N-1}(\psi(\widetilde{\mathcal{U}}_{\mathcal{B}})).$$

Hence, using (6.19), it follows that

$$\begin{aligned} h(\widetilde{\mathcal{U}}_{\mathcal{B}}) &= \lim_{N\to\infty} \frac{1}{N} \ln \Gamma_N(\widetilde{\mathcal{U}}_{\mathcal{B}}) \\ &= \lim_{N\to\infty} \left(\frac{N-1}{N}\right)\left(\frac{1}{N-1}\right) \ln \Gamma_{N-1}(\psi(\widetilde{\mathcal{U}}_{\mathcal{B}})) \\ &= h(\psi(\widetilde{\mathcal{U}}_{\mathcal{B}})) \\ &= \ln \lambda, \end{aligned}$$

where $\lambda \geqslant 0$ is the largest eigenvalue of the matrix M. By taking $\widetilde{\mathcal{U}}_{\mathcal{B}} = \mathcal{S}_{L^*}(\alpha, \beta, \gamma)$, or $\mathcal{S}_A(\alpha, \beta, \gamma)$ it is now possible to calculate the spatial entropy of these sets explicitly.

6.3 *Spatial entropy of Lyapunov stable solutions*

We will compute the spatial entropy when $\widetilde{\mathcal{U}}_{\mathcal{B}} = \mathcal{S}_L^*(\alpha, \beta, \gamma)$, from Definition 6.1.

The parameters α and γ can be treated as scaling values and so it is only their sign that will affect the spatial entropy calculations. Thus, to compute the spatial entropy for all values (α, β, γ) in parameter space it is sufficient to consider the four cases where $\alpha = \pm 1$ and $\gamma = \pm 1$. In each case, we divide the real line β into regions of β values. The boundaries for these regions arise naturally from the conditions in Definition 6.1, and correspond to the value of β for which each admissible five-tuple $\widetilde{u} \in \mathcal{B}$ satisfies (4.13). For $\gamma < 0$ the regions are defined as shown in Fig. 5 in the Appendix.

When $\gamma > 0$, the signs of β are reversed, and so we denote the resulting regions as $-R_i$ for $i \in [0, \ldots, 12]$.

We note from Section 2 that we only have global existence of solutions $u(t) \in [-1, 1]^{\mathbb{Z}}$ for all initial conditions in $[-1, 1]^{\mathbb{Z}}$ for some parameter values. However, it follows from Lemmas 4.6 and 4.7 that we do have global existence of solutions in the neighbourhood of a stable mosaic solution for all parameter values.

To compute the spatial entropy in the case $\widetilde{\mathcal{U}}_{\mathcal{B}} = \mathcal{S}_L^*(\alpha, \beta, \gamma)$, we consider each of the regions given by Fig. 5 in each of the cases $\alpha = \pm 1$, $\gamma = \pm 1$ in turn. In each region of parameter space, we check which of the 243 five-tuples are admissible and generate an 81×81 transition matrix M with $M_{i,j} = 1$ if the ith and jth four-tuples overlap to form an admissible five-tuple, and $M_{i,j} = 0$ otherwise. The eigenvalues of this matrix and hence the entropy can be calculated numerically.

Condensed transition matrices for $\mathcal{S}_{L^*}(\alpha, \beta, \gamma)$. Each of the different parameter regions in Fig. 5 results in a different 81×81 transition matrix M. However, these matrices are all very sparse and have only a few non-zero eigenvalues, and moreover the transition matrices M of many of the regions have the same eigenvalues and eigenvectors, despite the matrices themselves being different.

To understand this, consider that the transition matrix M can be thought of as a map, with $M_{i,j} = 1$ if the ith and jth four tuples overlap to form an available five-tuple. Given two such four-tuples, suppose we now want to extend the resulting five-tuple into a six-tuple, as the next step to constructing a mosaic solution. To do so, we require a four-tuple, which overlaps with the jth four-tuple to form a five-tuple. Such a four-tuple exists if and only if $M_{j,k} = 1$ for some k. If $M_{j,k} = 0$ for all k, then even though $M_{i,j} = 1$ no mosaic solution will exist that contains the corresponding admissible five-tuple.

We find that over all the regions considered, only seventeen of the 81 four-tuples combine to form available pairs of four-tuples which result in a five-tuple combination which is not only admissible, but which can also be used to form part of a translation invariant mosaic solution. Thus, rather than displaying the full 81×81 transition matrix M, to present our results it is sufficient to display a 17×17 matrix $\widetilde{M}$ which we refer to as a condensed transition matrix. Note also that M and $\widetilde{M}$ have the same eigenvalues, as the rows and columns of M deleted to form $\widetilde{M}$ always fall in the null-space of M.

The condensed transition matrix $\widetilde{M}$ can be used to construct chains of available pairs of four-tuples, whose combination is an admissible i-tuple, where $i \in [5, \infty)$ is an integer. These chains are mosaic solutions defined on a subset of length i of the lattice, or more precisely the projection of a mosaic solution onto a subset of length i of the lattice. It is also possible to use powers of the condensed transition matrices to calculate the number of mosaic solutions on any subset of $\mathbb{Z}$ using Theorem 6.4. Since the total number of words of length $k + 1$ is given by summing the elements of M^k, and in this case each 'letter' in a word represents a four-tuple, the number of mosaic solutions on a subset of $\mathbb{Z}$ of length j are found by summing all the entries of the matrix $\widetilde{M}^{j-4}$ for $j \geqslant 5$.

The seventeen four-tuples which can appear in mosaic solutions are listed in Table 2 in the Appendix. The general form of the condensed transition matrix $\widetilde{M}$, indicating the possible non-zero entries, is given in Table 3.

Although the transition matrices M for the regions in Fig. 5 are in general different, we find that the condensed transition matrices for many of the regions are the same, so that some of these regions have the same spatial entropy and admit the same mosaic solutions.

In total, considering all the regions for both $\alpha > 0$ and $\alpha < 0$ only 10 different condensed transition matrices occur. We label these $\widetilde{M}_1, \ldots, \widetilde{M}_{10}$ and the regions of phase space in which they occur are indicated in Figs 6 and 7.

Table 4 gives the spatial entropy of each of the transition matrices $\widetilde{M}_i$ in order of increasing spatial entropy. The columns headed $\Gamma_N(S_{L^*})$ give the values of $\Gamma_N(S_{L^*}(\alpha, \beta, \gamma))$ for each of the transition matrices; that is, the number of different N-tuples which arise as projections of mosaic solutions onto a subset of $\mathbb{Z}$ of length j. Note from above that $\Gamma_5(S_{L^*})$ is the same as the number of non-zero entries of $\widetilde{M}$ whilst for $N > 5$ this is given by summing all the entries of $\widetilde{M}^{N-4}$. Note that $\widetilde{M}_1$ has no non-zero entries. In this case $S_{L^*}(\alpha, \beta, \gamma)$ is empty, that is, there are no Lyapunov stable mosaic solutions in $S_{L^*}(\alpha, \beta, \gamma)$ for the corresponding parameter values, and accordingly the spatial entropy is undefined. The condensed transition matrices $\widetilde{M}_2$, $\widetilde{M}_3$ and $\widetilde{M}_4$ do have non-zero elements, but in each case the spatial entropy is zero and there is a fixed number of spatially periodic Lyapunov stable solutions in $S_{L^*}(\alpha, \beta, \gamma)$ for each case. The remaining cases $\widetilde{M}_i$, $i \geqslant 5$, are more interesting with positive entropy and spatial chaos, as illustrated by the rapid growth of $\Gamma_N(S_{L^*}(\alpha, \beta, \gamma))$ as N is increased. We now study these cases in more detail.

I: *No stable mosaic solution.* $\boldsymbol{\widetilde{M}_1}$: It is clear from Figs 6 and 7 that the condensed transition matrix $\widetilde{M}_1$ arises when

$$\alpha\beta < 0 < \alpha\gamma,$$

$$2\alpha\beta < \alpha\gamma < 0$$

or

$$\alpha < 0, \text{ and } 4\gamma < \beta < 2\gamma < 0.$$

In these cases $(\widetilde{M}_1)_{ij} = 0$ for all i, j and there are no Lyapunov stable mosaic solutions in $S_{L^*}(\alpha, \beta, \gamma)$.

II: *Pattern formation.* $\boldsymbol{\widetilde{M}_2}$: From Figs 6 and 7 we see that $\widetilde{M}_2$ arises when

$$0 < 4\alpha\beta < \alpha\gamma.$$

In this case $M_{1,1} = M_{9,9} = M_{17,17} = 1$, $M_{i,j} = 0$ otherwise and $\widetilde{M}_2$ has eigenvalues 1 and 0. From Table 2 it follows that the corresponding admissible five-tuples are $(-1\,-1\,-1\,-1\,-1)$, (00000) and (11111) which gives us just the three constant mosaic solutions $u_i = -1$, $u_i = 0$ or $u_i = 1$ for all $i \in \mathbb{Z}$ which we have already shown to be Lyapunov stable for these parameter values in Theorem 4.8.

$\boldsymbol{\widetilde{M}_3}$: This arises when

$$\alpha > 0, \text{ and } 0 < \gamma/4 < \beta < \gamma/3.$$

In this case $M_{6,12} = M_{12,6} = 1$ and $M_{i,j} = 0$ otherwise; so that $\widetilde{M}_3$ has eigenvalues 1, -1 and 0. The admissible five-tuples are $(-11\,-11\,-1)$ and $(1\,-11\,-11)$ leading to the alternating or saw-tooth mosaic solution $u_i = (-1)^i$ for all $i \in \mathbb{Z}$ and its translate $u_i = (-1)^{(i+1)}$ for all $i \in \mathbb{Z}$.

$\widetilde{\boldsymbol{M}}_{\mathbf{4}}$: This arises when

$$\begin{gathered}\alpha\gamma/2 < \alpha\beta < \alpha\gamma/3 < 0,\\ \alpha < 0,\ \text{and}\ \beta < 0 < \gamma,\\ \alpha < 0,\ \text{and}\ 2\beta < \gamma < 0.\end{gathered}$$

In this case $M_{4,7} = M_{7,14} = M_{14,11} = M_{11,4} = 1$ and $M_{i,j} = 0$ otherwise; so that $\widetilde{M}_4$ has eigenvalues $1, \pm i, -1$ and 0. There is one Lyapunov stable mosaic solution $(\ldots -1-111-1-111-1-111\ldots)$ which is period four in space and which we refer to as the double alternating solution, together with its three translates.

III: Spatial chaos. For

$$\begin{gathered}\alpha > 0 \text{ and } 0 < \gamma < 3\beta,\\ \alpha > 0, \quad \gamma < 0 \text{ and } \gamma < 3\beta\end{gathered}$$

or

$$\alpha < 0,\ \text{and}\ 0 < 3\beta < \gamma,$$

spatial chaos arises. The corresponding condensed transition matrices $\widetilde{M}_5, \ldots, \widetilde{M}_{10}$ are given in Table 5 and their corresponding parameter ranges and eigenvalues are given in Table 1. In Table 6 we list the admissible five-tuples for each condensed transition matrix $\widetilde{M}_i$. Note that by our definition of the condensed transition matrix, these are all the admissible five-tuples which occur in mosaic solutions, but not necessarily all the admissible five-tuples for the corresponding region of parameter space, since some will not lead to mosaic solutions. In Table 7 we list some simple spatially-periodic mosaic solutions which can occur. Not surprisingly, just as there are more admissible five-tuples as the spatial entropy increases, so there are more periodic solutions. Note that the stable periodic solution $(\ldots -11-111\ldots)$ corresponds to one period of -11 followed by -111 and perhaps not surprisingly arises in the parameter regions where both the alternating solution $(\ldots -11\ldots)$ and $(\ldots -111\ldots)$ are stable. Thus, being careful to check that all the five-tuples arising at the interfaces are admissible we can weld different solutions together to form additional solutions.

So far we have only exhibited periodic solutions, but it is easy to weld together different numbers of periods of different solutions to create non-periodic solutions. Alternatively we can create non-periodic solutions by taking a periodic solution and altering its values at some point in the sequence. For example, it is easily verified that

$$(\ldots -11-11-11-11)(1-11-11-11\ldots)$$

occurs for $\widetilde{M}_5$, $\widetilde{M}_8$ and $\widetilde{M}_{10}$ and it is clearly non-periodic. For $\widetilde{M}_{10}$ we also have

$$(\ldots -11-11-11-11)1(1-11-11-11\ldots)$$

TABLE 1
Parameter ranges, spatial entropy and eigenvalues of the condensed transition matrices

Matrix	Parameter range	Entropy	Eigenvalues
$\widetilde{M}_1$	$\alpha\beta < 0 < \alpha\gamma$, $2\alpha\beta < \alpha\gamma < 0$, $\alpha < 0$, and $4\gamma < \beta < 2\gamma < 0$.	$\ln 0$	0
$\widetilde{M}_2$	$0 < 4\alpha\beta < \alpha\gamma$	$\ln 1 = 0$	$1, 0$
$\widetilde{M}_3$	$\alpha > 0$, and $0 < \gamma/4 < \beta < \gamma/3$	$\ln 1 = 0$	$1, -1$ and 0
$\widetilde{M}_4$	$\alpha\gamma/2 < \alpha\beta < \alpha\gamma/3 < 0$, $\alpha < 0$, and $\beta < 0 < \gamma$, $\alpha < 0$, and $2\beta < \gamma < 0$.	$\ln 1 = 0$	$1, \pm i, -1$ and 0
$\widetilde{M}_5$	$\alpha > 0$ and $0 < \gamma < 3\beta < \gamma/2$	$\ln 1{\cdot}4656$	$1{\cdot}4656, 0{\cdot}7549, -0{\cdot}2328 \pm 0{\cdot}7926i$, $-0{\cdot}8774 \pm 0{\cdot}7449i$ and 0
$\widetilde{M}_6$	$\alpha < 0$ and $0 < \beta < \gamma/3$	$\ln 1{\cdot}4656$	$1{\cdot}4656, 0{\cdot}5474 \pm 1{\cdot}1209i$, $-0{\cdot}2328 \pm 0{\cdot}7926i$, $-0{\cdot}5474 \pm 0{\cdot}5857i, -1$ and 0
$\widetilde{M}_7$	$\gamma/3 < \beta < 2\gamma/7 < 0 < \alpha$	$\ln 1{\cdot}5552$	$1{\cdot}5552, 0{\cdot}9118, 0{\cdot}8612$, $-0{\cdot}4378 \pm 1{\cdot}0584i$, $-0{\cdot}6341 \pm 0{\cdot}3997i$, $-0{\cdot}7126 \pm 0{\cdot}7422i, -0{\cdot}8036$ and 0
$\widetilde{M}_8$	$\alpha > 0$ and $\beta > 0 > \gamma$ $\alpha > 0$ and $\beta > \gamma/2 > 0$	$\ln 1{\cdot}6180$	$1{\cdot}618, -0{\cdot}5 \pm 0{\cdot}866i$, $-0{\cdot}618$ and 0
$\widetilde{M}_9$	$2\gamma/7 < \beta < \gamma/4 < 0 < \alpha$	$\ln 1{\cdot}6180$	$1{\cdot}618, 1, 0{\cdot}6808 \pm 0{\cdot}3931i$, $0{\cdot}636 \pm 1{\cdot}016i, \pm i$, $-0{\cdot}5 \pm 0{\cdot}66i, -1$ and 0
$\widetilde{M}_{10}$	$\gamma/4 < \beta < 0 < \alpha$	$\ln 1{\cdot}9276$	$1{\cdot}9276\,, 0{\cdot}309 \pm 0{\cdot}9511i$, $-0{\cdot}0764 \pm 0{\cdot}8147i$, $-0{\cdot}809 \pm 0{\cdot}5878i, -0{\cdot}7748$ and 0

and

$$(\ldots -11-11-11-11)11(1-11-11-11\ldots).$$

Note that there are no Lyapunov-stable mosaic solutions with five equal consecutive values of u_i apart from the constant mosaic solutions. This follows from Table 6 in the Appendix, since the five-tuples that allow the constant mosaic solutions only occur for $\widetilde{M}_2$ and no other five-tuples arise in this case.

Finally note that we have always taken our parameter regions to be open and have not considered what happens on the boundaries of parameter regions. If we were interested in the boundaries of the parameter regions, then we should consider the class of mosaic solutions $\mathcal{S}_L(\alpha, \beta, \gamma)$ defined in Definition 4.11 rather than $\mathcal{S}_{L^*}(\alpha, \beta, \gamma)$, because in forming $\mathcal{S}_{L^*}(\alpha, \beta, \gamma)$ as a subset of $\mathcal{S}_L(\alpha, \beta, \gamma)$ we removed the possibility of some solutions occurring on the parameter boundaries. However, since the spatially discrete Cahn–Hilliard equation models certain physical processes, we are interested in solutions which are not only stable, but structurally stable to perturbations of the parameters; thus we are not interested in stable solutions which occur for isolated parameter values.

6.4 *Spatial entropy of asymptotically stable solutions*

Comparing Definitions 4.9 and 6.1 we see that $\mathcal{S}_A(\alpha, \beta, \gamma) \subset \mathcal{S}_{L^*}(\alpha, \beta, \gamma)$, and moreover that the only mosaic solutions which are in $\mathcal{S}_{L^*}(\alpha, \beta, \gamma)$ but not in $\mathcal{S}_A(\alpha, \beta, \gamma)$ must be solutions for which u_i satisfies Definition 6.1(i) for some $i \in \mathbb{Z}$. However, the only mosaic solutions arising from Definition 6.1(i) in Section 6.3 are the constant mosaic solutions which only arise for $\widetilde{M}_2$. Thus, all the other mosaic solutions identified in Section 6.3 are weakly asymptotically stable, and spatial entropy results for $\mathcal{S}_A(\alpha, \beta, \gamma)$ are identical to those for $\mathcal{S}_{L^*}(\alpha, \beta, \gamma)$, except every occurrence of $\widetilde{M}_2$ for $\mathcal{S}_{L^*}(\alpha, \beta, \gamma)$ is replaced by $\widetilde{M}_1$ for $\mathcal{S}_A(\alpha, \beta, \gamma)$.

Thus in particular, Figs 6 and 7 are unchanged for $\mathcal{S}_A(\alpha, \beta, \gamma)$, except all occurrences of $\widetilde{M}_2$ are replaced by $\widetilde{M}_1$, and $\widetilde{M}_2$ never arises, so that only nine different condensed transition matrices remain. Thus the region of parameter space on which there is no asymptotically stable mosaic solution is given by

$$\begin{gathered} 2\alpha\beta < \alpha\gamma < 0, \\ \alpha > 0, \quad \gamma > 0 \text{ and } 4\beta \leqslant \gamma, \end{gathered}$$

and

$$\alpha < 0, \quad \gamma < 0 \text{ and } 2\beta \geqslant \gamma.$$

Pattern formation occurs for

$$\begin{gathered} \alpha\gamma/2 < \alpha\beta < \alpha\gamma/3 < 0, \\ \alpha > 0, \text{ and } 0 < \gamma/4 < \beta < \gamma/3, \\ \alpha < 0, \text{ and } \beta < 0 < \gamma, \end{gathered}$$

and

$$\alpha < 0, \text{ and } 2\beta < \gamma < 0.$$

Finally, the spatially chaotic solutions and their parameter ranges for $\mathcal{S}_A(\alpha, \beta, \gamma)$ are the same as for $\mathcal{S}_{L^*}(\alpha, \beta, \gamma)$.

7. Numerical simulations

We illustrate the results from previous sections by means of numerical simulations of a discrete Cahn–Hilliard equation. The differential inclusion (1.5), (1.4) is difficult to simulate directly due to the set-valued nonlinearity and obstacles, so instead we simulate

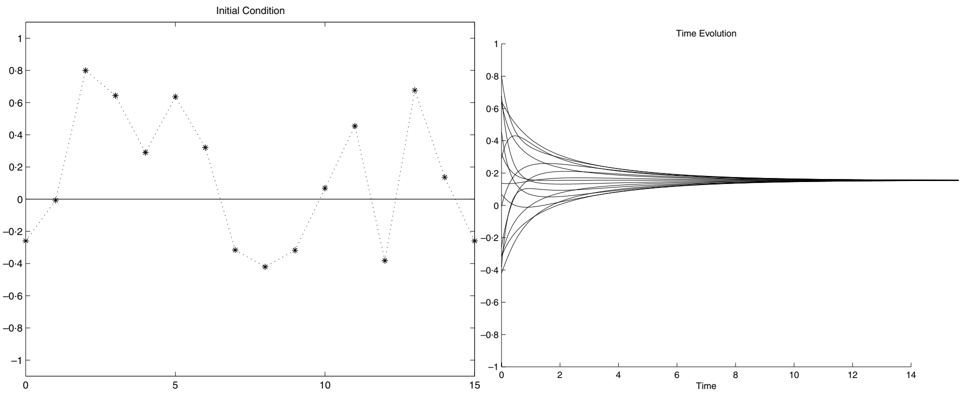

FIG. 1. Solution for $\widetilde{M}_2$ with random initial condition converging to a constant non-mosaic solution.

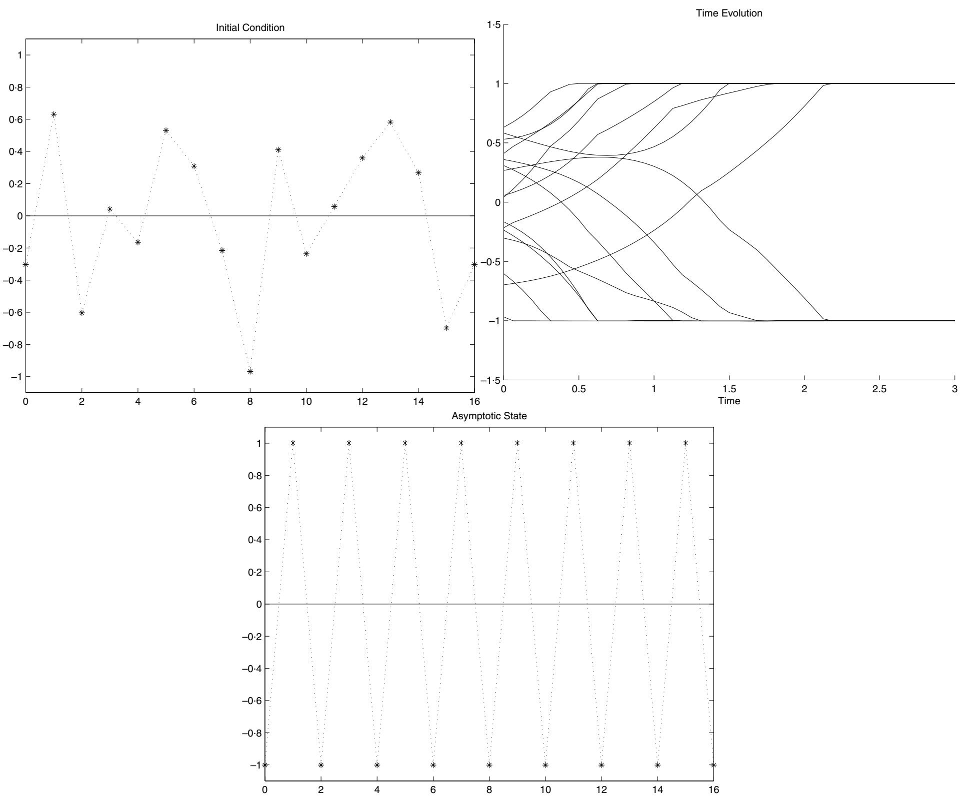

FIG. 2. Solution for $\widetilde{M}_5$ with initial condition such that $\sum_{i=1}^{N} u_i = 0$ converging to the alternating mosaic solution.

the ε-problem considered in Section 2 where we replace f by f^{ε} as defined by (2.12), and take small values of ε, typically $\varepsilon = 2{\cdot}5 \times 10^{-3}$ or $\varepsilon = 1{\cdot}25 \times 10^{-3}$. All computations

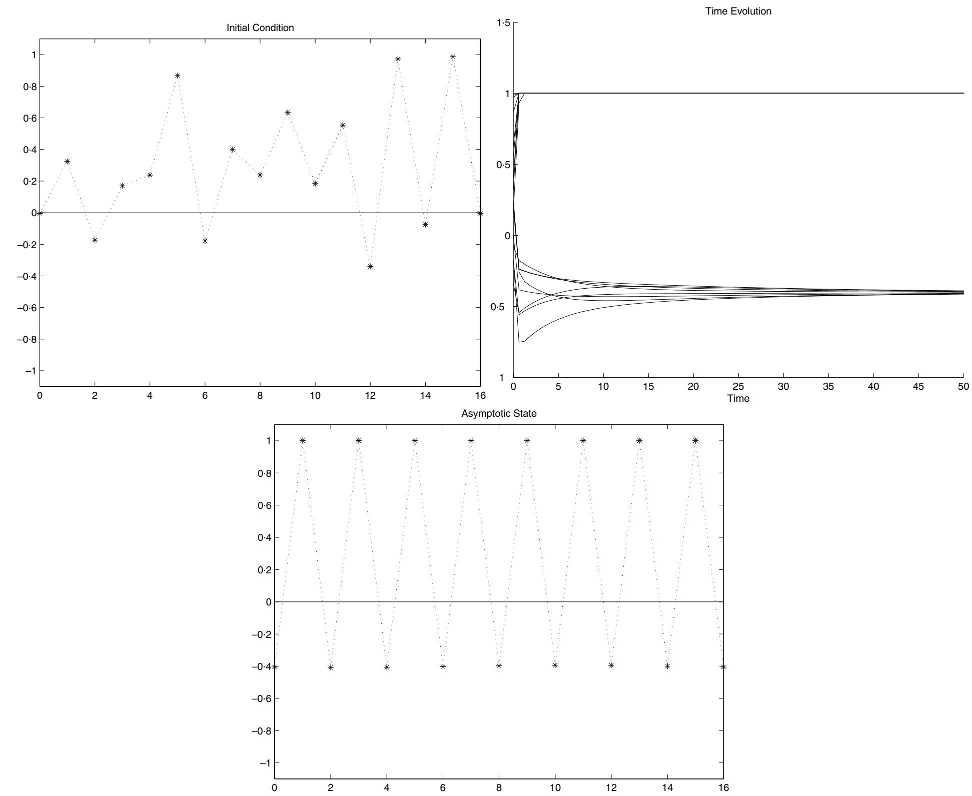

FIG. 3. Solution for $\widetilde{M}_5$ with random initial condition converging to a semi-mosaic alternating solution.

were performed on one-dimensional lattices $[-1, 1]^{\mathbb{Z}}$ with periodic boundary conditions $u_i = u_{i+N}$. Computations were performed with a second-order explicit Runge–Kutta method with a small step-size to avoid numerical instabilities—we acknowledge this not to be the most efficient method, but it is sufficient for our purposes.

In each of Figs 1–4, the first graph gives the initial condition $u(0)$ on a periodic window. Note here that $u_i(0)$ and indeed $u_i(t)$ are only defined at the integers $i \in \mathbb{Z}$; nevertheless on the graphs we join adjacent components u_i by a dotted line to give a continuous graph, as we find this makes the graphs more easily readable. The second graph in each figure gives the time evolution of all the components $u_i(t)$, and where it is non-trivial a third graph gives the asymptotic state $\lim_{t\to\infty} u_i(t)$.

In Fig. 1 we present a computation with random initial condition with $\alpha = \gamma = 1$ and $\beta = 0{\cdot}12$ corresponding to the condensed transition matrix $\widetilde{M}_2$ in Fig. 6. We note from Section 6.3 that the only Lyapunov stable mosaic solutions in this case are the constant mosaic solutions. However it follows easily from (1.5), (2.12) that $\sum_{i=1}^{N} \dot{u}_i = 0$ and hence the mass $m = \sum_{i=1}^{N} u_i(t)$ is conserved, so unless $m/N = 1$, 0, or -1 it is not possible for the solution to converge to a stable mosaic solution. In the example, $m/N = 0{\cdot}1549$

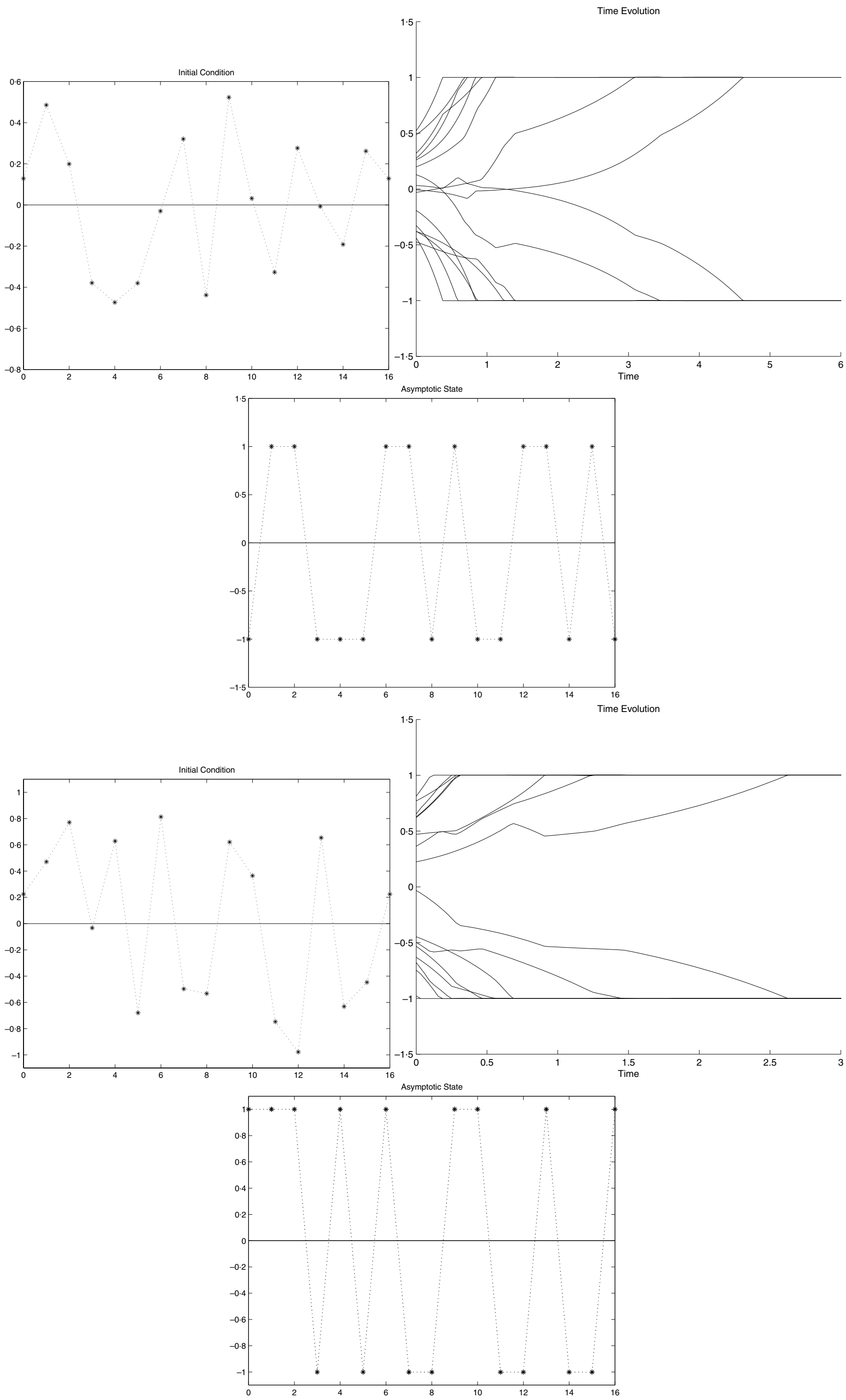

FIG. 4. Two examples for $\widetilde{M}_{10}$ with different initial conditions such that $\sum_{i=1}^{N} u_i = 0$ converging to the different mosaic solutions which are not simple periodic orbits.

and we see that the asymptotic state is the constant non-mosaic solution $u_i = m/N$ for all $i = 1, \ldots, N$ which we already identified as an equilibrium solution in Section 3.

Computations with parameters corresponding to the condensed transition matrix $\widetilde{M}_1$ in Fig. 6, for which there are no stable mosaic solutions, also reveal that solutions converge to the constant non-mosaic solution $u_i = m/N$ for all $i = 1, \ldots, N$.

In Fig. 2 we present a computation with $\alpha = \gamma = 1$ and $\beta = 0{\cdot}42$ corresponding to the condensed transition matrix $\widetilde{M}_5$ in Fig. 6. We take a random initial condition such that $m = \sum_{i=1}^{N} u_i(0) = 0$, which is achieved by first taking a truly random initial condition, then scaling and translating it to achieve $m = 0$ whilst still ensuring that $|u_i| \leqslant 1$. We note from Section 6.3 that this transition matrix has positive entropy and that spatial chaos can arise. Nevertheless in this case we see that the solution converges to the alternating mosaic solution $u_{i+1} = -u_i = \pm 1$. Similar results are seen with the condensed transition matrix $\widetilde{M}_3$ for which this is the only stable mosaic solution.

Initial conditions which are perturbations of the other simple spatially periodic solutions given in Table 7 for parameter values which correspond to an appropriate condensed transition matrix give solutions which converge to the given spatially periodic solution confirming their stability. However, note from Table 7 that for $\widetilde{M}_8$ and $\widetilde{M}_{10}$ both the alternating $\ldots -11 \ldots$ and double alternating $\ldots -1-111 \ldots$ solutions are stable mosaic solutions with mass $m = 0$; so specifying the mass of an initial condition is not enough alone to predict the asymptotic state.

Moreover, we see in Fig. 3 that not all initial conditions give solutions which converge to mosaic solutions. In this example we take the same parameter values corresponding to $\widetilde{M}_5$ as in Fig. 2, but now take a random initial condition with random initial mass. The resulting asymptotic state is a semi-mosaic solution of the form (3.8).

Finally in Fig. 4 we give a glimpse of spatial chaos by presenting computations in the region with the highest spatial entropy: $\widetilde{M}_{10}$. In both the computations random initial conditions corresponding to $m = 0$ are taken. The computations converge to different mosaic solutions, neither of which is a simple mosaic solution; note that both the so-called alternating and double alternating mosaic solutions have $m = 0$ and are stable for this parameter range.

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References

ABELL, K. A. 2000 Ph.D. Thesis, University of Sussex, In preparation.

BLOWEY, J. F. & ELLIOTT, C. M. 1991 The Cahn–Hilliard gradient theory for phase separation with non-smooth free energy, part I: Mathematical analysis. *Eur. J. Appl. Math.* **2**, 233–280.

CAHN, J. W., CHOW, S.-N., & VAN VLECK, E. S. 1995 Spatially discrete nonlinear diffusion equations. *Rocky Mountain J. Math.* **25**, 87–118.

CAHN, J. W. & HILLIARD, J. E. 1958 Free energy of a nonuniform system I. Interfacial free energy. *J. Chem. Phys.* **28**, 258–267.

CAHN, J. W., MALLET-PARET, J., & VAN VLECK, E. S. 1999 Travelling wave solutions for systems of ODEs on a two-dimensional spatial lattice. *SIAM J. Appl. Math.* **59**, 455–493.

CHOW, S.-N., MALLET-PARET, J., & VAN VLECK, E. S. 1996 Dynamics of lattice differential equations. *Int. J. Bifurcation and Chaos* **6**, 1605–1622.

CHOW, S.-N., MALLET-PARET, J., & VAN VLECK, E. S. 1996 Pattern formation and spatial chaos in spatially discrete evolution equations. *Random and Computational Dynamics* **4**, 109–178.

COOK, H. E., DE FONTAINE, D., & HILLIARD, J. E. 1969 A model for diffusion on cubic lattices and its application to the early stages of ordering. *Acta Met.* **17**, 765–773.

COPPEL, W. A. 1965 *Stability and Asymptotic Behavior of Differential Equations*. Boston: D. C. Heath.

ELLIOTT, C. M. 1989 The Cahn–Hilliard model for the kinetics of phase separation. *Mathematical Models for Phase Change Problems,*, Vol. 88, (J. F. Rogrigues ed). Basel: Birkhauser, pp. 97–128.

ELLIOTT, C. M. & FRENCH, D. A. 1987 Numerical studies of the Cahn–Hilliard equation for phase separation. *IMA J. Appl. Math.* **38**, 97–128.

ELLIOTT, C. M., GARDINER, A. R., KOSTIN, I., & LU, B. 1994 Mathematical and numerical analysis of a mean-field equation for the Ising model with Glauber dynamics. *Contemporary Math.* **172**, 217–241.

HILLERT, M. 1961 A solid-solution model for inhomogeneous systems. *Acta Met.* **9**, 525–535.

KLOEDEN, P. E. 1978 General control systems. *Mathematical Control Theory,* Lecture Notes in Mathematics*, 680*. (W. A. Coppel ed). Berlin: Springer, pp. 119–137.

KREYSZIG, E. 1978 *Introductory Functional Analysis with Applications*. New York: Wiley.

LAPLANTE, J. P. & ERNEUX, T. 1992 Propagation failure in arrays of coupled bistable chemical reactors. *J. Phys. Chem.* **96**, 4931–4934.

NOVICK, A. & SEGAL, L. A. 1984 Nonlinear aspects of the Cahn–Hilliard equation. *Physica* D **10**, 277–298.

OONO, Y. & PURI, S. 1988 Study of phase-separation dynamics by use of cell dynamical systems. I. modeling. *Phys. Rev.* A **38**, 434–453.

PAZY, A. 1983 *Semi-groups of Linear Operators and Applications to Partial Differential Equations*. Berlin: Springer.

ROBINSON, C. 1995 *Dynamical Systems*. Boca Raton: CRC Press.

SENECHAL, M. 1990 *Crystalline Symmetries An Informal Mathematical Introduction*. Bristol: Hilger.

Appendix

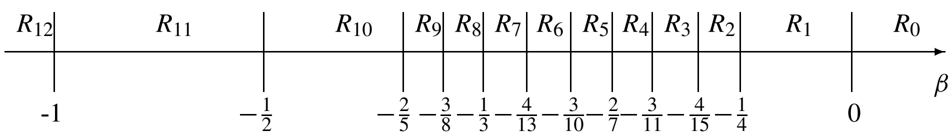

FIG. 5. Spatial entropy β-regions for $\gamma < 0$.

$\underline{\alpha > 0}$

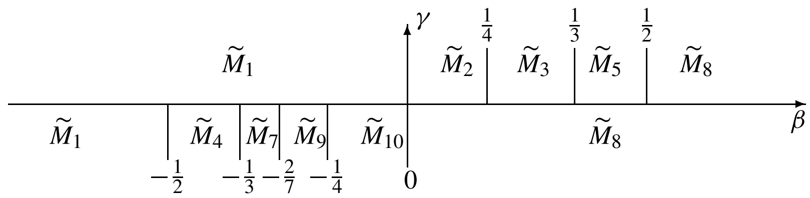

FIG. 6. Parameter value regions for condensed transition matrices with $\alpha > 0$ and $\gamma = \pm 1$.

$\underline{\alpha < 0}$

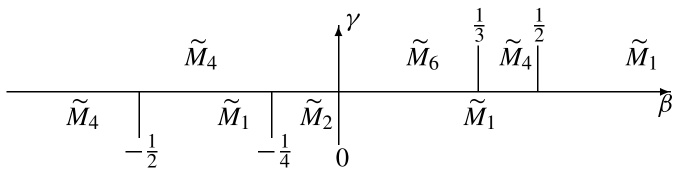

FIG. 7. Parameter value regions for condensed transition matrices with $\alpha < 0$ and $\gamma = \pm 1$.

TABLE 2

Four-tuples which can appear in $\mathcal{S}_{L^*}(\alpha, \beta, \gamma)$.

Label	Four-tuple			
1	(−1	−1	−1	−1)
2	(−1	−1	−1	1)
3	(−1	−1	1	−1)
4	(−1	−1	1	1)
5	(−1	1	−1	−1)
6	(−1	1	−1	1)
7	(−1	1	1	−1)
8	(−1	1	1	1)
9	(0	0	0	0)
10	(1	−1	−1	−1)
11	(1	−1	−1	1)
12	(1	−1	1	−1)
13	(1	−1	1	1)
14	(1	1	−1	−1)
15	(1	1	−1	1)
16	(1	1	1	−1)
17	(1	1	1	1)

TABLE 3
General form of the transition matrix

$$
\begin{pmatrix}
M_{1,1} & M_{1,2} & & & & & & & & & & & & & & & \\
0 & 0 & M_{2,3} & M_{2,4} & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & M_{3,5} & M_{3,6} & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & M_{4,7} & M_{4,8} & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{5,10} & M_{5,11} & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{6,12} & M_{6,13} & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{7,14} & M_{7,15} & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{8,16} & M_{8,17} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{9,9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
M_{10,1} & M_{10,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & M_{11,3} & M_{11,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & M_{12,5} & M_{12,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & M_{13,7} & M_{13,8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & M_{14,10} & M_{14,11} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & M_{15,12} & M_{15,13} & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & M_{16,14} & M_{16,15} & 0 & 0 \\
 & & & & & & & & & & & & & & & M_{17,16} & M_{17,17}
\end{pmatrix}
$$

TABLE 4
Spatial entropy and number of mosaic solutions in a subset of $\mathbb{Z}$ for each condensed transition matrix

Case	Entropy	$\Gamma_5(S_{L^*})$	$\Gamma_{10}(S_{L^*})$	$\Gamma_{50}(S_{L^*})$	$\Gamma_{100}(S_{L^*})$	$\Gamma_{1000}(S_{L^*})$
$\widetilde{M}_1$	ln 0	0	00	0	0	0
$\widetilde{M}_2$	$\ln 1 = 0$	3	03	3	3	3
$\widetilde{M}_3$	$\ln 1 = 0$	2	02	2	2	2
$\widetilde{M}_4$	$\ln 1 = 0$	4	04	4	4	4
$\widetilde{M}_5$	ln 1·4656	12	82	$3{\cdot}5791 \times 10^{8}$	$7{\cdot}1470 \times 10^{16}$	$1{\cdot}8212 \times 10^{166}$
$\widetilde{M}_6$	ln 1·4656	14	94	$4{\cdot}1084 \times 10^{8}$	$8{\cdot}2040 \times 10^{16}$	$2{\cdot}0906 \times 10^{166}$
$\widetilde{M}_7$	ln 1·5552	20	180	$8{\cdot}4872 \times 10^{9}$	$3{\cdot}2953 \times 10^{19}$	$1{\cdot}3253 \times 10^{192}$
$\widetilde{M}_8$	ln 1·6180	16	176	$4{\cdot}0730 \times 10^{10}$	$1{\cdot}1463 \times 10^{21}$	$1{\cdot}4066 \times 10^{209}$
$\widetilde{M}_9$	ln 1·6180	24	256	$5{\cdot}8945 \times 10^{10}$	$1{\cdot}6589 \times 10^{21}$	$2{\cdot}0357 \times 10^{209}$
$\widetilde{M}_{10}$	ln 1·9276	30	806	$2{\cdot}0162 \times 10^{14}$	$3{\cdot}5888 \times 10^{28}$	$1{\cdot}1547 \times 10^{285}$

TABLE 5
Condensed transition matrices corresponding to spatial chaos

$$
\widetilde{M}_5 = \left(\begin{array}{*{17}{c}}
0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & 1 & 1 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & & & 0 & 0
\end{array}\right)
\qquad
\widetilde{M}_6 = \left(\begin{array}{*{17}{c}}
0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & 1 & 0 & 0 & 0 \\
& & & & & & & & & & & & & & & 1 & 0
\end{array}\right)
$$

$$
\widetilde{M}_7 = \left(\begin{array}{*{17}{c}}
0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & 0 & 1 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & 1 & 0 & 0 & 0 \\
& & & & & & & & & & & & & & & 1 & 0
\end{array}\right)
\qquad
\widetilde{M}_8 = \left(\begin{array}{*{17}{c}}
0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & 1 & 1 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & & & 0 & 0
\end{array}\right)
$$

$$
\widetilde{M}_9 \left(\begin{array}{*{17}{c}}
0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & 1 & 1 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & 1 & 0 & 0 & 0 \\
& & & & & & & & & & & & & & & 1 & 0
\end{array}\right)
\qquad
\widetilde{M}_{10} \left(\begin{array}{*{17}{c}}
0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & 1 & 1 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & 1 & 1 & 0 & 0 \\
& & & & & & & & & & & & & & & 1 & 0
\end{array}\right)
$$

TABLE 6
Admissible five-tuples for each condensed transition matrix

Five-tuple					$\widetilde{M}_1$	$\widetilde{M}_2$	$\widetilde{M}_3$	$\widetilde{M}_4$	$\widetilde{M}_5$	$\widetilde{M}_6$	$\widetilde{M}_7$	$\widetilde{M}_8$	$\widetilde{M}_9$	$\widetilde{M}_{10}$
$(-1$	-1	-1	-1	$-1)$		•								
$(-1$	-1	-1	-1	$1)$						•	•		•	•
$(-1$	-1	-1	1	$-1)$										•
$(-1$	-1	-1	1	$1)$						•	•		•	•
$(-1$	-1	1	-1	$-1)$					•		•	•	•	•
$(-1$	-1	1	-1	$1)$					•			•	•	•
$(-1$	-1	1	1	$-1)$				•		•	•	•	•	•
$(-1$	-1	1	1	$1)$						•	•		•	•
$(-1$	1	-1	-1	$-1)$										•
$(-1$	1	-1	-1	$1)$					•		•	•	•	•
$(-1$	1	-1	1	$-1)$			•		•			•		•
$(-1$	1	-1	1	$1)$					•			•	•	•
$(-1$	1	1	-1	$-1)$				•		•	•	•	•	•
$(-1$	1	1	-1	$1)$					•		•	•	•	•
$(-1$	1	1	1	$-1)$						•	•		•	•
$(-1$	1	1	1	$1)$						•	•		•	•
$(0$	0	0	0	$0)$		•								
$(1$	-1	-1	-1	$-1)$						•	•		•	•
$(1$	-1	-1	-1	$1)$						•	•		•	•
$(1$	-1	-1	1	$-1)$					•		•	•	•	•
$(1$	-1	-1	1	$1)$				•		•	•	•	•	•
$(1$	-1	1	-1	$-1)$					•			•	•	•
$(1$	-1	1	-1	$1)$			•		•			•		•
$(1$	-1	1	1	$-1)$					•		•	•	•	•
$(1$	-1	1	1	$1)$										•
$(1$	1	-1	-1	$-1)$						•	•		•	•
$(1$	1	-1	-1	$1)$				•		•	•	•	•	•
$(1$	1	-1	1	$-1)$					•			•	•	•
$(1$	1	-1	1	$1)$					•		•	•	•	•
$(1$	1	1	-1	$-1)$						•	•		•	•
$(1$	1	1	-1	$1)$										•
$(1$	1	1	1	$-1)$						•	•		•	•
$(1$	1	1	1	$1)$		•								

TABLE 7
Simple periodic solutions arising for the different transition matrices

Mosaic Solution	$\widetilde{M}_1$	$\widetilde{M}_2$	$\widetilde{M}_3$	$\widetilde{M}_4$	$\widetilde{M}_5$	$\widetilde{M}_6$	$\widetilde{M}_7$	$\widetilde{M}_8$	$\widetilde{M}_9$	$\widetilde{M}_{10}$
$\ldots -1 \ldots$		•								
$\ldots 0 \ldots$		•								
$\ldots 1 \ldots$		•								
$\ldots -11 \ldots$			•		•			•		•
$\ldots -1-11 \ldots$					•		•	•	•	•
$\ldots -111 \ldots$					•		•	•	•	•
$\ldots -1-1-11 \ldots$										•
$\ldots -1-111 \ldots$				•		•	•	•	•	•
$\ldots -1111 \ldots$										•
$\ldots -1-1-1-11 \ldots$										•
$\ldots -1-1-111 \ldots$						•	•		•	•
$\ldots -1-1111 \ldots$						•	•		•	•
$\ldots -11111 \ldots$										•
$\ldots -1-11-11 \ldots$					•			•		•
$\ldots -11-111 \ldots$					•			•		•
$\ldots -1-1-1111 \ldots$						•	•		•	•
$\ldots -1-1-1-11111 \ldots$						•	•		•	•