REGULARITY OF THE GEODESIC EQUATION IN THE SPACE OF SASAKIAN METRICS

PENGFEI GUAN AND XI ZHANG

Abstract. In this paper, we study a geodesic equation in the space of Sasakian metrics $\mathcal{H}$. The equation leads to the Dirichlet problem of a complex Monge-Ampère type equation on the Kähler cone. This equation differs from the standard complex Monge-Ampère equation in a significant way, with gradient terms involved in the $(1,1)$ symmetric tensor of the operator. We establish appropriate regularity estimates for this complex Monge-Ampère type equation. As geometric application, we show that the space of Sasakian metrics $\mathcal{H}$ is a metric space, and the constant transversal scalar curvature metric realizes the global minimum of $K$-energy if the first basic Chern class $c_1^{3H} \leq 0$. We also prove that the constant transversal scalar curvature metric is unique in each basic Kähler class if the first basic Chern class is either strictly negative or zero.

AMS subjects: 35J96, 53C25.

1. Introduction

There has been renewed interest in Sasakian manifolds recently. Sasakian geometry provides a wealthy source for constructing new Einstein manifolds [2] and it also plays important role in superstring theory [29, 30]. Sasakian-Einstein manifolds are naturally connected to Kähler-Einstein manifolds. For example, the anti-canonical line bundle $K^{-1}$ over a Kähler-Einstein manifold admits a natural Sasaki-Einstein metric.

This paper is devoted to a complex Monge-Ampère type equation arising from Sasakian geometry. The equation is related to a geodesic equation in the space of Sasakian metrics $\mathcal{H}$ which is defined in (1.2). This infinite dimensional space $\mathcal{H}$ contains rich geometric information of the underlying manifold. The geodesic equation is a natural way to understand $\mathcal{H}$ and Sasakian geometry in general. This geodesic approach is modeled in the Kähler case, where the equation was considered by Mabuchi, Semmes and Donaldson in [28, 36, 13]. Donaldson proposed a far reaching program [13, 14] relating the geometry of the space of Kähler metrics to the existence and uniqueness of constant scalar curvature Kähler metrics. Extensive studies have been carried out in recent years, we refer [6, 5, 32, 33, 9, 7] and references listed there. The $C^2_w$ (see definition 1) regularity proved by Chen [6] for the geodesic equation in the space of Kähler metrics has significant geometric consequences in Kähler geometry, and recent work on geodesic rays by Phong-Strum [32] provides the further evidence of the importance of the geodesic equation.

The first author was supported in part by an NSERC Discovery grant, the second author was supported in part by NSF in China, No.11071212, No.11131007 and No.10831008.
Let us switch attention to the Sasakian case. A Sasakian manifold \((M, g)\) is a 
\((2n+1)\)-dimensional Riemannian manifold with the property that the cone manifold 
\((C(M), \bar{g}) := (M \times \mathbb{R}^+, r^2g + dr^2)\)
is Kähler. The Sasakian manifolds can be considered as odd dimensional counter-
part of Kähler manifolds. General properties of Sasakian manifolds can be found in 
the book [1] by Boyer-Galicki. A Sasakian structure on \(M\) consists of a Reeb field 
\(\xi\) of unit length on \(M\), a \((1,1)\) type tensor field \(\Phi(X) = \nabla_X \xi\) and a contact 1-form 
\(\eta\) (which is the dual 1-form of \(\xi\) with respect to \(g\)). We denote such a Sasakian 
structure by \((\xi, \eta, \Phi, g)\). \(\Phi\) defines a complex structure on the contact sub-bundle 
\(\mathcal{D} = \text{ker}\{\eta\}\). \((\mathcal{D}, \Phi|_\mathcal{D}, d\eta)\) provides \(M\) a transverse Kähler structure with Kähler 
form \(\frac{1}{2}d\eta\) and metric \(g^\omega\) defined by \(g^\omega (\cdot, \cdot) = \frac{1}{2}d\eta(\cdot, \Phi\cdot)\). The complexification \(\mathcal{D}^C\) 
of the sub-bundle \(\mathcal{D}\) can be decomposed it into its eigenspaces with respect to \(\Phi|_\mathcal{D}\) as 
\(\mathcal{D}^C = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}\).

A \(p\)-form \(\theta\) on Sasakian manifold \((M, g)\) is called \textit{basic} if \(i_\xi \theta = 0\), \(L_\xi \theta = 0\), 
where \(i_\xi\) is the contraction with the Reeb field \(\xi\) and \(L_\xi\) is the Lie derivative with 
respect to \(\xi\). The exterior differential preserves \textit{basic} forms. There is a natural 
splitting of the complexification of the bundle of the sheaf of germs of \textit{basic} \(p\)-forms 
\(\Lambda^p_B(M)\) on \(M\), 
\[\Lambda^p_B(M) \otimes C = \oplus_{i+j=p} \Lambda^{i,j}_B(M),\]
where \(\Lambda^{i,j}_B(M)\) denotes the bundle of basic forms of type \((i,j)\). Accordingly, \(\partial_B\) 
and \(\bar{\partial}_B\) can be defined. Set 
\(d_B^c = \frac{1}{2\sqrt{-1}}(\bar{\partial}_B - \partial_B)\) and \(d_B = d|_{\Lambda^p_B}\). The following 
relations hold 
\[d_B = \bar{\partial}_B + \partial_B, \quad d_B d_B^c = \sqrt{-1} \partial_B \bar{\partial}_B, \quad d_B^2 = (d_B^c)^2 = 0.\]

Denote the space of all smooth \textit{basic} real functions on \(M\) by \(C^\infty_B(M)\). Set 
\[(1.2) \quad \mathcal{H} = \{ \varphi \in C^\infty_B(M) : \eta_\varphi \wedge (d\eta_\varphi)^n \neq 0 \},\]
where 
\[(1.3) \quad \eta_\varphi = \eta + d_B^c \varphi, \quad d\eta_\varphi = d\eta + \sqrt{-1} \partial_B \bar{\partial}_B \varphi.\]

The space \(\mathcal{H}\) is contractible. Given a Sasakian structure \((\xi, \eta, \Phi, g)\), for \(\varphi \in \mathcal{H}\), set 
\[(1.4) \quad \Phi_\varphi = \Phi - \xi \otimes (d_B^c \varphi) \circ \Phi, \quad g_\varphi = \frac{1}{2} d\eta_\varphi \circ (Id \otimes \Phi_\varphi) + \eta_\varphi \otimes \eta_\varphi.\]
\((\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)\) is also a Sasakian structure on \(M\). \((\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)\) and 
\((\xi, \eta, \Phi, g)\) have the same transversely holomorphic structure on \(\nu(\mathcal{F}_\xi)\) and the same holomorphic 
structure on the cone \(C(M)\) (Proposition 4.2 in [18], also [1]). Conversely, if 
\((\xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g})\) is another Sasakian structure with the same Reeb field \(\xi\) and the same 
transversely holomorphic structure on \(\nu(\mathcal{F}_\xi)\), then \([d\tilde{\eta}]_B\) and \([d\tilde{g}]_B\) belong to the 
same cohomology class in \(H^1_B(M)\). There exists a unique basic function (e.g., 
[16]), \(\tilde{\varphi} \in \mathcal{H}\) up to a constant such that 
\[(1.5) \quad d\tilde{\eta} = d\eta + \sqrt{-1} \partial_B \bar{\partial}_B \tilde{\varphi}.\]

If \((\xi, \eta, \Phi, g)\) and \((\xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g})\) induce the same holomorphic structure on the cone 
\(C(M)\), then there must exist a unique function \(\varphi \in \mathcal{H}\) up to a constant such that 
\(\tilde{\eta} = \eta_\varphi, \tilde{\Phi} = \Phi_\varphi\) and \(\tilde{g} = g_\varphi\). \(\mathcal{H}\) will be called the space of Sasakian metrics. 
\(\forall \varphi \in \mathcal{H}, (\xi, \eta_\varphi, \Phi_\varphi, g_\varphi)\) defined in (1.3) and (1.4) is a Sasakian structure on \(M\),
and it has the same transversely holomorphic structure on $\nu(\mathcal{F}_\xi)$ and the same holomorphic structure on the cone $C(M)$ as $(\xi, \eta, \Phi, g)$. Their transverse Kähler forms are in the same basic $(1, 1)$ class $[d\eta]_B$ (Proposition 4.2 in [18]). This class is called the basic Kähler class of the Sasakian manifold $(M, \xi, \eta, \Phi, g)$.

In [24], we introduced a geodesic equation in $H$. The measure $d\mu = \eta \phi \wedge (d\eta \phi)^n$ in $H$ induces a Weil-Peterson metric in the space $H$ defined as

$$\langle \psi_1, \psi_2 \rangle_\phi = \int_M \psi_1 \cdot \psi_2 d\mu, \quad \forall \psi_1, \psi_2 \in \mathcal{T}H, \quad (1.6)$$

Since the tangent space $T\mathcal{H}$ can be identified as $\mathcal{C}^\infty(M)$, the corresponding geodesic equation can be expressed as

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{4} |d_B \frac{\partial \phi}{\partial t}|^2_{g_\phi} = 0,$$

where $g_\phi$ is the Sasakian metric determined by $\phi$. A natural connection of the metric can be deduced from the geodesic equation. In [24], we proved that this natural connection is torsion free and compatible with the metric, there is a splitting $\mathcal{H} \cong \mathcal{H}_0 \times \mathbb{R}$, $\mathcal{H}_0$ (defined in (6.5)) is totally geodesic and totally convex, the corresponding sectional curvature of $\mathcal{H}$ is non-positive.

The objective of this paper is to address the following fundamental question raised in [24]: can any two points in $\mathcal{H}$ be connected by a geodesic path?

The geodesic problem can be formulated as the following Dirichlet problem,

$$\left\{ \begin{array}{l}
\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{4} |d_B \frac{\partial \phi}{\partial t}|^2_{g_\phi} = 0, \quad M \times (0, 1) \\
\phi|_{t=0} = \phi_0 \\
\phi|_{t=1} = \phi_1
\end{array} \right. \quad (1.7)$$

It was discussed in [24], when $n = 1$, equation (1.7) is equivalent to the corresponding geodesic equation introduced by Donaldson [15] for the space of volume forms on Riemannian manifold with fixed volume. Recent work of Chen and He [8] implies the existence of a $C^2_\text{w}$ (see Definition 1) geodesic. In higher dimensions, there is no such simple connection. We establish the existence and regularity of solutions to geodesic equation (1.7) in any dimension in this paper.

Our first step is to reduce the geodesic equation (1.7) on $\mathcal{H}$ to the Dirichlet problem of complex Monge-Ampère type equation on the Kähler cone $C(M) = M \times \mathbb{R}^+$. Let $\tilde{\phi}_t : M \times [0, 1] \to \mathbb{R}$ be a path in the metric space $\mathcal{H}$. Define a function $\psi$ on $M \times [1, \frac{3}{2}] \subset C(M)$ by converting the time variable $t$ to the radial variable $r$ as follow,

$$\psi(\cdot, r) = \tilde{\phi}_{2(r-1)}(\cdot) + 4 \log r, \quad (1.8)$$

Set a $(1, 1)$ form on $M \times [1, \frac{3}{2}]$ by

$$\Omega_\phi = \tilde{\omega} + \frac{r^2}{2} \sqrt{-1}(\partial \bar{\partial} \psi - \frac{\partial \tilde{\phi}}{\partial r} \partial \bar{\partial} r), \quad (1.9)$$

where $\tilde{\omega}$ is the fundamental form of the Kähler metric $\tilde{g}$.

The key observation (Proposition 1 in section 2) is that the Dirichlet problem (1.7) is equivalent to the following Dirichlet problem of a degenerate Monge-Ampère
type equation
\[
\begin{cases}
(\Omega_\psi)^{n+1} = 0, & M \times (1, \frac{3}{2}), \\
\psi|_{r=1} = \psi_1, \\
\psi|_{r=\frac{3}{2}} = \psi_{\frac{3}{2}}.
\end{cases}
\]
(1.10)

Equation (1.10) is a degenerate elliptic complex Monge-Ampère type equation and it differs from the standard complex Monge-Ampère equation in [39] on Kähler manifolds in a significant way. The study of the Dirichlet problem for standard homogeneous complex Monge-Ampère equation was initiated by Chern-Levine-Nirenberg in [11] in connection to holomorphic norms. The regularity of the Dirichlet problem of the complex Monge-Ampère equation for strongly pseudoconvex domains in $\mathbb{C}^n$ was proved by Caffarelli-Kohn-Nirenberg-Spruck in [4]. In general, $C^{1,1}$ regularity is optimal for degenerate complex Monge-Ampère equations (e.g., [20, 6, 22]). We note that $\Omega_\psi$ in equation (1.10) involves the first order derivative term, which is much more complicated than the standard one. A similar type complex Monge-Ampère equation also appeared in Fu-Yau’s recent work in superstring theory in [17]. We believe the analysis developed in this paper for equation (1.10) will be useful for treating general type of complex Monge-Ampère equations.

In order to solve the degenerate equation (1.10), consider the following perturbation equation
\[
\begin{cases}
(\Omega_\psi)^{n+1} = \epsilon f \bar{\omega}^{n+1}, & M \times (1, \frac{3}{2}), \\
\psi|_{r=1} = \psi_1, \\
\psi|_{r=\frac{3}{2}} = \psi_{\frac{3}{2}}.
\end{cases}
\]
(1.11)

where $0 < \epsilon \leq 1$ and $f$ is a positive basic function. Also, consider the following approximation for (1.7)
\[
\begin{cases}
(\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{4}|d_B \frac{\partial}{\partial \nu}|^2_{\omega_\nu})\eta_\phi \wedge (d\eta_\phi)^n = \epsilon \eta \wedge (d\eta)^n, & M \times (0, 1) \\
\phi|_{t=0} = \phi_0, \\
\phi|_{t=1} = \phi_1.
\end{cases}
\]
(1.12)

In fact, the above equations (1.11) and (1.12) are equivalent (see Proposition 1). Let us introduce some necessary notation.

**Definition 1.** Let $C^2_w(M \times [1, \frac{3}{2}])$ be the closure of smooth functions on $M \times [1, \frac{3}{2}]$ under the norm
\[
\|\psi\|_{C^2_w(M \times [1, \frac{3}{2}])} = \|\psi\|_{C^1(M \times [1, \frac{3}{2}])} + \sup_{M \times [1, \frac{3}{2}]} |\Delta \psi|.
\]

$\psi$ is called a $C^2_w$ solution of equation (1.10) if $\psi \in C^2_w(M \times [1, \frac{3}{2}])$ such that $\Omega_\psi \geq 0, \Omega_\psi^{n+1} = 0, \text{a.e.}$ Define
\[
\mathcal{H} = \{\text{completion of } \mathcal{H} \text{ under the norm } \|\|_{C^2_w}\}.
\]

For any two points $\varphi_0, \varphi_1 \in \mathcal{H}$, $\varphi$ is called a $C^2_w$ geodesic segment connecting $\varphi_0, \varphi_1$, if $\psi$ defined in (1.8) is a $C^2_w$ solution of equation (1.10). That is, there is a $C^2_w$ geodesic path in $\mathcal{H}$ connecting them.

The main result of this paper is the following regularity estimates.

**Theorem 1.** Fix a Sasakian structure $(\xi, \eta, \Phi, g)$ on a compact Sasakian manifold $M$. For any positive basic smooth function $f$ and for any given smooth boundary data in $\mathcal{H}$, there is a unique smooth solution $\psi$ to the equation (1.11). Moreover, $\psi$ is
basic and there is a constant $C > 0$ depending only on $(\xi, \eta, \Phi, g)$, $\|f^{\frac{1}{2n}}\|_{C^2(M \times [1, 3])}$, $\|\psi_1\|_{C^2}$ and $\|\psi_2\|_{C^2}$, such that

$$\|\psi\|_{C^2(M \times [1, 3])} \leq C.$$ 

For any two functions $\varphi_0, \varphi_1 \in \mathcal{H}$, there exists a unique $C^2_w$ geodesic path in $\mathcal{H}$ connecting them. And this path is a $C^2_w$ weak limit of solutions of $\varphi_\varepsilon$ of equation (1.12) such that $\Omega_{\varphi_\varepsilon + 4\log r}$ is positive and bounded.

There are some immediate geometric applications of Theorem 1. A direct consequence of it is that the infinite dimensional space $(\mathcal{H}, d)$ is a metric space.

**Definition 2.** For any $\varphi_0, \varphi_1 \in \mathcal{H}$, let $\varphi_t : [0, 1] \to \mathcal{H}$ be the unique $C^2_w$ geodesic connecting these two points (guaranteed by Theorem 1). Define the geodesic distance between $\varphi_0$ and $\varphi_1$ as

$$(1.15) \quad d(\varphi_0, \varphi_1) = \int_0^1 dt \sqrt{\int_M |\dot{\varphi}_t|^2 \eta_{\varphi_t} \land (d\eta_{\varphi_t})^n}.$$ 

**Theorem 2.** Let $C = \varphi(s), s \in [0, 1]$ be a smooth path in $\mathcal{H}$, and $\varphi^* \in \mathcal{H}$ be a point. Then, for any $s$,

$$d(\varphi^*, \varphi(0)) \leq d(\varphi^*, \varphi(s)) + d_c(\varphi(0), \varphi(s)),$$

where $d_c$ denotes the length along the curve $C$. In particular, the following triangle inequality is true

$$d(\varphi^*, \varphi(0)) \leq d(\varphi^*, \varphi(1)) + d_c(\varphi(0), \varphi(1)).$$

The space $(\mathcal{H}, d)$ is a metric space.

As in the Kähler case, one may define $K$-energy map $\mu : \mathcal{H} \to \mathbb{R}$ (we defer the precise definition of $\mu$ in Section 6). Theorem 1 implies $\mu$ is convex in $\mathcal{H}$.

**Theorem 3.** Let $(M, \xi, \eta, \Phi, g)$ be a compact Sasakian manifold with $C^1_B(M) \leq 0$. Then a constant scalar curvature transverse Kähler metric, if it exists, realizes the global minimum of the $K$ energy functional $\mu$ in each basic Kähler class. In addition, if either $C^1_B(M) = 0$ or $C^1_B(M) < 0$, then the constant scalar curvature transverse Kähler metric, if it exists, in any basic Kähler class must be unique.

There are several recent development on Sasakian geometry. The study of toric Sasaki-Einstein metrics was carried out by Futaki-Ono-Wang [18]. The role of geodesic equation for the uniqueness of constant transversely scalar curvature metric on toric Sasakian manifolds was discussed in [12]. We would also like to call attention to recent papers [31, 35, 37] on the uniqueness of Sasakian-Einstein metrics and Sasaki-Ricci flow.

The organization of the paper is as follow: we derive the complex Monge-Ampère type equation on Kähler cone in the next section; and sections 3-5 are devoted to the a priori estimates of the equation, they are the core of this paper; the regularity of the geodesics will be used to prove $\mathcal{H}$ is a metric space in section 6, along with other geometric applications there. The proofs of the technical lemmas in section 6 are given in the appendix.
2. A Complex Monge-Ampère Type Equation on Kähler Cone

This section is devoted to converting geodesic equation (1.7) in $\mathcal{H}$ to the Dirichlet problem of complex Monge-Ampère type equation (1.10) on the Kähler cone. Let $C(M) = M \times \mathbb{R}^+$, $\bar{g} = dr^2 + r^2 g$, and $(\xi, \eta, \Phi, g)$ be a Sasakian structure on the manifold $M$. The almost complex structure on $C(M)$ defined by

\begin{equation}
J(Y) = \Phi(Y) - \eta(Y)\frac{\partial}{\partial r}, \quad J(r\frac{\partial}{\partial r}) = \xi,
\end{equation}

makes $(C(M), \bar{g}, J)$ a Kähler manifold. In what follows, the pull back forms $p^*\eta$ and $p^*(dh)$ will be also denoted by $\eta$ and $dh$ if there is no confusion, where $p : C(M) \to M$ is the projective map. The following lemma can be found in [1] and [18].

**Lemma 1.** The fundamental form $\bar{\omega}$ of the Kähler cone $(C(M), \bar{g})$ can be expressed by

\begin{equation}
\bar{\omega} = \frac{1}{2}r^2d\eta + r dr \wedge \eta = \frac{1}{2}d(r^2\eta) = \frac{1}{2}dd^c r^2.
\end{equation}

As in the Kähler case, the Sasakian metric can locally be generated by a free real function of $2n$ variables [19]. This function is a Sasakian analogue of the Kähler potential for the Kähler geometry. More precisely, for any point $q$ in $M$, there is a local basic function $h$ and a local coordinate chart $(x, z^1, z^2, \cdots, z^n) \in \mathbb{R} \times \mathbb{C}^n$ on a small neighborhood $U$ around $q$, such that

\[ \eta = dx - \sqrt{-1}(h_1 dz^1 - h_j dz^j), \quad g = \eta \otimes \eta + 2h_{ij} dz^i d\bar{z}^j, \]

where $h_i = \frac{\partial h}{\partial x_i}$, $h_{ij} = \frac{\partial^2 h}{\partial x_i \partial x_j}$. Through a local change of coordinates, one may further assume that

\[ h_i(q) = 0, \quad h_{ij}(q) = \delta_{ij}, \quad d(h_{ij})|_q = 0. \]

This type local coordinate $(x, z^1, \cdots, z^n)$ is called a normal coordinate on Sasakian manifold. This can be achieved by setting

\[ y = x - \sqrt{-1}(h_i(q) z^i - h_j(q) \bar{z}^j), \quad u^k = z^k, \quad \forall k = 1, \cdots, n; \]
\[ h^* = h - h_i(q) u^i - h_j(q) \bar{u}^j. \]

For a normal local coordinate chart $(x, z^1, z^2, \cdots, z^n)$, set

\begin{equation}
(z^1, z^2, \cdots, z^n, w), \quad \text{on } U \times \mathbb{R}^+ \subset C(M), \quad \text{where } w = r + \sqrt{-1}x.
\end{equation}

It should be pointed out that $(z^1, z^2, \cdots, z^n, w)$ is not a holomorphic local coordinate of the complex manifold $C(M)$. Set

\begin{equation}
\begin{cases}
X_j = \frac{\partial}{\partial z^j} + \sqrt{-1}h_j\frac{\partial}{\partial x}, \\
X_{n+1} = \frac{1}{2}\left(\frac{\partial}{\partial r} - \sqrt{-1}\frac{\partial}{\partial x}\right), \\
\theta^i = dz^i, \quad \theta^{n+1} = dr + \sqrt{-1}r\eta.
\end{cases}
\end{equation}

In this local coordinate chart, $\mathcal{D} \otimes \mathbb{C}$ is spanned by $X_i$ and $\bar{X}_i$, $i = 1, \cdots, n$, and

\begin{equation}
\begin{cases}
\xi = \frac{\partial}{\partial r}; \\
\eta = dx - \sqrt{-1}(h_i dz^i - h_j d\bar{z}^j); \\
\Phi = \sqrt{-1}(X_j \otimes dz^j - X_j \otimes d\bar{z}^j); \\
g = \eta \otimes \eta + 2h_{ij} dz^i d\bar{z}^j.
\end{cases}
\end{equation}
\[ \Phi X_i = \sqrt{-1}X_i, \quad \Phi X_i = -\sqrt{-1}X_i, \]

\[ [X_i, X_j] = [X_i, X_j] = [\xi, X_i] = [\xi, X_i] = 0, \]

\[ [X_i, X_j] = -2\sqrt{-1}h_{ij}\xi. \]

\( \{\eta, dz^i, dz^j\} \) is the dual basis of \( \{\frac{\partial}{\partial z^i}, X_i, X_j\} \), and

\[ g^T = 2g_{ij}^T dz^i dz^j = 2h_{ij} dz^i dz^j, \quad d\eta = 2\sqrt{-1}h_{ij} dz^i \wedge dz^j. \]

**Proposition 1.** The path \( \varphi_t \) connects \( \varphi_0, \varphi_1 \in \mathcal{H} \) and satisfies equation (1.12) for some \( \varepsilon \geq 0 \) if and only if \( \psi \) satisfies equation (1.11), where \( f = r^2, \psi \) and \( \Omega_\psi \) defined as in (1.9), \( \Omega_\psi|_{D^c} \) is positive and \( \psi|_{M \times \{1\}} = \varphi_0, \psi|_{M \times \{\frac{3}{2}\}} = \varphi_1 + 4\log \frac{3}{2}. \)

**Proof.** For any point \( p \), pick a local coordinate chart \((z_1, \ldots, z_n, w)\) as in (2.3) with properties (2.5)-(2.7). It is straightforward to check that

\[ J(X_i) = \sqrt{-1}X_i, \quad J(X_i) = -\sqrt{-1}X_i, \]

\[ J(X_{n+1}) = -\sqrt{-1}X_{n+1}. \]

\( \{dz^i, dz^j, dr + \sqrt{-1}\eta r, dr - \sqrt{-1}\eta r\} \) is the dual basis of \( \{X_i, X_j, X_{n+1}, X_{n+1}\} \).

For \( F(\cdot, r) \in C^\infty(M \times \mathbb{R}^+), \)

\[ \partial F = (X_i F) dz^i + (X_{n+1} F)(dr - \sqrt{-1}\eta), \]

\[ \partial \partial r = -\sqrt{-1}\frac{1}{2} d\eta + \frac{1}{4r}(dr + \sqrt{-1}\eta) \wedge (dr - \sqrt{-1}\eta). \]

From above, \( -\partial \partial r \) is a positive (1,1)-form on \( M \times \mathbb{R}^+ \). If \( F \) is basic, i.e., \( \frac{\partial F}{\partial z^i} = 0, \)

\[ \partial \partial F - \frac{\partial F}{\partial z^i} \partial \partial r = \frac{\partial^2 F}{\partial z^i \partial r}(dr + \sqrt{-1}\eta) \wedge dz^j + \frac{1}{4r}(dr + \sqrt{-1}\eta) \wedge dz^j \]

\[ + \frac{1}{2r}(dr + \sqrt{-1}\eta) \wedge (dr - \sqrt{-1}\eta). \]

Let \( \varphi_t : M \times [0, 1] \rightarrow R \) be a path in the space \( \mathcal{H} \), and the function \( \psi \) on \( M \times [1, \frac{3}{2}] = M \times [1, \frac{3}{2}] \subset C(M) \) defined as in (1.8). Since \( \Psi \equiv 0 \), for \( \Omega_\psi \) defined as in (1.9),

\[ \Omega_\psi = \sqrt{-1}r^2\{(h_{ij} + \frac{1}{2}b_{ij}) dz^i \wedge dz^j + \frac{1}{4r} \partial^2 \phi \partial z^i \partial \eta \wedge dz^j + (dr - \sqrt{-1}\eta) \]

\[ + \frac{1}{4r} \partial^2 \phi \partial z^j \partial \eta \wedge dz^i + (dr + \sqrt{-1}\eta) \wedge dz^j \]

\[ + \frac{1}{2} \partial^2 \phi \partial z^i \partial z^j \wedge (dr - \sqrt{-1}\eta) \]

\[ + \frac{1}{2} \partial^2 \phi \partial z^j \partial z^i \wedge (dr + \sqrt{-1}\eta) \}

\[ = \sqrt{-1}r^2\{(h_{ij} + \frac{1}{2}b_{ij}) dz^i \wedge dz^j + \frac{1}{4r} \partial^2 \phi \partial z^i \partial \eta \wedge dz^j + (dr - \sqrt{-1}\eta) \]

\[ + \frac{1}{4r} \partial^2 \phi \partial z^j \partial \eta \wedge dz^i + (dr + \sqrt{-1}\eta) \wedge dz^j \]

\[ + \frac{1}{2} \partial^2 \phi \partial z^i \partial z^j \wedge (dr - \sqrt{-1}\eta) \]

\[ + \frac{1}{2} \partial^2 \phi \partial z^j \partial z^i \wedge (dr + \sqrt{-1}\eta) \}

\[ \Omega_\psi^{n+1} = 2^{-n}(n+1)r^{2n+3}\left(\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{4r} \partial^2 \phi \partial \eta \partial \eta \right) (dr \wedge \eta \wedge (d\eta)^n). \]
Lemma 2. Let $f$ be the boundary data of $\psi$.

(2.16) $f_0 = \frac{\Omega_{\psi_0}}{\omega^{n+1}} > 0$.

The proposition follows directly from (2.14). □

In order to solve (1.11), we need to find a appropriate subsolution. Let $\psi_1, \psi_2 \in H$ be the given boundary data on $M \times \{1\}$ and $M \times \{\frac{3}{2}\}$ respectively, set $\psi \in C^\infty(M \times [1, \frac{3}{2}])$ by

(2.15) $\psi_0(\cdot, r) = 2(\frac{3}{2} - r)\psi_1(\cdot) + 2(r - 1)\psi_2(\cdot) + m((2(r - 1) - \frac{1}{2})^2 - \frac{1}{4})$,

where the positive constant $m$ is chosen sufficiently large such that $\Omega_{\psi_0}$ is positive. Let

(2.17) $f_0 = \frac{\Omega_{\psi_0}}{\omega^{n+1}} > 0$.

$b$ yields $b_0 \equiv 0$. Fix $\psi_1, \psi_2 \in H$, for any positive basic function $f$, and set $f_s = sf + (1 - s)f_0$ for each $0 \leq s \leq 1$. We consider the following Dirichlet problem

(2.17) $\begin{cases} (\Omega_{\psi})^{n+1} = f_s\omega^{n+1}, & M \times (1, \frac{3}{2}), \\ \psi|_{r=1} = \psi_1, \\ \psi|_{r=\frac{3}{2}} = \psi_2. \end{cases}$

In local coordinates, (2.17) can be written as

$$\det(\bar{g}_{\alpha\beta}) = f_s \det(\bar{g}_{\alpha\beta}), \quad \text{where} \quad \bar{g}_{\alpha\beta} = \bar{g}_{\alpha\beta} + \frac{r^2}{2} \psi_{\alpha\beta} - \frac{r^2}{2} \frac{\partial \psi_{\alpha\beta}}{\partial r}.$$  

Remark 1. Note that for any $B \in \mathbb{R}$, $\Omega_{\psi + Br} = \Omega_{\psi}, \forall \psi \in H$. Therefore, $f_0$ can be chosen as large as we wish by picking $m$ sufficiently large (leaving the boundary data unchanged at the same time). For any given $f$, we may assume $f_0(Z) \geq f(Z), \forall Z \in M \times [1, \frac{3}{2}]$. $\psi_0$ is the unique solution to the equation (2.17) at $s = 0$. Also note that $\psi_0$ is a subsolution of (2.17) for each $0 \leq s \leq 1$.

We will apply the method of continuity to solve (2.17). By Remark 1, we will assume (2.17) has a subsolution $\psi_0$. For the simplicity of notation, we will write $f$ in place of $f_s$ in (2.17).

We conclude this section with the following lemma.

Lemma 2. Let $\psi$ be a solution of the equation (2.17), and $\Omega_{\psi}$ is positive. If the boundary data of $\psi$ is basic, then $\xi_\psi \equiv 0$ on $M \times [1, \frac{3}{2}]$. Moreover, the kernel of the linearized operator of equation (1.11) with null boundary data is trivial.

Proof. Choose the same local coordinate $(z_1, \ldots, z^n, w)$ as in (2.3) with properties (2.5)-(2.7). $T^{1,0}M \times (1, \frac{3}{2})$ is spanned by $X_\alpha, \theta^n (\alpha = 1, \cdots, n + 1)$ defined as in (2.4). Set

(2.18) $\begin{aligned} \Omega_{\psi} &= \sqrt{-1}\bar{g}_{\alpha\beta}\theta^n \land \bar{\theta}^\beta, \\ \end{aligned}$

where $i, j = 1, \cdots, n$, and $\alpha, \beta = 1, \cdots, n + 1$. $\psi$ is not assumed to be basic. We have

$$\begin{aligned} \Omega_{\psi} &= \frac{1}{2} \sqrt{-1} \left( 2h_j z_i + X_i \bar{X}_j \psi + \sqrt{-1} \frac{\partial \psi}{\partial r} dz_i \land dz_j \right. \\
&\quad + X_i \bar{X}_{i+1} \psi dz_i \land (dr - \sqrt{-1} r \eta) + X_{i+1} \bar{X}_j \psi (dr + \sqrt{-1} r \eta) \land dz_j \\
&\quad \left. + (X_{i+1} \bar{X}_{i+1} \psi + r^{-2} + \frac{1}{2} \frac{\partial \psi}{\partial r} + \sqrt{-1} \bar{\psi}^{\alpha\beta}) (dr - \sqrt{-1} r \eta) \land (dr - \sqrt{-1} r \eta) \right). \\ \end{aligned}$$
and,
\[
[X_i, \bar{X}_j] = -2\sqrt{-\Omega} \frac{\partial}{\partial r}, \quad [X_i, \bar{X}_{n+1}] = 0, \quad [X_{n+1}, \bar{X}_j] = 0,
\]
\[
[X_{n+1}, \bar{X}_{n+1}] = -\frac{1}{2}\sqrt{-\Omega} r^{-2} \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial r} \hat{g}_{\alpha \beta} = \sqrt{-\Omega}(X_{\alpha} \bar{X}_{\beta} \frac{\partial}{\partial r} - \frac{1}{2}[X_{\alpha}, \bar{X}_{\beta}] \frac{\partial}{\partial r}).
\]

Let \( \tilde{\nabla} \) be the Chern connection of Hermitian metric \( \tilde{g}(\cdot, \cdot) = \Omega_{\psi}(\cdot, J \cdot) \). Note that \( \tilde{\nabla} \tilde{g} = 0, \tilde{\nabla} J = 0 \), and the (1, 1) part of the torsion vanishes,
\[
\tilde{\nabla}_{X_{\alpha}} \bar{X}_{\beta} - \tilde{\nabla}_{\bar{X}_{\beta}} X_{\alpha} = [X_{\alpha}, \bar{X}_{\beta}], \quad \tilde{\nabla}_{X_{\alpha}} \bar{X}_{\beta} + \tilde{\nabla}_{\bar{X}_{\beta}} X_{\alpha} = \sqrt{-\Omega} J([X_{\alpha}, \bar{X}_{\beta}])
\]
Set \( \omega^* = \frac{1}{2} r^2 p^* \, d\eta + \frac{1}{2} r \, dr \wedge \eta = \sqrt{-\Omega} \partial_0 \partial r \) and \( g^*(\cdot, \cdot) = \omega^*(\cdot, J \cdot) \), we have
\[
(2.19) \quad \tilde{\nabla}_{X_{\alpha}} \bar{X}_{\beta} + \tilde{\nabla}_{\bar{X}_{\beta}} X_{\alpha} = -2 r^{-1} \, g_{\alpha \beta} \frac{\partial}{\partial r}.
\]

It is straightforward to compute that
\[
(2.20) \quad 0 = \frac{\partial}{\partial r} (\log(f \det(\tilde{g}_{\alpha \beta}))) = \frac{r^2}{4} \tilde{\Delta} (\xi \psi) + 2 r^{-1} \, \tilde{g}^{\alpha \beta} g^*_{\alpha \beta} \frac{\partial}{\partial r} (\xi \psi),
\]
where \( \tilde{\Delta} \) is the Laplacian of the Chern connection \( \tilde{\nabla} \). It is known that the Laplacian \( \tilde{\Delta} \) differs from the standard Laplacian of the Levi-Civita connection of \( \tilde{g} \) by a linear first order term ([26], p237, Remark). Therefore, \( \xi \psi \) satisfies homogeneous linear elliptic equation (2.20) with vanishing boundary data. By the Maximum principle ([38], Chapter 5, section 2), it follows \( \xi \psi \equiv 0 \). The last assertion of the lemma follows from the same arguments. \( \square \)

3. \( C^1 \) estimate

This section and the next two sections will be devoted to the a priori estimates of solutions of equation (2.17). We start from \( C^0 \) estimate. We already have a subsolution to (1.11). We now construct a supersolution.

Let \( \rho \) be a smooth function on \( M \times [1, \frac{3}{2}] \) such that
\[
(3.1) \quad \frac{r^2}{4} \Delta_{\tilde{g}} \rho - \frac{r^2}{4} \Delta_{\tilde{g}} \rho \frac{\partial}{\partial r} + n + 1 = 0,
\]
and satisfies the boundary condition \( \rho(\cdot, 1) = \psi_1(\cdot), \rho(\cdot, \frac{3}{2}) = \psi_2(\cdot) \). The solvability of the above boundary value problem can be found in [38] (Chapter 5, Proposition 1.9). Therefore, \( \psi_0 \) and \( \rho \) are a subsolution and a supersolution of (2.17) respectively. The \( C^0 \) estimate is direct
\[
(3.2) \quad \psi_0 \leq \psi \leq \rho.
\]

The next lemma provides estimates for \( |\frac{\partial \psi}{\partial r}(Z)| \) on the whole of \( M \times [1, \frac{3}{2}] \) and boundary gradient estimates of \( \psi \).

Lemma 3. Let \( \psi \) be a solution of the equation (2.17) and coincides with \( \psi_0 \) at the boundary \( \partial (M \times [1, \frac{3}{2}]) \). Then there exists a constant \( C^* \) which depends only on \( \psi_0 \) and the metric \( \tilde{g} \) such that
\[
(3.3) \quad |\frac{\partial \psi}{\partial r}(Z)| \leq C^*, \forall Z \in M \times [1, \frac{3}{2}]; \quad |\partial \psi|^2(p) \leq C^*, \forall p \in \partial (M \times [1, \frac{3}{2}]).
\]

Proof. Since \( \Omega_{\psi} \) is positive definite, if the boundary data of \( \psi \) is basic, it follows from (2.13) that \( \frac{\partial^2 \psi}{\partial r^2} > -4r^{-2} \) on \( M \times [1, \frac{3}{2}] \). Together with (3.2), we obtain
\[
(3.4) \quad \frac{\partial \psi_0}{\partial r}(\cdot, 1) - \frac{4}{3} \leq \frac{\partial \psi}{\partial r}(\cdot, r) \leq \frac{\partial \psi_0}{\partial r}(\cdot, \frac{3}{2}) + \frac{4}{3}.
\]
As $|d\psi|^2(p) = |d\psi|^2 - (\frac{\partial \phi}{\partial \theta})^2 + (\frac{\partial \psi}{\partial r})^2$, hence $|d\psi|^2(p)$ is under control.

The following is the global gradient estimate.

**Proposition 2.** Suppose $\psi$ is a solution of equation (2.17). Let $\phi = \psi - Br$, $B = \sup_{M \times [\frac{1}{2}, \frac{3}{2}]} \frac{\partial \phi}{\partial r}$, $W = |\phi|^2$, $L = \sup_{M \times [\frac{1}{2}, \frac{3}{2}]} |\phi|$. There exist positive constants $A$ and $C$ depending only on $L$, $\inf_{M \times [\frac{1}{2}, \frac{3}{2}]} \|f_{\psi, r}^2\|_{C^1(M \times [\frac{1}{2}, \frac{3}{2}])}$, $\|r\|_{C^3}$, and $\text{osc}_{M \times [\frac{1}{2}, \frac{3}{2}]}$, if the maximum of $H = e^{Ae^{L-\phi}W}$ is achieved at an interior point $p$, then

(3.5)

$$H(p) \leq C.$$  

Combining with Lemma 3, there exist a positive constant $C_0$ depending only on $\rho$, $\psi_0$, $\inf_{M \times [\frac{1}{2}, \frac{3}{2}]} \|f_{\psi, r}^2\|_{C^1(M \times [\frac{1}{2}, \frac{3}{2}])}$, $\|r\|_{C^3}$ such that

(3.6)

$$|d\psi|^2(Z) \leq C_0, \quad \forall Z \in M \times [\frac{1}{2}, \frac{3}{2}].$$

**Proof.** As noted in Remark 1, for $\phi = \psi - Br$, $\Omega_\phi = \Omega_\psi$ for any constant $B$. Since $\frac{\partial \phi}{\partial r}$ is bounded, one may pick $B = \sup_{M \times [\frac{1}{2}, \frac{3}{2}]} \frac{\partial \phi}{\partial r}$ so that $\phi_r \leq 0$ and $\phi$ satisfies the same equation (2.17). We only need to prove (3.5). Pick a holomorphic normal coordinate system centered at $p$ such that $\bar{g}_{\alpha \beta}|_p = \delta_{\alpha \beta}$, $\bar{d}_{\alpha \beta}|_p = 0$, and $\bar{g}_{\alpha \beta}$ is diagonal at $p$, where $\bar{g}_{\alpha \beta} = \bar{g}_{\alpha \beta} + \frac{\partial^2}{\partial\alpha \partial\beta}r_{\alpha \beta} - \frac{\partial^2}{\partial\alpha \partial\beta} \rho_{\alpha \beta}$. We may assume that $W(p) \geq 1$.

Differentiate $\log H$ at $p$,

$$\frac{W_{\alpha}}{W} - Ae^{L-\phi} \phi_{\alpha} = 0,$$

$$\frac{W_\alpha}{W} - Ae^{L-\phi} \phi_\alpha = 0.$$  

(3.7)

We compute

$$W_{\alpha} = \phi_{\beta \alpha} \phi_\beta + \phi_{\beta \alpha} \phi_\beta,$$

$$W_\alpha = \phi_{\beta \alpha} \phi_\beta + \phi_{\beta \alpha} \phi_\beta,$$

$$W_{\alpha} = \bar{g}_{\alpha \alpha} \phi_\beta + \sum |\phi_{\beta \alpha}|^2 + \phi_{\beta \alpha} \phi_{\beta \alpha} + \phi_{\beta \alpha} \phi_{\beta \alpha},$$

$$W_{\alpha}^2 = \phi_{\beta \alpha} \phi_\beta + \phi_{\beta \alpha} \phi_\beta + \phi_{\beta \alpha} \phi_{\beta \alpha}.$$

Since $(\bar{g}_{\alpha \beta}) > 0$, by the assumption, at $p$,

$$0 \geq F_{\alpha \beta} (\log H)_{\alpha \beta} = F_{\alpha \beta} (\log H)_{\alpha \beta},$$

$$= F_{\alpha \beta} \left\{ W_{\alpha} W^{-1} - |W_{\alpha}|^2 W^{-2} - Ae^{L-\phi} (\phi_{\alpha \alpha} - |\phi_\alpha|^2) \right\},$$

$$= - Ae^{L-\phi} W^{-1} (\phi_{\beta \alpha} \phi_\beta + \phi_{\beta \alpha} \phi_\beta),$$

$$+ W^{-1} |\phi_{\beta \alpha}|^2 W^{-2} |\sum \phi_{\beta \alpha} \phi_{\beta \alpha}|^2,$$

(3.8)

$$+ W^{-1} |\phi_{\beta \alpha}|^2 W^{-2} |\sum \phi_{\beta \alpha} \phi_{\beta \alpha}|^2,$$

$$+ \sum W^{-1} (\phi_{\beta \alpha} \phi_{\beta \alpha} + \phi_{\beta \alpha} \phi_{\beta \alpha}).$$
where $F^{\alpha \beta}$ is the $(\alpha, \beta)$th cofactor of the matrix $(\bar{g}_{\alpha \beta})$. On the other hand,

$$
\begin{align*}
\phi_{\alpha \beta} &= \psi_{\alpha \beta} - Br_{\alpha \beta} = 2r^{-2}(\bar{g}_{\alpha \beta} - g_{\alpha \beta}) + \frac{\partial \phi}{\partial r} r_{\alpha \beta}, \\
\phi_{3\alpha} \phi_{\alpha} &= \phi_{3\beta} \phi_{\beta}(2r^{-2}(\bar{g}_{\alpha \beta} - g_{\alpha \beta}) + \frac{\partial \phi}{\partial r} r_{\alpha \beta}), \\
\phi_{\eta \alpha} \phi_{\alpha} &= \phi_{\eta \beta} \phi_{\beta}(2r^{-2}(\bar{g}_{\alpha \beta} - g_{\alpha \beta}) + \frac{\partial \phi}{\partial r} r_{\alpha \beta}),
\end{align*}
$$

and

$$
\begin{align*}
\sum_{\beta} W^{-1}(\phi_{3\beta} \phi_{3\alpha} + \phi_{3\beta} \phi_{3\alpha}) &= -4r^{-3}W^{-1}(\phi_{3 \beta} r_{\beta} + \phi_{3 \beta} r_{\beta})(\bar{g}_{\alpha \alpha} - g_{\alpha \alpha}) \\
&+ 2r^{-2}W^{-1}(\phi_{3 \beta} \bar{g}_{\alpha \alpha , \beta} + \phi_{3 \beta} g_{\alpha \alpha , \beta}) \\
&+ 4r^{-1}r_{\alpha \alpha}((\frac{\partial \phi}{\partial r})_{\beta} \phi_{\beta} + (\frac{\partial \phi}{\partial r})_{\beta} \phi_{\beta}).
\end{align*}
$$

To calculate terms in above equation, we need to commute $\frac{\partial}{\partial r}$ with other derivatives. For any cone type metric $\bar{g} = r^2 g + dr^2$ on $C(M) = \bar{M} \times \mathbb{R}^+$ (where $g$ is a Riemannian metric on $\bar{M}$) and for any local coordinate $\{x^i\}_{i=1}^m$ on $\bar{M}$, where $m = \text{dim} \bar{M}$, $\{x^1, \cdots, x^m, r\}$ is a local coordinate on the cone $C(M)$. By the definition of the cone metric $\bar{g}$,

$$
< \frac{\partial}{\partial x^i}, \frac{\partial}{\partial r} >_{\bar{g}} = 0, \quad < \frac{\partial}{\partial r}, \frac{\partial}{\partial r} >_{\bar{g}} = 1.
$$

For any vector field $Y = Y^i \frac{\partial}{\partial x^i} + Y^{m+1} \frac{\partial}{\partial r}$,

$$
< \frac{\partial}{\partial r}, Y >_{\bar{g}} = Y^{m+1} = dr(Y) = < \nabla^g r, Y >_{\bar{g}}.
$$

That is, $\frac{\partial}{\partial r} = \nabla^g r$. Therefore,

$$
\frac{\partial}{\partial r} = \nabla^g r = \bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha} \frac{\partial}{\partial z^\beta} + \bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha} \frac{\partial}{\partial z^\beta}.
$$

In other words, $\frac{\partial}{\partial r}$ is equal to the gradient of $r$ with respect to the Kähler cone metric $\bar{g}$.

By (3.11),

$$
\begin{align*}
W^{-1} r_{\alpha \alpha}((\frac{\partial \phi}{\partial r})_{\beta} \phi_{\beta} + (\frac{\partial \phi}{\partial r})_{\beta} \phi_{\beta}) &= W^{-1} r_{\alpha \alpha}((\frac{\partial \phi}{\partial r})_{\beta} \phi_{\beta} + (\frac{\partial \phi}{\partial r})_{\beta} \phi_{\beta}) \\
&= W^{-1} r_{\alpha \alpha}((\frac{\partial \phi}{\partial r})_{\beta} \phi_{\beta} + (\frac{\partial \phi}{\partial r})_{\beta} \phi_{\beta}) \\
&+ W^{-1} r_{\alpha \alpha}((\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta} + (\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta}) \\
&+ W^{-1} r_{\alpha \alpha}((\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta} + (\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta}) \\
&= A e^{-\phi} r_{\alpha \alpha} \\
&+ W^{-1} r_{\alpha \alpha}((\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta} + (\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta}) \\
&+ W^{-1} r_{\alpha \alpha}((\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta} + (\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta}).
\end{align*}
$$

Hence,

$$
\begin{align*}
-A e^{-\phi} r_{\alpha \alpha} + \sum_{\beta} W^{-1}(\phi_{3 \beta} \phi_{3 \alpha} + \phi_{3 \beta} \phi_{3 \alpha}) &= A e^{-\phi} r_{\alpha \alpha} \\
&- 4r^{-3} W^{-1}(\phi_{3 \beta} r_{\beta} + \phi_{3 \beta} r_{\beta})(\bar{g}_{\alpha \alpha} - g_{\alpha \alpha}) \\
&+ 2r^{-2} W^{-1}(\phi_{\beta} g_{\alpha \alpha , \beta} + \phi_{\beta} g_{\alpha \alpha , \beta}) \\
&+ W^{-1} \phi_{3 \beta} \phi_{3 \beta} + \phi_{3 \beta} \phi_{3 \beta} \\
&+ W^{-1} r_{\alpha \alpha}((\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta} + (\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta}) \\
&+ W^{-1} r_{\alpha \alpha}((\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta} + (\bar{g}^{\alpha \beta} \frac{\partial}{\partial z^\alpha})_{\beta} \phi_{\beta}).
\end{align*}
$$

One may pick $A \geq 1$ sufficiently large, so that

$$
A \frac{1}{2} r^{-2} - 4r^{-3} + 2r^{-1} ||C^2 ||_{C} \geq 0, \quad (\bar{g}^{\alpha \beta}_{\beta} \phi_{\gamma} W^{-1} + \frac{2}{9} A \bar{g}_{\alpha \beta}) \geq 0.
$$
such $A$ depends only on $r$, $\text{osc}(\psi^2)\|\psi\|_{C^3}$ and the lower bound of the holomorphic bisectional curvature of $(M \times [1, \frac{3}{2}], \tilde{g})$.

As $F^{\alpha\bar{\beta}}\tilde{g}_{\alpha\beta} = f_\beta$ and $\phi_\beta \leq 0$, at point $p$,

$$0 \geq F^{\alpha\bar{\beta}}(\log H)_{\alpha\beta}$$

$$= F^{\alpha\bar{\beta}}(\tilde{g}_{\alpha\beta}\phi_\beta\phi_\delta W^{-1} + Ae^{L - \phi}(2r^{-2}\tilde{g}_{\alpha\beta} + |\phi_\beta|^2)$$

$$- 2Ae^{L - \phi}r^{-2}\tilde{g}_{\alpha\beta}(1 + 2|\phi_\beta|^2) + 4Ae^{L - \phi}r^{-2}\tilde{g}_{\alpha\beta}W^{-1}|\phi_\beta|^2$$

$$- Ae^{L - \phi}W^{-1}\phi_\beta(\phi_\beta\phi_\delta\phi_\alpha + \phi_\gamma\phi_\eta\phi_\alpha)$$

$$+ |W^{-1}(\sum_\beta |\phi_\beta|^2) - W^{-2}(\sum_\beta \phi_\beta\phi_\beta\beta\bar{\alpha}|^2|)$$

$$+ W^{-1}|\phi_\beta|^2 + W^{-2}|\sum_\beta \phi_\beta\phi_\beta\beta\bar{\alpha}|^2$$

$$+ 2r^{-2}W^{-1}(\phi_\beta\tilde{g}_{\alpha\beta} + \phi_\beta\tilde{g}_{\alpha\beta})$$

$$- 4r^{-3}W^{-1}(\phi_\beta r_\bar{\beta} + \phi_\beta r_\alpha)(\tilde{g}_{\alpha\beta} - \bar{g}_{\alpha\beta})$$

$$W^{-1}\phi_\beta(\phi_\beta r_{\alpha\beta\bar{\beta}} + \phi_\beta r_{\alpha\bar{\beta}\beta})$$

$$+ W^{-1}r_{\alpha\beta}(\tilde{g}_{\alpha\beta}\phi_\delta + (\tilde{g}_{\alpha\beta}\phi_\beta)\phi_\delta + (\tilde{g}_{\alpha\beta}\phi_\beta)\phi_\delta)$$

$$(\tilde{g}_{\alpha\beta}\phi_\delta + (\tilde{g}_{\alpha\beta}\phi_\beta)\phi_\delta + (\tilde{g}_{\alpha\beta}\phi_\beta)\phi_\delta)$$

$$(\tilde{g}_{\alpha\beta})(\tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}(1 + W))$$

$$\geq (n + 1)|\tilde{g}_{\alpha\beta}|^{-1}(1 + W)^{\frac{2n}{n+1}}.$$

Note that $F^{\alpha\bar{\beta}}\tilde{g}_{\alpha\beta} = (n + 1)f$, the above inequality yields

$$0 \geq \frac{Ae^{L - \phi}r^{-2}(\sum_\alpha F^{\alpha\bar{\beta}}(\tilde{g}_{\alpha\beta} + |\phi_\beta|^2) - 10(n + 1)f - 2r^{-2}W^{-\frac{1}{2}}|\nabla f|}{(\sum_\alpha F^{\alpha\bar{\beta}}\tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}(1 + W))}$$

$$(\tilde{g}_{\alpha\beta}(1 + W)^{\frac{2n}{n+1}}).$$

(3.16)

Now (3.5) follows directly.

4. $C^{1,1}$ boundary estimate

$C^{1,1}$ boundary estimates will be proved in this section by careful construction of appropriate barrier functions. This type of construction of barriers follows from B. Guan [20] (in the real case, this method was introduced by Hoffman-Rosenberg-Spruck [25] and Guan-Spruck in [21]). Let $\psi$ be a solution of the equation (2.17), and we want to obtain second derivative estimates of $\psi$ on the boundary $\partial(M \times [1, \frac{3}{2}]) = M \times \{1\} \cup M \times \{\frac{3}{2}\}$. We will only consider the boundary estimate on $M \times \{1\}$, the treatment for the other piece of the boundary follows the same way.
For any point \( p = (q, 1) \in M \times \{1\} \), we may pick a local coordinate chart as in (2.3) with properties (2.5)-(2.7). Furthermore, we may assume

\[
\frac{1}{4} \delta_{ij} \leq h_{ij}(z) \leq \delta_{ij}, \quad \sum_{i=1}^{n} |h_i|^2(z) \leq 1, \quad \forall z \in U,
\]

where \( h \) is a local real basic function,

\[
\eta = dx - \sqrt{-1} (h_i dz^i - h_{ij} dz^j), \quad z^i = x^i + \sqrt{-1} y^i, \quad \forall i, j = 1, \ldots, n.
\]

Set \( V = U \times [1, 1 + \delta] \), there is a local coordinate chart \((r, x, z^1, \ldots, z^n)\) on \( V \) such that \( \partial(M \times [1, \frac{3}{2}]) \cap V = \{r = 1\} \). For \( X_\alpha, \theta^\beta \) defined in (2.4),

\[
\{X_1, \ldots, X_j, \ldots X_n, X_{n+1}\} \quad \text{is a local real basic function},
\]

\[
\theta^1, \ldots, \theta^{n+1} \quad \text{is its dual basis}.
\]

The Kähler form \( \tilde{\omega} \) of \((M \times [1, \frac{3}{2}], \tilde{g})\) can be written as

\[
\tilde{\omega} = \sqrt{-1} (r^2 h_{ij} \theta^i \wedge \tilde{\theta}^j + \frac{1}{2} \theta^{n+1} \wedge \tilde{\theta}^{n+1}) = \sqrt{-1} \tilde{g}_{\alpha \beta} \theta^\alpha \wedge \tilde{\theta}^\beta,
\]

and

\[
\Omega_\psi = \sqrt{-1} \left\{ (r^2 h_{ij} + \frac{r^2}{2} \frac{\partial^2 \psi}{\partial x_i \partial x_j}) \theta^i \wedge \tilde{\theta}^j + \frac{r^2}{4} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \theta^i \wedge \tilde{\theta}^{n+1}
\right.
\]

\[
+ \frac{r^2}{8} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \theta^{n+1} \wedge \tilde{\theta}^j + (\frac{r^2}{4} \frac{\partial^2 \psi}{\partial x_i \partial x_j} + \frac{1}{2}) \theta^{n+1} \wedge \tilde{\theta}^{n+1} \right\}
\]

\[
= \sqrt{-1} \left( \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right) \theta^i \wedge \tilde{\theta}^j + \frac{r^2}{2} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \theta^{n+1} \wedge \tilde{\theta}^j.
\]

Since \( 2(\tilde{g}_{\alpha \beta} + \frac{r^2}{2} X_\alpha X_\beta \psi_0) \theta^\alpha \wedge \tilde{\theta}^\beta \) is a Hermitian metric on \( M \times [1, \frac{3}{2}] \), there exists a constant \( 0 < a_0 < 1 \) such that

\[
a_0 \tilde{g}_{\alpha \beta} < \tilde{g}_{\alpha \beta} + \frac{r^2}{2} X_\alpha X_\beta \psi_0 < \frac{1}{a_0} \tilde{g}_{\alpha \beta}
\]

in \( M \times [1, \frac{3}{2}] \). In the neighborhood \( V \) of \( p \), we have

\[
\frac{1}{4} a_0 \delta_{\alpha \beta} \leq \tilde{g}_{\alpha \beta} + \frac{r^2}{2} X_\alpha X_\beta \psi_0 \leq \frac{9}{4} \frac{1}{a_0} \delta_{\alpha \beta}.
\]

Let \( \triangle_\psi \) be the canonical Laplacian corresponding with the Chern connection determined by the Hermitian metric \( \Omega_\psi = \sqrt{-1} \tilde{g}_{\alpha \beta} \theta^\alpha \wedge \tilde{\theta}^\beta \) on \( M \times [1, \frac{3}{2}] \). In Kähler case, this canonical Lapalacian is same as the standard Levi-Civita Laplacian. In general Hermitian case they are different, and the difference of two Laplacian is a first order linear differential operator. In the above local coordinates,

\[
\frac{1}{2} \triangle_\psi u = - \sqrt{-1} \Pi \partial \bar{\partial} u, \quad \Omega_\psi > \psi
\]

\[
= - \langle (X_\gamma X_\delta u) \theta^\gamma \wedge \tilde{\theta}^\delta + 2 (X_{n+1} u) \partial \bar{\partial} r, \tilde{g}_{\alpha \beta} \theta^\alpha \wedge \tilde{\theta}^\beta \rangle > \psi
\]

\[
= (\tilde{g})^{\alpha \beta} (X_\alpha X_\beta u) + (X_{n+1} u) \triangle_\psi r,
\]

where \((\tilde{g})^{\alpha \beta} \tilde{g}_{\gamma \delta} = \delta_{\alpha \gamma}\). Define differential operator \( L \) as

\[
L u = \frac{1}{2} \triangle_\psi u - \frac{1}{2} \frac{\partial u}{\partial r} \triangle_\psi r
\]

for all \( u \in C^\infty(M \times [1, \frac{3}{2}]) \).
Now assume $f^\pm \in C^{1,1}$. This implies $|\nabla f^\pm(Z)| \leq C f^\pm(Z), \forall Z \in M \times [1, \frac{3}{2}].$

Since $\sum_{\alpha=1}^{n+1}(\bar{g})^{\alpha\alpha} \geq (n+1)f^{-\frac{n}{n+1}}$, we have

\[
\frac{|\nabla f^\pm|}{f^\pm}(Z) \leq C f^\pm \leq C \sum_{\alpha=1}^{n+1}(\bar{g})^{\alpha\alpha}(Z), \forall Z \in M \times [1, \frac{3}{2}].
\]

Let $D$ be any locally defined constant linear first order operator (with respect to the coordinate chart we chosen) near the boundary (e.g., $D = \pm \frac{\partial}{\partial r}, \pm \frac{\partial}{\partial y}$ for any $1 \leq i \leq n$). Differentiating both sides of equation (2.17) by $D$, by (4.8),

\[
LD(\psi - \psi_0) = \frac{1}{r} \Delta_{\bar{g}} D(\psi - \psi_0) - \frac{1}{r} \frac{\partial D(\psi - \psi_0)}{\partial r} \Delta_{\bar{g}} r
\]

\[
= (\bar{g})^{\alpha\beta} X_\alpha X_\beta D(\psi - \psi_0)
\]

\[
= 2r^{-2}(\bar{g})^{\alpha\beta} D\left(\frac{\partial}{\partial r} X_\alpha X_\beta(\psi - \psi_0)\right)
\]

\[
= 2r^{-2}(\bar{g})^{\alpha\beta} \left(\bar{g}_{\alpha\beta} - (\bar{g}_{\alpha\beta} + \frac{\partial}{\partial r} X_\alpha X_\beta(\psi_0))\right)
\]

\[
= 2r^{-2}(\bar{g})^{\alpha\beta} D\left(\log f + \log \det(\bar{g}_{\alpha\beta})\right)
\]

\[
- 2r^{-2}(\bar{g})^{\alpha\beta} D\left(\bar{g}_{\alpha\beta} + \frac{\partial}{\partial r} X_\alpha X_\beta(\psi_0)\right)
\]

\[
\leq C_1 (1 + \sum_{\alpha=1}^{n+1}(\bar{g})^{\alpha\alpha})
\]

where constant $C_1$ depends only on $\psi_0$, $\|f^\pm\|_{C^{1,1}}$ and the metric $\bar{g}$. Here we have used the properties that $\psi$ and $\psi_0$ are basic, $[D, X_{n+1}] = 0$.

Set

\[
v = (\psi - \psi_0) + b(\rho - \psi_0) - N(r-1)^2
\]

as a barrier function.

**Lemma 4.** For $N$ sufficiently large and $b, \delta_0$ sufficiently small,

\[
Lv \leq -\frac{a_0}{g} (1 + \sum_{\alpha=1}^{n+1}(\bar{g})^{\alpha\alpha})
\]

in $U \times [1, \frac{3}{2}]$, and $v \geq 0$ in $M \times [1, 1+\delta_0]$, where constants $N, b, \delta_0, a_0$ depend only on $\psi_0, \rho, \|f^\pm\|_{C^{1,1}}$, and $\bar{g}$.

**Proof.** By assumption,

\[
L(\psi - \psi_0) = (\bar{g})^{\alpha\beta} X_\alpha X_\beta(\psi - \psi_0)
\]

\[
= 2r^{-2}(\bar{g})^{\alpha\beta}\left(\bar{g}_{\alpha\beta} - (\bar{g}_{\alpha\beta} + \frac{\partial}{\partial r} X_\alpha X_\beta(\psi_0))\right)
\]

\[
\leq C_2 (1 + \sum_{\alpha=1}^{n+1}(\bar{g})^{\alpha\alpha})
\]

where constant $C_2$ depends only on $\rho$ and the metric $\bar{g}$. Therefore,

\[
Lv = L(\psi - \psi_0) + bL(\rho - \psi_0) - \frac{1}{2} N(\bar{g})^{n+1}\nabla r
\]

\[
\leq 2r^{-2}(n+1-\frac{a_0}{g} \sum_{\alpha=1}^{n+1}(\bar{g})^{\alpha\alpha}) + bc_2 (1 + \sum_{\alpha=1}^{n+1}(\bar{g})^{\alpha\alpha})
\]

\[
- \frac{1}{2} N(\bar{g})^{n+1}\nabla r.
\]
Suppose $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n+1}$ are eigenvalues of $(\bar{g}_{\alpha \beta})$. It follows that

\begin{equation}
\sum_{\alpha=1}^{n+1} (\bar{g})^{\alpha \alpha} = \sum_{\alpha=1}^{n+1} \lambda_{n+1}^{-1} \geq \lambda_{n+1}^{-1}.
\end{equation}

Thus

\begin{equation}
\frac{\partial}{\partial r} \sum_{\alpha=1}^{n+1} (\bar{g})^{\alpha \alpha} + \frac{\partial}{\partial r} (\bar{g})^{n+1, n+1} \geq \frac{\partial}{\partial r} \sum_{\alpha=1}^{n+1} \lambda_{n+1}^{-1} + \frac{1}{2} N \lambda_{n+1}^{-1} \\
\geq (n+1)(\frac{\partial}{\partial r})^{\frac{n+1}{n+1}} N^{\frac{n+1}{n+1}} (\lambda_1 \cdots \lambda_{n+1})^{\frac{n}{n+1}} \geq C_3 N^{\frac{n}{n+1}},
\end{equation}

where positive constant $C_3$ depends only on $f$ and $(M \times [1, \frac{3}{2}], \bar{g})$. Choose $N$ large enough so that

\begin{equation}
-C_3 N^{\frac{n}{n+1}} + 2r^{-2}(n+1) + bC_2 \leq \frac{a_0}{9},
\end{equation}

and choose $b$ small enough so that $bC_2 \leq \frac{a_0}{9}$. Then, on $U \times [1, \frac{3}{2}]$,

\[ \mathbf{L}v \leq -\frac{a_0}{9} (1 + \sum_{\alpha=1}^{n+1} (\bar{g})^{\alpha \alpha}). \]

By the definition of function $\rho$,

\begin{equation}
\begin{aligned}
\Delta \bar{g}(\rho - \psi_0) &= \Delta \bar{g} \cdot \frac{\partial}{\partial r} (\rho - \psi_0) \\
&= (\Delta \bar{g} \rho - \Delta \bar{g} r \cdot \frac{\partial}{\partial r} \rho) - (\Delta \bar{g} \psi_0 - \Delta \bar{g} r \cdot \frac{\partial}{\partial r} \psi_0) \\
&= -4r^{-2} \{ n + 1 + \frac{r^2}{2} \bar{g} X_\alpha X_\beta \psi_0 \} \\
&= -4r^{-2} \bar{g}^{\alpha \beta} (\bar{g}_{\alpha \beta} + \frac{\partial}{\partial r} \psi_0) \\
&\leq -4r^{-2}(n+1)a_0.
\end{aligned}
\end{equation}

On the boundary $\partial M \times [1, \frac{3}{2}]$, since $\rho$ coincides with $\psi_0$,

\begin{equation}
\frac{\partial^2}{\partial r^2} (\rho - \psi_0) = 2 \bar{g}^{\alpha \beta} X_\alpha X_\beta (\rho - \psi_0) = \Delta \bar{g} (\rho - \psi_0) - \Delta \bar{g} r \cdot \frac{\partial}{\partial r} (\rho - \psi_0) \leq -4r^{-2}(n+1)a_0 < 0.
\end{equation}

As $\psi_0 \leq \rho$ on $M \times [1, \frac{3}{2}]$, it’s easy to show that

\[ \frac{\partial}{\partial r} (\rho - \psi_0) (q, 1) > 0, \quad \frac{\partial}{\partial r} (\rho - \psi_0) (q, \frac{3}{2}) > 0, \forall q \in M. \]

Therefore, there exists a positive constant $C_4$ depending only on $\rho$, $\psi_0$ and $\bar{g}$ such that $\rho - \psi_0 > C_4(r - 1)$ near $M \times \{1\}$ and $\rho - \psi_0 > C_4(\frac{3}{2} - r)$ near $M \times \{\frac{3}{2}\}$. Fix $N$, and choose $\delta_0$ small enough so that

\begin{equation}
b(\rho - \psi_0) - N(r - 1)^2 \geq (bC_4 - N\delta)(r - 1) \geq 0,
\end{equation}

on $M \times [1, 1 + \delta_0]$. Then $v \geq 0$ in $M \times [1, 1 + \delta_0]$. } \hfill \Box

**Lemma 5.** There exists a constant $C_5$ depending only on $(M \times [1, \frac{3}{2}], \bar{g})$, $\psi_0$, $\|f^{\perp}\|_{C^{1,1}}$, and $\rho$ such that

\begin{equation}
\frac{\partial^2 \psi}{\partial z^2 \partial r} (\rho) \leq C_5 \max_{M \times [1, \frac{3}{2}]} (|d\psi|_{\bar{g}} + 1), \quad \forall \rho \in \partial (M \times [1, \frac{3}{2}]).
\end{equation}
The Maximum principle implies that
\[ B_\delta(0) = \{(x, z^1, \cdots, z^n) : x^2 + \sum |z|^2 \leq 4\delta \} \subset U, \quad \text{and} \quad 2\delta \leq \delta_0. \]

The constant \( \delta \) depends only on \((M \times [1, \frac{3}{2}], \tilde{g})\) \( \psi_0 \) and \( p \). Set
\[ V_\delta = \{(r, x, z^1, \cdots, z^n) : (r - 1)^2 + x^2 + \sum |z|^2 \leq \delta \} \cap M \times [1, \frac{3}{2}]. \]

Let \( A = \max_{M \times [1, \frac{3}{2}]} (|d\psi|_\tilde{g} + 1) \). Choose \( d_1, d_2 \) as big multiples of \( A \) such that \( d_2\delta^2 - |D(\psi - \psi_0)| > 0 \). Consider \( \mu = d_1 v + d_2 (x^2 + (r - 1)^2 + \sum |z|^2) + D(\psi - \psi_0) \) (or \( \mu = d_1 v + d_2 (x^2 + (\frac{3}{2} - r)^2 + \sum |z|^2) + D(\psi - \psi_0) \)). Then \( \mu \geq 0 \) in \( \partial V_\delta \) and \( \mu(p) = 0 \). Moreover,
\[ (4.22) \quad L(v) = (\sum |z|^2 + (r - 1)^2) = \sum_{i=1}^{n} (\tilde{g})_{ij} + \frac{1}{2} (\tilde{g})^{n+1}_{n+1} \alpha, \]
\[ L\psi = \frac{1}{2} \Delta_{\psi} x^2 = (\tilde{g})^{\alpha\beta} X_\alpha X_\beta x^2 + \Delta_{\psi} r X_{n+1} x^2 \]
\[ = -2\sqrt{-1} \rho (\tilde{g})^{ij} h_{ij} + 2 (\tilde{g})^{ij} h_{ij} - (\tilde{g})^{n+1}_{n+1} h_{i} r^{-1} - (\tilde{g})^{n+1}_{n+1} h_{i} r^{-1} \]
\[ + \frac{1}{2} (\tilde{g})^{n+1}_{n+1} h_{i} r^{-1} - \frac{1}{2} \sqrt{-1} (\tilde{g})^{n+1}_{n+1} r^{-1} x + \sqrt{-1} (\Delta_{\psi} r) r^{-1} x \]
\[ = 2 (\tilde{g})^{h_{ij} - h_{ij} - (\tilde{g})^{n+1}_{n+1} h_{i} r^{-1} - (\tilde{g})^{n+1}_{n+1} h_{i} r^{-1} + \frac{1}{2} (\tilde{g})^{n+1}_{n+1} r^{-1} \]
\[ \leq 3 \sum_{i=1}^{n} (\tilde{g})^{h_{ij}} (\tilde{g})^{n+1}_{n+1} r^{-1}. \]

Pick \( d_1 \) large, by (4.9) and Lemma 4,
\[ (4.24) \quad L_\mu \leq -\frac{a_0}{9} (d_1 + 4d_2 + C_1) (1 + \sum_{i=1}^{n} (\tilde{g})^{\alpha\beta}) < 0. \]

The Maximum principle implies that \( \mu \geq 0 \) in \( V_\delta \). Since \( \mu(p) = 0 \), we have \( \frac{\partial \mu}{\partial r} \geq 0 \)
when \( p = (q, 1) \). In other words, there is constant \( C_5 \) depending only on \( \psi_0, \rho \) and \( \tilde{g} \) such that
\[ (4.25) \quad -D \frac{\partial \psi}{\partial r} (p) \leq C_5 A. \]

Since \( D \) is any local first order constant differential operator, replacing \( D \) with \(-D\),
\[ (4.26) \quad D \frac{\partial \psi}{\partial r} (p) \leq C_5 A. \]

Therefore,
\[ (4.27) \quad |\frac{\partial^2 \psi}{\partial r \partial z^i} (p)| \leq C_5 A. \]

\[ \blacksquare \]

**Proposition 3.** If \( \psi \) is a solution of equation (2.17), then there exists a constant \( C_6 \) which depends only on \( \rho, \psi_0, \| f \|_{C^{1,1}} \) and \((M \times [1, \frac{3}{2}], \tilde{g})\) such that for any unit vectors \( T_i, T_j \) on \( M \times [1, \frac{3}{2}] \)
\[ (4.28) \quad \max_{\partial(M \times [1, \frac{3}{2}])} |T_i T_j \psi| \leq C_6 \max_{M \times [1, \frac{3}{2}]} (|d\psi|^2_\tilde{g} + 1). \]
In particular,

\[ \max_{\partial(M \times [1, \frac{1}{2}])} |\Delta_\psi \psi| \leq C_0 \max_{M \times [1, \frac{1}{2}]} (|d\psi|_g^2 + 1). \]

**Proof.** We only need to get double normal derivative estimate. At point \( p \in \partial(M \times [1, \frac{1}{2}]) \), choose a local coordinate centered at \( p \) as above, then equation (2.17) reduces to

\[ \det(\tilde{g}_{\alpha \beta} + \frac{r^2}{2} X_\alpha \tilde{X}_\beta \psi) = 2^{-(n+1)} f r^{2n}, \]

where \( \tilde{g}_{ij} = \frac{1}{4} r^2 \delta_{ij}, \tilde{g}_{n+1 \eta} = \frac{1}{2}, \tilde{g}_{n+1 \bar{\eta}} = \tilde{g}_{n+1} = 0 \). Denote \( E_{ij} = g_{ij} + \frac{1}{4} \bar{X}_i \bar{X}_j \psi_0 \) and \( E^{ik} E_{jk} = \delta_{ij} \). By assumption (4.5) on the local coordinate, \( \frac{1}{4} \partial_0 \delta_{ij} \leq E_{ij} \leq \frac{9}{2} \delta_{ij} \). Hence,

\[ 0 < \frac{r^2}{8} \bar{\partial}_r \partial_r \partial_{\eta} (p) + \frac{1}{2} = det(E_{ij})^{-1} 2^{-(n+1)} f r^{2n} + \frac{1}{16} \bar{\partial}_r \partial_r \partial_{\eta} E_{ij} \partial_r \partial_{\eta} \]

\[ \leq 2^{n-1} a_0^2 n f r^{2n} + a_0^{-1} (\sum_{i=1}^n |\bar{\partial}_r \partial_{\eta} \partial_r (p)|). \]

By Lemma 5, we may pick a uniform constant \( C_7 \) such that

\[ |\frac{\partial^2 \psi}{\partial r^2} (p)| \leq C_7 \max_{M \times [1, \frac{1}{2}]} (|d\psi|_g^2 + 1). \]

\[ \square \]

5. \( C^2 \) estimate

We want to establish global \( C^2 \) estimate in this section. For the standard complex Monge-Ampère equation on Kähler manifolds, \( C^2 \) a priori estimate was proved by Yau for compact Kähler manifolds without boundary in [39] independent of the gradient estimate; in the presence of boundaries, gradient estimates cannot be bypassed (see references [6], [3] and [23]). For equation (2.17), the gradient estimate plays a crucial role. The global \( C^2 \) estimate will depend on the gradient estimate on \( \psi \). By (4.8), \( \|f \|_{C^1} \leq C \|f \|_{C^{1,1}} \). By Proposition 2, we may assume \( \|\psi\|_{C^1} \) is bounded.

Since \( \sqrt{-1} \partial \bar{\partial} \psi \) is a positive \((1,1)\) form, it determines a Kähler metric \( K \) as \((2.3) \) on \( M \times [1, \frac{1}{2}] \). Choose a local coordinate \((z^1, \ldots, z^n, w)\) as in (2.3) on \( M \times [1, \frac{1}{2}] \), where \((x, z^1, \ldots, z^n)\) is a local Sasakian coordinates on \( M \), and \( \{X_\alpha \}_{\alpha=1}^{n+1}, \{\theta_\alpha \}_{\alpha=1}^{n+1} \) defined as in (2.4). It’s easy to check that

\[ \sqrt{-1} \partial \bar{\partial} \psi = \frac{i}{2} \partial \eta + \frac{i}{2} \partial r \wedge \eta \]
\[ = r^{-1} \bar{\omega} - \frac{i}{2} \partial r \wedge \eta \]
\[ = \sqrt{-1} r h_{ij} \partial^i \wedge \partial^j + \sqrt{-1} (4r)^{-1} \theta^{n+1} \wedge \bar{\theta}^{n+1}, \]

where \( i, j = 1, \ldots, n \). Therefore,

\[ K = rg + (2r)^{-1} dr^2 - \frac{r}{2} \eta \otimes \eta = r^{-1} \bar{g} - (2r)^{-1} dr^2 - \frac{r}{2} \eta \otimes \eta, \]

and

\[ K_{ij} = rh_{ij}, \quad K_{i\eta} = K_{n+1} = 0, \quad K_{n+1} = (4r)^{-1}, \]
where \( K_{\alpha\beta} = \langle X_\alpha, \bar{X}_\beta \rangle > K \). For any vector \( Y = Y^\alpha X_\alpha + \bar{Y}^\beta \bar{X}_\beta \),

\[
\begin{align*}
\langle \frac{\partial}{\partial r}, Y \rangle > K &= Y^\alpha < \frac{\partial}{\partial r}, X_\alpha > K + \bar{Y}^\beta < \frac{\partial}{\partial r}, \bar{X}_\beta > K \\
&= Y^{n+1} < \frac{\partial}{\partial r}, X_{n+1} > K + \bar{Y}^{n+1} < \frac{\partial}{\partial r}, \bar{X}_{n+1} > K \\
&= (4r)^{-1}(Y^{n+1} + \bar{Y}^{n+1}) = (2r)^{-1} dr(Y) \\
&= (2r)^{-1} < \nabla^K r, Y > K,
\end{align*}
\]

(5.4)

and

\[
\frac{\partial}{\partial r} = (2r)^{-1} \nabla^K r,
\]

(5.5)

where \( \nabla^K r \) is the gradient of \( r \) corresponding to the metric \( K \).

Recall

\[
[X_i, \bar{X}_j] = -2\sqrt{-1}h_{ij}\frac{\partial}{\partial x}, \quad [X_{n+1}, \bar{X}_{n+1}] = -\frac{1}{2} \sqrt{-1} r^{-2} \frac{\partial}{\partial x},
\]

\[
\begin{align*}
\nabla^K_{X_\alpha} \bar{X}_\beta - \nabla^K_{\bar{X}_\beta} X_\alpha &= [X_\alpha, \bar{X}_\beta], \\
\nabla^K_{X_\alpha} \bar{X}_\beta + \nabla^K_{\bar{X}_\beta} X_\alpha &= \sqrt{-1} J([X_\alpha, \bar{X}_\beta]).
\end{align*}
\]

and

\[
\nabla^K_{X_\alpha} \bar{X}_\beta = \frac{1}{2} ([X_\alpha, \bar{X}_\beta] + \sqrt{-1} J([X_\alpha, \bar{X}_\beta])).
\]

By above, give any smooth function \( \varphi \) on \( M \times [1, \frac{3}{2}] \), we have

\[
\begin{align*}
\frac{1}{2}\Delta_K \varphi &= K^{\alpha\beta} \nabla^K d\varphi(X_\alpha, \bar{X}_\beta) \\
&= K^{\alpha\beta} X_\alpha \bar{X}_\beta \varphi - K^{\alpha\beta} d\varphi(\nabla^K_{X_\alpha} \bar{X}_\beta) \\
&= K^{\alpha\beta} X_\alpha \bar{X}_\beta \varphi + (n+1) d\varphi(\frac{\partial}{\partial r} + \sqrt{-1} r^{-1} \frac{\partial}{\partial x}),
\end{align*}
\]

(5.6)

where \( K^{\alpha\beta} \) satisfies \( K^{\alpha\beta} K_{\gamma\beta} = \delta_{\alpha\gamma} \).

Note that \( \Delta_{\beta} \psi \) and \( \Delta_{K} \psi \) are equivalent as \( \|\psi\|_{C^1} \) is bounded.

**Lemma 6.** Let \( \psi \) be a smooth function on \( M \times [1, \frac{3}{2}] \) and satisfy \( \xi \psi \equiv 0 \), then

\[
\Delta_{K} \left( \frac{\partial \psi}{\partial r} \right) = \frac{\partial}{\partial r} (\Delta_{K} \psi) + r^{-1} \Delta_{K} \psi - 2(n+1) r^{-1} \frac{\partial \psi}{\partial r} - \frac{1}{2} \frac{\partial^2 \psi}{\partial r^2}.
\]

**Proof.** It is straightforward to check that

\[
\begin{align*}
K^{ij} &= r^{-1} h^{ij}, \quad K^{n+1+n+1} = 4r, K^{n+1} = K^{n+1} = 0, \\
\frac{\partial}{\partial r} X_i &= X_i \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial r} \bar{X}_i = \bar{X}_i \frac{\partial}{\partial r}, \\
\frac{\partial}{\partial r} X_{n+1} &= X_{n+1} \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial r} X_{n+1} = X_{n+1} \frac{\partial}{\partial r} + \sqrt{-1} r^{-2} \frac{\partial}{\partial x}, \\
\frac{\partial}{\partial r} \bar{X}_{n+1} &= \bar{X}_{n+1} \frac{\partial}{\partial r} + \sqrt{-1} r^{-2} \frac{\partial}{\partial x},
\end{align*}
\]

(5.8)

and

\[
\begin{align*}
\frac{\partial}{\partial r} (K^{\alpha\beta} X_\alpha \bar{X}_\beta \psi) &= \frac{\partial}{\partial r} (K^{\alpha\beta} X_\alpha \bar{X}_\beta \psi) + K^{\alpha\beta} \frac{\partial}{\partial r} (X_\alpha \bar{X}_\beta \psi) \\
&= -r^{-2} h^{ij} X_i \bar{X}_j \psi + 4X_{n+1} \bar{X}_{n+1} \psi + K^{ij} X_i \bar{X}_j \psi \\
&+ K^{n+1+n+1} \frac{\partial}{\partial r} (X_{n+1} \bar{X}_{n+1} \psi) \\
&= -r^{-1} K^{\alpha\beta} X_\alpha \bar{X}_\beta \psi + 2 \frac{\partial^2 \psi}{\partial x^2} + K^{\alpha\beta} X_\alpha \bar{X}_\beta \left( \frac{\partial}{\partial r} \right),
\end{align*}
\]

(5.9)

where the condition \( \xi \psi \equiv 0 \) has been used. Thus,

\[
\frac{\partial}{\partial r} (\Delta_{K} \psi) = 2 \frac{\partial}{\partial r} (K^{\alpha\beta} X_\alpha \bar{X}_\beta \psi) + 2(n+1) \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2}.
\]

\[
= \Delta_{K} (\frac{\partial \psi}{\partial r}) - r^{-1} \Delta_{K} \psi + 2(n+1) r^{-1} \frac{\partial \psi}{\partial r} + 4 \frac{\partial^2 \psi}{\partial x^2}.
\]
Suppose \( \psi \) is a solution of equation (2.17) and \( \Omega_\psi \) is positive. Let \( \tilde{g} \) be the Hermitian metric induced by positive (1, 1) form \( \Omega_\psi \). From above,
\[
\tilde{g}(X_\alpha, \bar{X}_\beta) = \tilde{g}(X_\alpha, \bar{X}_\beta) + \frac{1}{2} r^2 X_\alpha \bar{X}_\beta \psi.
\]
Hence,
\[
\frac{1}{2} Tr_K \tilde{g} = \tilde{g}(X_\alpha, \bar{X}_\beta) K_{\alpha\beta}^\gamma > \tilde{g}(X_{n+1}, \bar{X}_{n+1}) K^{n+1\beta} = \tilde{g}(X_{n+1}, \bar{X}_{n+1}) K^{n+1\beta} + r \tilde{g} + \frac{1}{2} r^2 \tilde{g} \psi.
\]
In what follows, the Kähler metric \( K \) will be considered as the background metric. Let \( p \) be a point of \( M \times [1, \frac{3}{2}] \), choose a normal holomorphic local coordinate \( (z^1, \cdots, z^{n+1}) \) centered at \( p \), and such that \( K_{\alpha\beta}(p) = \delta_{\alpha\beta}, \quad dK_{\alpha\beta}(p) = 0 \). By the definition, \( K_{\alpha\beta} = r \delta_{\alpha\beta} \) and \( \tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{2}{r} \tilde{g}_{\alpha\beta} - \frac{r^2}{2} \partial_{z^\beta} K_{\alpha\beta} \). We may also assume that \( \{\tilde{g}_{\alpha\beta}\} \) is diagonal at the point \( p \). For two fixed metrics \( K \) and \( \tilde{g} \), there exist two positive constants \( d_1 \) and \( d_2 \) such that
\[
d_1 \tilde{g} \leq K \leq d_2 \tilde{g}.
\]
By direct calculation,
\[
0 < 2 r^{-2} Tr_K \tilde{g} = 2 r^{-2} Tr_K \tilde{g} + \Delta_K \psi - 2(n + 1) \frac{\partial \psi}{\partial r}
\]
(5.12)
\[
\leq r^{-2} \frac{1}{d_1}(n + 1) + \Delta_K \psi - 2(n + 1) \frac{\partial \psi}{\partial r}
\]
\[
\leq \frac{4}{d_1}(n + 1) + \Delta_K \psi - 2(n + 1) \frac{\partial \psi}{\partial r}
\]
Set
\[
\zeta = 2 + \frac{4}{d_1}(n + 1) + \Delta_K \psi - 2(n + 1) \frac{\partial \psi}{\partial r}
\]
and
\[
u = \log \zeta + A_1|\partial \psi|_{K}^2 - A_2 \psi,
\]
where constants \( A_1 \) and \( A_2 \) are chosen sufficiently large. Denote the Chern connection of the Hermitian metric \( \tilde{g} \) by \( \nabla \), and the canonical Laplacian corresponding with the connection \( \nabla \) by \( \Delta \).

**Lemma 7.** There exist positive constants \( B_2, B_3 \) and \( B_4 \) depending only on \( r, \max_{M \times [1, \frac{3}{2}]} |\partial \psi|_{K}^2, \|\nabla \psi\|_{C^1(M \times [1, \frac{3}{2}])} \), metric \( K \) and metric \( \tilde{g} \) such that
\[
\frac{1}{2} \Delta u \geq -\frac{1}{2} A_2 \Delta \psi - B_2 + Tr_{\tilde{g}} K([-B_1(1 + \zeta^{-1} \frac{\partial \psi}{\partial r}) - (n + 3) \zeta^{-1} \frac{\partial^2 \psi}{\partial r^2}]
\]
- \( A_1 B_3 - B_4 + \frac{1}{2} \zeta^{-2} \frac{\partial^2 \psi}{\partial r^2} \Delta_K \psi
\]
+ \( A_1 - 4(n + 1) - \frac{1}{4} n^2 \sum_{\alpha, \gamma} (\tilde{g}^{\gamma\gamma} |\psi_{\alpha\gamma}|^2 + \tilde{g}^{\gamma\gamma} |\psi_{\alpha\gamma}|^2)\).

**Proof.** With the local coordinate picked above,
\[
\frac{1}{2} \Delta u = \frac{1}{2} \Delta (\log \zeta + A_1|\partial \psi|_{K}^2 - A_2 \psi)
\]
(5.16)
\[
= \tilde{g}^{\gamma\delta} (\log \zeta + A_1|\partial \psi|_{K}^2 - A_2 \psi) \gamma_{\delta}
\]
\[
= \tilde{g}^{\gamma\delta} (\log \zeta) \gamma_{\delta} + A_1 \tilde{g}^{\gamma\delta} (K^{\alpha\beta} \psi_{\alpha\beta}) \gamma_{\delta} - A_2 \tilde{g}^{\gamma\delta} \psi_{\delta}
\]
\[
= \zeta^{-1} \tilde{g}^{\gamma\delta} \gamma_{\delta} - \zeta^{-2} \tilde{g}^{\gamma\delta} \zeta_{\delta} - A_1 \tilde{g}^{\gamma\delta} (K^{\alpha\beta} \psi_{\alpha\beta}) \gamma_{\delta} - A_2 \tilde{g}^{\gamma\delta} \psi_{\delta}.
\]
At the point $p$,

\begin{equation}
(5.17)
\zeta^{-1} \tilde{g}^{\delta} \zeta_{\delta} = 2 \zeta^{-1} \tilde{g}^{\delta}(K^{\alpha \beta} \psi_{\alpha \beta} - (n + 1) \frac{\partial \psi}{\partial r})_{\alpha \beta},
\end{equation}

\begin{equation}
(5.18)
= 2 \zeta^{-1} \tilde{g}^{\delta} \{K^{\alpha \beta} \psi_{\alpha \beta} + K^{\alpha \beta} \psi_{\alpha \beta \gamma \delta} - (n + 1) \frac{\partial \psi}{\partial r})_{\alpha \beta} \}
\end{equation}

By Lemma 6,

\begin{equation}
(5.19)
2 \zeta^{-1} \tilde{g}^{\delta} K^{\alpha \beta} r_{\gamma \delta}(\frac{\partial \psi}{\partial r})_{\alpha \beta} = \frac{1}{2} \zeta^{-1}(Tr_{K} K) \Delta_{K} (\frac{\partial \psi}{\partial r}) + r^{-1} \Delta_{K} \psi - 2(n + 1) r^{-1} \frac{\partial \psi}{\partial r} - 4 \Omega_{\psi},
\end{equation}

It follows from equation (2.17),

\begin{equation}
(5.20)
4 r^{-2} \zeta^{-1} \tilde{g}^{\delta} K^{\alpha \beta} \tilde{g}_{\gamma \delta} \psi_{\alpha \beta} = 4 r^{-2} \zeta^{-1} K^{\alpha \beta} (\tilde{g}^{\delta} \tilde{g}_{\gamma \delta})_{\alpha \beta} - 4 r^{-2} \zeta^{-1} K^{\alpha \beta} (\tilde{g}^{\delta})_{\alpha \beta} \tilde{g}_{\gamma \delta},
\end{equation}

and

\begin{equation}
(5.21)
A_{1} \tilde{g}^{\delta} (K^{\alpha \beta} \psi_{\alpha \beta})_{\gamma \delta} = A_{1} \tilde{g}^{\delta} K^{\alpha \beta} \psi_{\alpha \beta} + A_{1} \tilde{g}^{\delta} K^{\alpha \beta} (\psi_{\alpha \gamma} \psi_{\beta \delta} + \psi_{\alpha \delta} \psi_{\beta \gamma}) + A_{1} \tilde{g}^{\delta} K^{\alpha \beta} (\psi_{\alpha \gamma} \psi_{\beta \delta} + \psi_{\alpha \delta} \psi_{\beta \gamma})
\end{equation}

Note that $Tr_{K} \geq (Tr_{K} \tilde{g}) \frac{1}{n+1} f^{\frac{1}{n+1}} \geq \frac{1}{2} (Tr_{K} \tilde{g}) f^{\frac{1}{n+1}}$. Since $f^{\frac{1}{n+1}} \in C^{1,1}$,

\begin{equation}
(5.22)
| \frac{f_{\psi}}{f}(\beta) |_{\beta} (Z) | + | \frac{f_{\psi}}{f}(Z) |^{2} \leq C f^{\frac{1}{2}} (Z) \leq 2 C \frac{Tr_{K} (Z) \alpha \beta}{\alpha \beta} \forall Z \in M \times [1, \frac{3}{2}].
\end{equation}
On the other hand,

\[
\zeta_\gamma = (\Delta K \psi - 2(n + 1) \frac{\partial \psi}{\partial r})_\gamma \\
= 2K^{\alpha \beta}(\psi_\alpha - \frac{\partial \psi}{\partial r} r_\alpha)_\gamma \\
= 2\{K^{\alpha \beta}(\psi_\alpha - \frac{\partial \psi}{\partial r} r_\alpha)\}_\gamma \\
= 2\{K^{\alpha \beta}(\psi_\alpha - \frac{\partial \psi}{\partial r} r_\alpha) + K^{\alpha \beta}(\frac{\partial \psi}{\partial r})_\alpha r_\beta - (n + 1)(\frac{\partial \psi}{\partial r})_\gamma \} \\
= 2\{K^{\alpha \beta}(2r^{-2}(\bar{g}_{\gamma \beta} - \bar{g}_{\gamma \beta}) - n(\frac{\partial \psi}{\partial r})_\gamma \} \\
= 4r^{-2}K^{\alpha \beta} \bar{g}_{\gamma \beta, \alpha} - 4r^{-2}K^{\alpha \beta} \bar{g}_{\gamma \beta, \alpha} - 8r^{-3}K^{\alpha \beta} \bar{g}_{\gamma \beta} r_\alpha \\
+ 8r^{-3}K^{\alpha \beta} \bar{g}_{\gamma \beta} r_\alpha - n(\frac{\partial \psi}{\partial r}).
\]

By the Schwarz inequality, at point \( p \),

\[
-c^{-2}g^{\gamma \delta} \zeta_\gamma \zeta_\delta = -c^{-2}g^{\gamma \delta} \zeta_\gamma \zeta_\delta \\
\geq -16(1 + \sigma)r^{-4} \zeta^{-2}(\sum \bar{g}_{\gamma \alpha, \alpha} | \sum \bar{g}_{\gamma \alpha, \alpha} |^2) \\
-64(1 + \sigma^{-1})r^{-6} \zeta^{-2}(\sum \bar{g}_{\gamma \alpha, \alpha} | \sum \bar{g}_{\gamma \alpha, \alpha} |^2) \\
-64(1 + \sigma^{-1})r^{-6} \zeta^{-2}(\sum \bar{g}_{\gamma \alpha, \alpha} | \sum \bar{g}_{\gamma \alpha, \alpha} |^2) \\
-64(1 + \sigma^{-1})n^{-2} \zeta^{-2}(\sum \bar{g}_{\gamma \alpha, \alpha} | \sum \bar{g}_{\gamma \alpha, \alpha} |^2) \\
-4(1 + \sigma^{-1})n^{-2} \zeta^{-2}(\sum \bar{g}_{\gamma \alpha, \alpha} | \sum \bar{g}_{\gamma \alpha, \alpha} |^2).
\]

and

\[
| \sum \bar{g}_{\gamma \alpha, \alpha} |^2 = \sum (\frac{1}{\sqrt{g_{\alpha, \alpha}}}) | \sum \bar{g}_{\gamma \alpha, \alpha} |^2 \\
\leq (\sum \bar{g}_{\gamma \alpha, \alpha} | \sum \bar{g}_{\gamma \alpha, \alpha} |^2)(\sum \bar{g}_{\gamma \alpha, \alpha}) \\
= \frac{1}{2}(Tr K \bar{g})(\sum \bar{g}_{\gamma \alpha, \alpha})^2.
\]

In turn,

\[
4r^{-2} \zeta^{-1}K^{\alpha \beta} \bar{g}_{\gamma \beta} \bar{g}_{\gamma \beta, \alpha} \\
= 4r^{-2} \zeta^{-1} \sum \alpha, \gamma, \delta | \bar{g}_{\gamma \alpha, \alpha} |^2 \\
= 4r^{-2} \zeta^{-1} \sum \gamma, \delta | \bar{g}_{\gamma \alpha} |^2 \\
\geq 8r^{-2}(\zeta(Tr \bar{g}))^{-1}(\sum \gamma | \bar{g}_{\gamma \alpha} |^2) \\
= 16r^{-4} \zeta^{-2}(1 + \frac{1}{\sqrt{2} \sqrt{Tr \bar{g}}})(\sum \gamma | \bar{g}_{\gamma \alpha} |^2).
\]

In local holomorphic coordinates, from (5.5),

\[
\frac{\partial}{\partial r} = (2r)^{-1} \nabla K r = (2r)^{-1} (K^{\tau \eta} \frac{\partial}{\partial z^\tau} \frac{\partial}{\partial z^\eta} + K^{\tau \eta} \frac{\partial}{\partial z^\tau} \frac{\partial}{\partial z^\eta}).
\]

Thus,

\[
\frac{\partial}{\partial r} = \frac{\partial \psi}{\partial r} \psi + ((2r)^{-1} K^{\tau \eta} r_\tau) \psi_\eta + (((2r)^{-1} K^{\tau \eta} r_\eta) \psi_\tau \\
= (2r)^{-1} (K^{\tau \eta} r_\tau \psi_\eta + K^{\tau \eta} r_\eta \psi_\tau) \\
+ (((2r)^{-1} K^{\tau \eta} r_\eta) \psi_\eta + ((2r)^{-1} K^{\tau \eta} r_\eta) \psi_\tau),
\]

and

\[
\frac{\partial}{\partial r} = \frac{\partial \psi}{\partial r} \psi \delta + ((2r)^{-1} K^{\tau \eta} r_\tau) \psi_\eta + ((2r)^{-1} K^{\tau \eta} r_\eta) \psi_\tau \\
= + ((2r)^{-1} K^{\tau \eta} r_\eta) \psi_\eta + ((2r)^{-1} K^{\tau \eta} r_\eta) \psi_\tau.
\]
Combine (5.16)–(5.21), (5.28) and (5.29),

\[
\frac{1}{2} \Delta u \\
\geq -\frac{1}{2} A_2 \Delta \psi - B_1 (1 + \frac{\alpha}{\beta}) \Delta K K - B_2 - (n + 3) \zeta^{-1} \frac{\partial \psi}{\partial r} \partial_r K \\
\geq -A_1 B_2 \Delta r K + \frac{1}{2} A_1 (\Delta r K K) K^{\alpha \beta} \left[ \frac{\partial \psi}{\partial r} \psi_\beta + \frac{\partial \psi}{\partial r} \psi_\alpha \right] \\
+ \frac{1}{2} \zeta^{-1} (\Delta K \psi) + 4r^{-2} \zeta^{-1} K \zeta g^{-\gamma} g_r \partial_r \zeta g^{-\gamma} g_r \partial_r \zeta \\
+ \left( A_1 - 4(n + 1) \right) \left\{ \sum_{\alpha, \gamma} (\zeta g^{-\gamma} \psi_\alpha)^2 + (\bar{g}^{-\gamma} \psi_\alpha)^2 \right\},
\]

where positive constants \( B_1, B_2, B_3 \) depend only on \( r, \max_{M \times [1, \frac{R}{2}]} |d\psi|^2, \|f^+\|_{C^{1,1}}, \) metric \( K \) and metric \( \bar{g} \). From (5.24), (5.26) and (5.28), we may pick a constant \( B_4 \) depending only on \( r, \) metric \( K \) and metric \( \bar{g} \), such that

\[
0 \geq -B_4 \Delta r K - \frac{1}{2} n^2 \left\{ \sum_{\alpha, \gamma} (\bar{g}^{-\gamma} \psi_\alpha)^2 + (\bar{g}^{-\gamma} \psi_\alpha)^2 \right\}.
\]

The lemma now follows from (5.30) and (5.31).

We are ready to prove the following estimate.

**Proposition 4.** Let \( \psi \) be a solution of (1.11) for some \( 0 < \epsilon \leq 1 \) with \( \Omega_\psi > 0 \). Let \( \zeta \) be defined as in (5.13). There exist constants \( A_1, A_2 \) and \( A_3 \) depending only on \( r, \|f^+\|_{C^{1,1}(M \times [1, \frac{R}{2}])}, \max_{M \times [1, \frac{R}{2}]} |\psi|, \max_{M \times [1, \frac{R}{2}]} |d\psi|^2, \) metric \( K \) and metric \( \bar{g} \), such that if the maximum value of \( u \) defined in (5.14) is achieved at an interior point \( p \), then \( u(p) \leq A_3 \). As a consequence, for any \( 0 < f \in C^\infty_b(M \times [1, \frac{R}{2}]) \) and basic boundary value \( \psi_0 \), there exists constant \( C \) depending only on \( \|f^+\|_{C^{1,1}(M \times [1, \frac{R}{2}])}, \|\psi_0\|_{C^{2,1}}, \) and metric \( \bar{g} \), such that

\[
\|\psi\|_{C^3_b} \leq C.
\]

**Proof.** At interior \( p \) where \( u \) attains maximum value,

\[
0 = \frac{\partial u}{\partial r} = \zeta^{-1} \frac{\partial \psi}{\partial r} + A_1 \zeta K \zeta \left[ \frac{\partial \psi}{\partial r} \psi_\beta + \frac{\partial \psi}{\partial r} \psi_\alpha \right] - A_2 \frac{\partial \psi}{\partial r} \\
= -2(n + 1) \frac{\partial \psi}{\partial r} - \frac{1}{2} A_2 \frac{\partial \psi}{\partial r} \partial_r K \\
= -2(n + 1) \frac{\partial \psi}{\partial r} \partial_r K - 2(n + 1) A_2 \partial^2 \psi \partial_r \partial_r K \\
+ \left\{ \sum_{\alpha, \gamma} (\zeta g^{-\gamma} \psi_\alpha)^2 + (\bar{g}^{-\gamma} \psi_\alpha)^2 \right\}.
\]

By (5.10),

\[
\zeta \geq 2 + 2r^{-2} \partial_r K \bar{g} > 2 \frac{\partial^2 \psi}{\partial r^2} + 2.
\]

From (5.15), at point \( p \),

\[
0 \geq \frac{1}{2} \Delta u \\
\geq A_2 r^{-2} \partial_r \bar{g} - 2(n + 1) A_2 r^{-2} - \frac{1}{2} A_2 \frac{\partial \psi}{\partial r} \partial_r K \\
+ \left\{ \sum_{\alpha, \gamma} (\zeta g^{-\gamma} \psi_\alpha)^2 + (\bar{g}^{-\gamma} \psi_\alpha)^2 \right\}.
\]
Pick $A_1 = 4(n+1) + \frac{1}{2}n^2$ and $A_2 = \frac{n}{4}(2+B_1+B_4+A_1B_3)d_2$, the above inequality yields at point $p$,

$$(5.36) \quad Tr_\gamma K \leq B_2 + 2(n+1)A_2.$$ 

On the other hand,

$$(5.37) \quad \left(\frac{1}{2} Tr_\gamma K\right)^n \geq \frac{1}{2} \left(Tr_\gamma \tilde{g} \frac{\det(K_{\alpha\beta})}{\det(g_{\alpha\beta})}\right) = \frac{1}{2} \left(Tr_\gamma \tilde{g} \frac{\det(K_{\alpha\beta})}{\det(g_{\alpha\beta})}\right) \geq \frac{1}{2} \left(Tr_\gamma \tilde{g}\right) f^{-1}(d_1)^{n+1},$$

and

$$2Tr_\gamma \tilde{g} \geq 2r^{-2}Tr_\gamma \tilde{g} = 2r^{-2}Tr_\gamma \tilde{g} + \Delta_K \psi - 2(n+1)\frac{\partial \psi}{\partial r}$$

$$(5.38) \quad \geq 4r^{-2} \frac{1}{\delta^2}(n+1) + \Delta_K \psi - 2(n+1)\frac{\partial \psi}{\partial r} = \delta - 2 - 4r^{-2} \frac{1}{\delta^2}(n+1)$$

$$\geq \delta - 2 - \frac{4}{\delta}(n+1) + \frac{16}{9\delta^2}(n+1).$$

Since we already have estimated $|\psi|_{C^1}$, $|\Delta g \psi|$, $Tr_\gamma \tilde{g}$ and $|\Delta_K \psi|$ are all equivalent. $C^2_w$ bound follows directly. \(\square\)

We have established $C^2_w$ bound for any smooth solution $\psi$ of equation (2.17). For each $f > 0$, equation (2.17) is strictly elliptic and concave. For this point, the theory of Evans and Krylov can be applied. In fact, with sufficient smooth boundary data, for a uniformly elliptic and concave fully nonlinear equation, the assumption of $u \in C^{1,\gamma}$ for some $\gamma > 0$ is sufficient to get global $C^{2,\gamma}$ regularity (e.g., see Theorem 7.3 in [10]). The higher order estimates follow from the standard elliptic theory. By Lemma 2, the kernel of the linearized operator of (2.17) with null boundary data is trivial. The linearized equation is solvable by the Fredholm alternative. It should be pointed out that higher regularity estimates depend on positive lower bound of $f$, while $C^2_w$ estimate is independent of it. The first part of Theorem 1 follows from the method of continuity.

We discuss the uniqueness of $C^2_w$ solutions of the Dirichlet problem (1.10) and prove the second part of Theorem 1.

**Lemma 8.** Suppose $\psi$ is a $C^2_w$ function defined on $M \times [1, \frac{1}{2}]$ with $\Omega_\psi \geq 0$ defined in (1.9). For any $\delta > 0$, there is a function $\psi_\delta \in C^\infty(M \times [1, \frac{1}{2}])$ such that $\delta \bar{\omega} \geq \Omega_{\psi_\delta} > 0$ and $\|\psi - \psi_\delta\|_{C^2(M \times [1, \frac{1}{2}])} \leq \delta$, where $\bar{\omega}$ is the Kähler form on $M$ and $||.||_{C^2(M \times [1, \frac{1}{2}])}$ is defined as in (1.13).

**Proof.** $\psi \in C^2_w(M)$ implies that $\Omega_\psi$ is bounded (as $||.||_{C^2_w}$ controls the complex hessian). For any $\epsilon > 0$, set $\psi_\epsilon = (1-\epsilon)\psi + \epsilon r$ where $r$ is a radial function in the Kähler cone $C(M)$. It is obvious $\Omega_{\psi_\epsilon} > 0$ and it is also bounded. We now approximate $\psi_\epsilon$ by a smooth function $\psi_\delta$ such that $\|\psi_\epsilon - \psi_\delta\|_{C^2(M)} \leq \epsilon^2$. It is clear that we can make $\Omega_{\psi_\delta} > 0$ and $|\Omega_{\psi_\delta} - \Omega_\psi|$ as small as we wish by shrinking $\epsilon$. \(\square\)

**Lemma 9.** $C^2_w$ solutions to the degenerate Monge-Ampère equation (1.10) with given boundary data are unique.
Proof. Suppose there are two such solutions $\psi_1, \psi_2$ with the same boundary data. For any $0 < \delta < 1$, pick any $0 < \delta_1, \delta_2 < \delta$ by Lemma 8, there exist two smooth functions $\psi_i'$ and $\psi_2'$ such that

$$\Omega_{\psi_i'}^{n+1} = f_i \omega^{n+1}$$

in $M \times [1, \frac{3}{2}]$, $\max_{M \times [1, \frac{3}{2}]} |\psi_i' - \psi_i| \leq \delta_i$ and $0 < f_i < \delta_i$ for $i = 1, 2$. Set $\tilde{\psi}_i' = (1 - \delta)\psi_i' + \delta r$, where $r$ is the radial function on $\bar{M}$. Since $\Omega_{\tilde{\psi}_i'}^{n+1} \geq \delta^{n+1} \omega^{n+1}$ and $\Omega_{\psi_2'}^{n+1} = 0$, a.e., we may choose $\delta_2$ sufficient small such that $0 < f_2 \omega^{n+1} \leq \Omega_{\bar{\psi}_1'}^{n+1}$. The maximum principle implies $\max_{M \times [1, \frac{3}{2}]} (\tilde{\psi}_1' - \psi_2') \leq \max_{\partial M \times [1, \frac{3}{2}]} (\tilde{\psi}_1' - \psi_2')$. Thus

$$\max_{M \times [1, \frac{3}{2}]} (\psi_1 - \psi_2) \leq \max_{\partial M \times [1, \frac{3}{2}]} (\psi_1 - \psi_2) + C\delta = C\delta,$$

where constant $C$ depends only on $C^0$ norm of $\psi_1$ and $\psi_2$. Interchange the role of $\psi_1$ and $\psi_2$, we have

$$\max_{M \times [1, \frac{3}{2}]} |\psi_1 - \psi_2| \leq C\delta.$$

Since $0 < \delta < 1$ is arbitrary, we conclude that $\psi_1 = \psi_2$. \qed

The proof of Theorem 1 is complete.

Remark 2. One may deal with geodesic equation (1.7) in the setting of transverse Kähler geometry. Complexifying time variable $t$ as in [28, 36, 13], one arrives a homogeneous complex Monge-Ampère equation in transverse Kähler setting. There is no problem to carry out interior estimates for this type of equation as in Kähler case [6]. But it is difficult to obtain the boundary regularity estimates including the direct gradient estimates. The approach via equation (2.17) puts the problem in the frame of degenerate elliptic complex Monge-Ampère. The analysis developed here should be useful to deal complex Monge-Ampère type equations in other contexts.

6. Applications

As in the case of the space of Kähler metrics [6], the regularity result of the geodesic equation has geometric implications on the Sasakian manifold $(M, g)$. One of them is the uniqueness of transverse Kähler metric with constant scalar curvature in the given basic Kähler class. The discussion here follows similar way as in [6]. The proofs of technical lemmas 10-14 can be found in the Appendix.

Let us recall the definition of the natural connection on the space $\mathcal{H}$ in [24].

Definition 3. Let $\varphi(t) : [0, 1] \to \mathcal{H}$ be any path in $\mathcal{H}$ and let $\psi(t)$ be another basic function on $M \times [1, \frac{3}{2}]$, which we regard as a vector field along the path $\varphi(t)$. Define the covariant derivative of $\psi$ along the path $\varphi$ by

$$D_{\tilde{\varphi}} \psi = \frac{\delta \psi}{\delta t} - \frac{1}{4} g_{\tilde{\varphi}} \langle d_B \psi, d_B \tilde{\varphi} \rangle_{g_{\tilde{\varphi}}},$$

where $\langle \cdot, \cdot \rangle_{g_{\tilde{\varphi}}}$ is the Riemannian inner product on co-tangent vectors to $(M, g_{\tilde{\varphi}})$, and $\tilde{\varphi} = \frac{\partial}{\partial t}$. 

The geodesic equation (1.7) can be written as
\begin{equation}
D \dot{\phi} = 0.
\end{equation}

It is shown in [24] that the connection \(D\) is compatible with the Weil-Peterson metric structure and torsion free; the sectional curvature of \(D\) is formally non-negative, and \(\mathcal{H}_0 \subset \mathcal{H}\) is totally geodesic and totally convex.

Let \(K\) be the space of all transverse Kähler form in the basic \((1,1)\) class \([d\eta]_B\), then the natural map
\begin{equation}
\mathcal{H} \rightarrow K, \quad \varphi \mapsto \frac{1}{2}(d\eta + \sqrt{-1}\partial_B \bar{\partial}_B \varphi)
\end{equation}
is surjective. Normalize \(\int_M \eta \wedge (d\eta)^n = 1\). Define a function \(I: \mathcal{H} \rightarrow \mathbb{R}\) by
\begin{equation}
I(\varphi) = \sum_{p=0}^{n} \frac{n!}{(p+1)!(n-p)!} \int_M \varphi \eta \wedge (d\eta)^{n-p} \wedge (\sqrt{-1}\partial_B \bar{\partial}_B \varphi)^p.
\end{equation}

Set
\begin{equation}
\mathcal{H}_0 = \{ \varphi \in \mathcal{H} | I(\varphi) = 0 \},
\end{equation}
then
\begin{equation}
\mathcal{H}_0 \cong K, \quad \varphi \mapsto \frac{1}{2}(d\eta + \sqrt{-1}\partial_B \bar{\partial}_B \varphi),
\end{equation}
and
\begin{equation}
\mathcal{H} \cong \mathcal{H}_0 \times \mathbb{R}, \quad \varphi \mapsto (\varphi - I(\varphi), I(\varphi)).
\end{equation}

Recall for a given Sasakian structure \((\xi, \eta, \Phi, g)\), the exact sequence of vector bundles,
\begin{equation}
0 \rightarrow L \xi \rightarrow TM \rightarrow \nu(F_\xi) \rightarrow 0,
\end{equation}
generates the Reeb foliation \(F_\xi\) (where \(L \xi\) is the trivial line bundle generated by the Reeb field \(\xi\) and \(\nu(F_\xi)\) is the normal bundle of the foliation \(F_\xi\)). The metric \(g\) gives a bundle isomorphism \(\sigma_{\eta}: \nu(F_\xi) \rightarrow \mathcal{D}\), where \(\mathcal{D} = \ker[\eta]\) is the contact subbundle. \(\Phi|_{\mathcal{D}}\) induces a complex structure \(\bar{\eta}\) on \(\nu(F_\xi)\). Since the Nijenhuis torsion tensor of \(\Phi\) satisfies
\begin{equation}
N_{\Phi}(X,Y) = -d\eta(X,Y) \otimes \xi.
\end{equation}

So, \((\nu(F_\xi), \bar{\eta}) \cong (\mathcal{D}, \Phi|_{\mathcal{D}})\) gives \(F_\xi\) a transverse holomorphic structure. Then \((\mathcal{D}, \Phi|_{\mathcal{D}}, d\eta)\) give \(M\) a transverse Kähler structure with transverse Kähler form \(\frac{1}{2}d\eta\) and transverse metric \(g^T\) defined by
\begin{equation}
g^T(\cdot, \cdot) = \frac{1}{2}d\eta(\cdot, \Phi\cdot)
\end{equation}
which is related to the Sasakian metric \(g\) by
\begin{equation}
g = g^T + \eta \otimes \eta.
\end{equation}

For simplicity, we will denote the bundle metric \(\sigma^*g^T\) by \(g^T\) if there is no confusion. We will identify \(\nu(F_\xi)\) and \(\mathcal{D}\) and \(\sigma_{\eta} = id\) if there is no confusion. The transverse metric \(g^T\) induces a transverse Levi-Civita connection on \(\nu(F_\xi)\) by
\begin{equation}
\nabla^\mathcal{D}_X Y = \begin{cases} 
(\nabla_X Y)^p, & X \in \mathcal{D}, \\
[\xi, Y]^p, & X = \xi,
\end{cases}
\end{equation}
where $Y$ is a section of $\mathcal{D}$ and $X^p$ the projection of $X$ onto $\mathcal{D}$, $\nabla$ is the Levi-Civita connection of metric $g$. It is easy to check that the connection satisfies
\[
\nabla^T_X Y - \nabla^T_Y X - [X,Y]^p = 0, \quad X g^T(Z,W) = g^T(\nabla^T_X Z, W) + g^T(Z, \nabla^T_X W),
\]
$\forall X, Y \in TM, Z, W \in \mathcal{D}$. This means that the transverse Levi-Civita connection is torsion-free and metric compatible. The transverse curvature relating with the above transverse connection is defined by
\[
(6.12) \quad R^T(V,W)Z = \nabla^T_V \nabla^T_W Z - \nabla^T_W \nabla^T_V Z - \nabla^T_{[V,W]} Z,
\]
where $V, W \in TM$ and $Z \in \mathcal{D}$. The transverse Ricci curvature is defined as
\[
(6.13) \quad \text{Ric}^T(X,Y) = \langle \text{R}^T(X, e_i) e_i, Y \rangle_g,
\]
where $e_i$ is an orthonormal basis of $\mathcal{D}$ and $X, Y \in \mathcal{D}$. The following is held
\[
(6.14) \quad \text{Ric}^T(X,Y) = \text{Ric}(X,Y) + 2g^T(X,Y), \quad X, Y \in \mathcal{D}.
\]
A Sasakian metric $g$ is said to be $\eta$-Einstein if $g$ satisfies
\[
(6.15) \quad \text{Ric}_g = \lambda g + \nu \eta \otimes \eta,
\]
for some constants $\lambda, \nu \in R$. It is equivalent to be transverse Einstein in the sense that
\[
(6.16) \quad \text{Ric}^T = c g^T,
\]
for certain constant $c$. The trace of transverse Ricci tensor is called the transverse scalar curvature and will be denoted by $S^T$.

Let $\rho^T(\cdot, \cdot) = \text{Ric}^T(\cdot, \cdot)$ and $\rho = \text{Ric}^T(\Phi \cdot, \cdot)$, $\rho^T$ is called the transverse Ricci form. They satisfy the relation
\[
(6.17) \quad \rho^T = \rho + d\eta.
\]
$\rho^T$ is a closed basic $(1,1)$ form and the basic cohomology class $[\frac{1}{\pi} \rho^T]_B = C^B_1(M)$ is the basic first Chern class. The basic first Chern class of $M$ is called positive (resp. negative, null) if $C^B_1(M)$ contains a positive (resp. negative, null) representation, and this condition is expressed by $C^B_1(M) > 0$ (resp. $C^B_1(M) < 0$, $C^B_1(M) = 0$).

**Definition 4.** Fixed a transverse holomorphic structure $(\nu(F^\xi), J)$ on the characteristic foliation $F^\xi$. A complex vector field $X$ on $M$ is called a transverse holomorphic vector field if it satisfies:

1. $\pi(\xi, X) = 0$;
2. $J(\pi(X)) = \sqrt{-1} \pi(X)$;
3. $\pi([Y, X]) = \sqrt{-1} J \pi([Y, X]) = 0$, $\forall Y$ satisfying $\tilde{J} \pi(Y) = -\sqrt{-1} \pi(Y)$,

where $\pi$ is the projection $\pi : TM \to \nu(F^\xi)$. Given a transverse Kähler form $d\eta$. Let $\psi$ be a complex valued basic function, then there is a unique vector field $V_\psi(\psi) \in \Gamma(T^* M)$ satisfies: (1) $\tilde{J}(\pi(V_\psi(\psi))) = -\sqrt{-1} \pi(V_\psi(\psi))$; (2) $\psi = \sqrt{-1} \eta(V_\psi(\psi))$; (3) $\bar{\partial}_B \psi = -\frac{\sqrt{-1}}{2} d\eta(V_\psi(\psi), \cdot)$. The vector field $V_\psi(\psi)$ is called the Hamiltonian vector field of $\psi$ corresponding to the transverse Kähler form $d\eta$.

With the local coordinate chart and the function $h$ chosen as in (2.5), the transverse Ricci form can be expressed by
\[
\rho^T = -\sqrt{-1} \bar{\partial}_B \partial_B \log \det(\bar{g}^T) = -\sqrt{-1} \bar{\partial}_B \frac{\partial^2}{\partial z^i \partial \bar{z}^j}(\log \det(h_{kl})) dz^i \wedge d\bar{z}^j.
\]
In this setting, \( \forall \varphi \in \mathcal{H}, \) we have
\[
\begin{align*}
\eta_{\varphi} &= dx - \sqrt{-1}((h_{ij} + \frac{1}{2}\varphi_{ij})dz^j - (h_{ij} + \frac{1}{2}\varphi_{ij})\bar{dz}^j); \\
\Phi_{\varphi} &= \sqrt{-1}(Y_j^i \otimes dz^i - \tilde{Y}_j^i \otimes d\bar{z}^i); \\
g_{\varphi} &= \eta \otimes \eta + 2(h + \frac{4}{3}\varphi)_{ij}dz^i \wedge d\bar{z}^j \\
d\eta_{\varphi} &= 2\sqrt{-1}(h + \frac{1}{2}\varphi)_{ij}dz^i \wedge d\bar{z}^j, \\
g_{\varphi} &= 2(h + \frac{1}{3}\varphi)_{ij}(dz^i \wedge d\bar{z}^j), \\
\rho^T &= -\sqrt{-1}\frac{\partial^2}{\partial x^2}(\log \det((h + \frac{1}{2}\varphi)_{k\bar{l}}))dz^i \wedge d\bar{z}^j.
\end{align*}
\]

where \( Y_j^i = \frac{\partial}{\partial x^i} + \sqrt{-1}(h_{ij} + \frac{1}{2}\varphi_{ij})\frac{\partial}{\partial \bar{z}^j} \) and \( \tilde{Y}_j^i = \frac{\partial}{\partial x^i} - \sqrt{-1}(h_{ij} + \frac{1}{2}\varphi_{ij})\frac{\partial}{\partial \bar{z}^j}. \)

**Remark 3.** A complex vector field \( X \) on the Sasakian manifold \((M, \xi, \eta, \Phi, g)\) is transverse holomorphic if and only if it satisfies:

1. \( \Phi(X - \eta(X)\xi) = \sqrt{-1}(X - \eta(X)\xi), \)
2. \( [X, \xi] = \eta([X, \xi]_\xi) \) or equivalently \( \nabla^T_X(X - \eta(X)\xi) = 0; \)
3. \( \nabla^T_{Y - \eta(Y)\xi}(X - \eta(X)\xi) = 0, \forall Y \) satisfying \( Y - \eta(Y)\xi \in \mathcal{D}^{0,1}. \)

In local coordinate \((x, z^1, \ldots, z^n)\) as in (2.3), the transverse holomorphic vector field \( X \) can be written as
\[
X = \eta(X)\frac{\partial}{\partial x} + \sum_{i=1}^n X^i\left(\frac{\partial}{\partial x^i} - \eta\left(\frac{\partial}{\partial x^i}\right)\frac{\partial}{\partial \bar{z}^i}\right),
\]
where \( X^i \) are local holomorphic basic functions, and \( \varphi \in \mathcal{H}. \)

The Hamiltonian vector field \( V_{\eta_{\varphi}}(\psi) \) of the basic function \( \psi \) with respect to the Sasakian structure \((\xi, \eta_{\varphi}, \Phi_{\varphi}, g_{\varphi})\) (i.e. with respect to the transverse Kähler form \( d\eta_{\varphi} \)) can be written as
\[
V_{\eta_{\varphi}}(\psi) = -\sqrt{-1}\psi\frac{\partial}{\partial x} + \sum_{i=1}^n h_{\varphi}^{ij}\psi\frac{\partial}{\partial z^j} - \eta_{\varphi}(\frac{\partial}{\partial z^j})\frac{\partial}{\partial \bar{z}^i},
\]
where \( h_{\varphi}^{ij}(h_{\varphi})_{kj} = \delta^j_k, (h_{\varphi})_{kj} = h_{kj} + \frac{1}{2}\varphi_{kj} \) and \( \varphi \in \mathcal{H}. \) In general, \( V_{\eta_{\varphi}}(\psi) \) is not transversely holomorphic. If define \( \partial_B V_{\eta_{\varphi}}(\psi) \in \Gamma(\wedge^B(M) \otimes (\nu_{\mathcal{F}_x})^{1,0}) \) by
\[
\partial_B V_{\eta_{\varphi}}(\psi) = (h_{\varphi}^{ij}\psi_j)_k dz^k \otimes \frac{\partial}{\partial z^j},
\]
\( V_{\eta}(\psi) \) is transversely holomorphic if and only if \( \partial_B V_{\eta}(\psi) = 0. \) In local coordinates (2.3), it is equivalent to
\[
\frac{\partial}{\partial z^i}(h_{\varphi}^{ij}\psi_j)_k dz^k = 0, \quad \forall i, k.
\]

**Lemma 10.** Let \((M, \xi, \eta, \Phi, g)\) be a Sasakian manifold and \( \psi \) be a real basic function on \( M. \) Assuming that \( V_{\eta_{\varphi}}(\psi) \) is transverse holomorphic for some \( \varphi \in \mathcal{H}, \) where \( V_{\eta_{\varphi}}(\psi) \) is the Hamiltonian vector field of \( \psi \) corresponding with the Sasakian structure \((\xi, \eta_{\varphi}, \Phi_{\varphi}, g_{\varphi})\). If the basic first Chern class \( C^B(M) \leq 0, \) then \( \psi \) must be a constant.

\( \forall \varphi \in \mathcal{H}, (\xi, \eta_{\varphi}, \Phi_{\varphi}, g_{\varphi}) \) defined in (1.3) and (1.4) is also a Sasakian structure on \( M. \) \((\xi, \eta_{\varphi}, \Phi_{\varphi}, g_{\varphi})\) and \((\xi, \eta, \Phi, g)\) have the same transversely holomorphic structure on \( \nu(\mathcal{F}_x) \) and the same holomorphic structure on the cone \( C(M), \) and their transverse Kähler forms are in the same basic \((1, 1)\) class \([d\eta]_B \) (Proposition 4.2 in [18] ). This class is called the **basic Kähler class** of the Sasakian
manifold \((M, \xi, \eta, \Phi, g)\). All these Sasakian metrics have the same volume, as 
\[ \int_M \eta \wedge (d\eta)^n = \int_M \eta \wedge (d\eta)^n = 1 \quad \text{(e.g., section 7 of [1])}. \]

Let \(\rho^\Sigma\) denote the transverse Ricci form of the Sasakian structure \((\xi, \eta_\Sigma, \Phi, g_\Sigma)\). 
\[ \int_M \rho^\Sigma \wedge (d\eta_\Sigma)^2 \wedge \eta_\Sigma \text{ is independent of the choice of } \varphi \in \mathcal{H} \quad \text{(e.g., Proposition 4.4 [18])}. \] This means that 
\[ (6.19) \quad \bar{S} = \frac{\int_M S^\Sigma (d\eta_\Sigma)^n \wedge \eta_\Sigma}{\int_M (d\eta_\Sigma)^n \wedge \eta_\Sigma} = \frac{\int_M 2n \rho^\Sigma \wedge (d\eta_\Sigma)^{n-1} \wedge \eta_\Sigma}{\int_M (d\eta_\Sigma)^n \wedge \eta_\Sigma}, \]
depends only on the basic Kähler class. As in the Kähler case (see [27]), we have the following lemma.

**Lemma 11.** Let \(\varphi'\) and \(\varphi''\) are two basic functions in \(\mathcal{H}\) and \(\varphi, (t \in [a, b])\) be a path in \(\mathcal{H}\) connecting \(\varphi'\) and \(\varphi''\). Then 
\[ (6.20) \quad \mathcal{M}(\varphi', \varphi'') = - \int_a^b \int_M \bar{\varphi}_t (S^\Sigma_t - \bar{S})(d\eta_t)^n \wedge \eta_t \frac{dt}{t} \]
is independent of the path \(\varphi_t\), where \(\bar{\varphi}_t = \frac{\partial}{\partial t} \varphi_t\), \(S^\Sigma_t\) is the transverse scalar curvature to the Sasakian structure \((\xi, \eta_t, \Phi_t, g_t)\) and \(\bar{S}\) is the average defined as in (6.19). Furthermore, \(\mathcal{M}\) satisfies the 1-cocycle condition and 
\[ (6.21) \quad \mathcal{M}(\varphi' + C', \varphi'' + C'') = \mathcal{M}(\varphi', \varphi'') \]
for any \(C', C'' \in \mathbb{R}\).

In view of (6.21), \(\mathcal{M} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}\) factors through \(\mathcal{H}_0 \times \mathcal{H}_0\). Hence we can define the mapping \(\mathcal{M} : \mathcal{K} \times \mathcal{K} \to \mathbb{R}\) by the identity \(\mathcal{K} \cong \mathcal{H}_0\) 
\[ (6.22) \quad \mathcal{M}(d\eta', d\eta'') := \mathcal{M}(\varphi', \varphi''), \]
where \(d\eta', d\eta'' \in \mathcal{K}\), and \(\varphi', \varphi'' \in \mathcal{H}\) such that 
\[ d\eta' = d\eta + \sqrt{-1} \partial_B \partial_B \varphi' \quad \text{and} \quad d\eta'' = d\eta + \sqrt{-1} \partial_B \partial_B \varphi'' . \]

**Definition 5.** The mapping 
\[ (6.23) \quad \mu : \mathcal{K} \to \mathbb{R}, \quad d\eta \mapsto \mu(d\eta) := \mathcal{M}(d\eta, d\eta) \]
is called the \(\mathcal{K}\)-energy map of the transverse Kähler class in \([d\eta]_B\). The mapping \(\mu : \mathcal{H} \to \mathbb{R}, \mu(\varphi) := \mathcal{M}(0, \varphi)\) is also called by the \(\mathcal{K}\)-energy map of the space \(\mathcal{H}\).

**Lemma 12.** For every smooth path \(\{\varphi_t|a \leq t \leq b\}\), we have 
\[ (6.24) \quad \frac{d^2}{dt^2} \mu(\varphi_t) = -(D^2 \varphi_t \bar{\varphi}_t, S^\Sigma_t - \bar{S}) \varphi_t + \int_M \frac{1}{2} (\partial_B V_{\varphi_t}(\varphi_t) |^2 \eta_t^n) \wedge \eta_t. \]
where \(\partial_B V_{\varphi_t}(\varphi_t) = (h^{ij}(\varphi_t) \bar{\varphi}_t) i^j \partial_k \eta_t \otimes \frac{\partial}{\partial x_k}\) in local coordinates. \(\mu : \mathcal{H} \to \mathbb{R}\) is a convex function, i.e. the Hessian of \(\mu\) is nonnegative everywhere on \(\mathcal{H}\).

If \(C^B_1(M) \leq 0\), by the transverse Calabi-Yau theorem in [16], there exists \(\tilde{\varphi} \in \mathcal{H}\) such that the transverse Ricci curvature \(\tilde{\text{Ric}}^T\) of the transverse Kähler metric \(g^T_\Sigma\) is nonpositive, where \(g^T_\Sigma\) is induced by the Sasakian structure \((\xi, \tilde{\eta}_\Sigma, \tilde{\Phi}_\Sigma, g_\Sigma)\). One may assume that \(\tilde{\text{Ric}}^T < 0\) if \(C^B_1(M) < 0\); and \(\tilde{\text{Ric}}^T \equiv 0\) if \(C^B_1(M) = 0\). We will take \(\eta_\Sigma\) as the background contact form, we write \(\eta\) for \(\eta_\Sigma\).
For any two points $\varphi_0, \varphi_1 \in \mathcal{H}$, by Theorem 1, there is an $\epsilon$-approximate geodesic $\varphi(t)$ satisfying (1.12). We have

$$
(6.25) \quad \rho^T_\varphi - \rho^T = \sqrt{\partial_B \partial_B \log Q},
$$

where $\rho^T_\varphi$ and $\rho^T$ are the transverse Ricci forms of $g^T_\varphi$ and $g^T$ respectively, and $Q = \varphi^n - \frac{1}{4} d_B \varphi^2 |_{g_\varphi}^2$. Then,

$$
\begin{align*}
(6.26) \quad & \int_M S^T_\varphi Q \eta_\varphi \wedge (\eta_\varphi)^n = \int_M 2nQ \rho^T_\varphi \wedge (\eta_\varphi)^{n-1} \wedge \eta_\varphi \\
& + \int_M 2nQ \rho^T \wedge (\eta_\varphi)^{n-1} \wedge \eta_\varphi \\
& = - \int_M \left| \partial_B Q \right|^2 (\eta_\varphi)^n \wedge \eta_\varphi + \int_M Q \text{tr}_{g_\varphi}(\tilde{\text{Ric}}^T)(\eta_\varphi)^n \wedge \eta_\varphi.
\end{align*}
$$

Consider the $\mathcal{K}$ energy map $\mu$ on $\mathcal{H}$, we have

$$
(6.27) \quad \frac{d}{dt} \mu(\varphi(t)) = - (\varphi', S^T_\varphi - \tilde{S})_{\varphi(t)}.
$$

By Lemma 12 and (6.25),

$$
\begin{align*}
(6.28) \quad & \frac{\partial^2}{dt^2} \mu(\varphi(t)) = - (D\varphi \varphi', S^T_\varphi - \tilde{S})_{\varphi} + \int_M \frac{1}{2} \partial_B V_{g_\varphi}(\varphi')(\eta_\varphi)^2 (d\eta_\varphi)^n \wedge \eta_t \\
& + \int_M \left| \partial_B Q \right|^2 (\eta_\varphi)^n \wedge \eta_\varphi - \int_M Q \text{tr}_{g_\varphi}(\tilde{\text{Ric}}^T)(\eta_\varphi)^n \wedge \eta_\varphi.
\end{align*}
$$

**Definition 6.** A smooth path $\varphi_1$ in the space $\mathcal{H}$ is called an $\epsilon$-approximate geodesic if the following holds:

$$
(6.29) \quad \frac{\partial^2 \varphi}{dt^2} - \frac{1}{4} d_B \frac{\partial \varphi}{\partial |_{g_\varphi}} \eta_\varphi \wedge (d\eta_\varphi)^n = f_e \eta \wedge (d\eta)^n,
$$

where $d\eta_\varphi = d\eta + \sqrt{\partial_B \partial_B \phi} > 0$ and $0 \leq f_e < \epsilon$.

Theorem 1 guarantees the existence of $\epsilon$-approximate geodesic for any two points $\varphi_0, \varphi_1 \in \mathcal{H}$.

**Lemma 13.** For any two different points $\varphi_0, \varphi_1 \in \mathcal{H}$, the geodesic distance between them is positive.

**Lemma 14.** Let $\varphi_i(s) : [0, 1] \to \mathcal{H}$ ($i = 0, 1$) are two smooth curves in $\mathcal{H}$. For any $0 < \epsilon \leq 1$, there exist two parameter families of smooth curves $C(t, s, \epsilon) : \varphi(t, s, \epsilon) : [0, 1] \times [0, 1] \times (0, 1] \to \mathcal{H}$ such that the following properties hold:

1. Let $\psi_{s, \epsilon}(r, \cdot) = \varphi(2(r - 1), s, \epsilon) + 4 \log r \in C^\infty(M \times [1, \frac{3}{2}])$ solving

$$
(\Omega_{s, \epsilon})^{n+1} = \epsilon \omega^{n+1}
$$

with boundary conditions: $\psi_{s, \epsilon}(1, \cdot) = \varphi_0(s, \cdot)$ and $\psi_{s, \epsilon}(\frac{3}{2}, \cdot) = \varphi_1(s, \cdot)$, and $\Omega_0 > 0$.

2. There exists a uniform constant $C$ which depends only on $\varphi_0$ and $\varphi_1$ such that

$$
|\psi| + \frac{\partial \varphi}{dt} + |\partial \varphi| \leq C; \quad 0 < \frac{\partial^2 \varphi}{dt^2} \leq C; \quad \frac{\partial^2 \varphi}{\partial s^2} \leq C.
$$

3. For fixed $s$, let $\epsilon \to 0$, the curve $C(s, \epsilon)$ converge to the unique weak geodesic connecting $\varphi_0(s)$ and $\varphi_1(s)$ in the weak $C^{1,1}$ topology.
(4) Define the energy element along \( \varphi(t, s, \epsilon) \in \mathcal{H} \) as

\[
E(t, s, \epsilon) = \int_M \frac{\partial \varphi}{\partial t}^2 \, dv(\varphi(t, s, \epsilon))
\]

where \( dv(\varphi) = \eta_\varphi \wedge (d\eta_\varphi)^n \). There exist a uniform constant \( C \) which independent of \( \epsilon \), such that

\[
|\frac{\partial E}{\partial t}| \leq C \epsilon.
\]

**Proof of Theorem 2.** For any \( \epsilon > 0 \), by Lemma 14 there exist two parameter families of smooth curves \( C(t, s, \epsilon) : \tilde{\varphi}(t, s, \epsilon) \in \mathcal{H} \) such that it satisfies \((\Omega_\psi)^{n+1} = \epsilon \omega^n \) or equivalently

\[
\varphi(0, s, \epsilon) = \varphi^* \quad \text{and} \quad \tilde{\varphi}(1, s, \epsilon) = \varphi(s).
\]

For each \( s \) fixed, denote the length of curve \( \tilde{\varphi}(t, s, \epsilon) \) form \( \varphi^* \) to \( \varphi(s) \) by \( L(s, \epsilon) \), and denote the length from \( \varphi(0) \) to \( \varphi(s) \) along curve \( C \) by \( l(s) \). In what follows, we assume that energy element \( E > 0 \) (we may replace \( \sqrt{E} \) by \( \sqrt{E + \delta} \) and let \( \delta \to 0 \)). We compute

\[
\frac{dL(s, \epsilon)}{ds} = \int_0^1 \frac{dt}{\sqrt{E(t, s, \epsilon)}} \left( D_s \frac{\partial \tilde{\varphi}}{\partial t} + \frac{\partial \tilde{\varphi}}{\partial s} \right) \tilde{\varphi} = \int_0^1 \frac{dt}{\sqrt{E(t, s, \epsilon)}} \left( D_s \frac{\partial \tilde{\varphi}}{\partial t} + \frac{\partial \tilde{\varphi}}{\partial s} \right) \tilde{\varphi} = \int_0^1 \frac{dt}{\sqrt{E(t, s, \epsilon)}} \sqrt{\left( D_s \frac{\partial \tilde{\varphi}}{\partial t} + \frac{\partial \tilde{\varphi}}{\partial s} \right)^2} \tilde{\varphi} = \int_0^1 \frac{dt}{\sqrt{E(t, s, \epsilon)}} \left( D_s \frac{\partial \tilde{\varphi}}{\partial t} + \frac{\partial \tilde{\varphi}}{\partial s} \right) \tilde{\varphi} \geq \int_0^1 \frac{dt}{\sqrt{E(t, s, \epsilon)}} \left( D_s \frac{\partial \tilde{\varphi}}{\partial t} + \frac{\partial \tilde{\varphi}}{\partial s} \right) \tilde{\varphi} |_{t=1} = C \epsilon,
\]

and

\[
\frac{dl(s)}{ds} = \sqrt{\left( \frac{\partial \tilde{\varphi}}{\partial s} \right)^2} \tilde{\varphi} = \sqrt{\left( \frac{\partial \tilde{\varphi}}{\partial s} \right)^2} \tilde{\varphi} |_{t=1} = 1.
\]

Set \( F(s, \epsilon) = L(s, \epsilon) + l(s) \). By the Schwartz inequality, \( \frac{dF(s, \epsilon)}{ds} \geq -C \epsilon \). In turn, \( F(s, \epsilon) - F(0, \epsilon) \geq -C \epsilon \). Letting \( \epsilon \to 0 \),

\[
d(\varphi^*, \varphi(0)) \leq d(\varphi^*, \varphi(s)) + d(\varphi(0), \varphi(s)).
\]

The triangle inequality in the Theorem can be deduced from the above inequality by choosing appropriate \( \epsilon \)-approximate geodesics.

We now verify the second part of the theorem. By taking \( \varphi^* = \varphi(1) \) in the triangle inequality, we know that the geodesic distance is no greater than the length of any curve connecting the two end points. Then, Lemma 13 implies that \((\mathcal{H}, d)\) is a metric space. We only need to show the differentiability of the distance function. Suppose \( \varphi^* \neq \varphi(s_0) \), from (6.30), we have

\[
\left| \frac{dL(s, \epsilon)}{ds} - \frac{1}{\sqrt{E(1, s, \epsilon)}} \left( \frac{\partial \tilde{\varphi}}{\partial s} \frac{\partial \tilde{\varphi}}{\partial t} \right) \tilde{\varphi} |_{t=1} \right| \leq C \epsilon
\]

Let \( \epsilon \to 0 \), it follows that

\[
\frac{d}{ds} d(\varphi^*, \varphi(s)) |_{s=s_0} = \lim_{\epsilon \to 0} \frac{1}{\sqrt{E(1, s_0, \epsilon)}} \left( \frac{\partial \tilde{\varphi}}{\partial s} \frac{\partial \tilde{\varphi}}{\partial t} \right) \tilde{\varphi} |_{t=1, s=s_0}.
\]
Proof of Theorem 3. Let $\mathcal{K}$ be the space of all transverse Kähler metrics in the same basic Kähler class, we know $\mathcal{K} \equiv \mathcal{H}_0 \subset \mathcal{H}$. Suppose $\varphi_0 \in \mathcal{H}$ satisfy $S^T_{\varphi_0} \equiv \text{constant}$. For any point $\varphi_1 \in \mathcal{H}$, let $\varphi(t)$ be an $\epsilon$-approximate geodesic as defined in (1.12). Since $\text{Ric}^T$ is nonpositive by the assumption, (6.28) implies,

\begin{equation}
\frac{d^2}{dt^2} \mu(\varphi(t)) < -\epsilon C,
\end{equation}

where $C$ is a uniform constant. On the other hand, since $S^T_{\varphi_0} \equiv \text{constant}$,

\begin{equation}
\frac{d}{dt} \mu(\varphi(t))|_{t=0} = 0.
\end{equation}

Hence

\begin{equation}
\mu(\varphi(t)) - \mu(\varphi(0)) \geq -\epsilon C t^2.
\end{equation}

Let $t = 1$ and $\epsilon \to 0$, we have $\mu(\varphi_1) \geq \mu(\varphi_0)$. The first part of the theorem is proved since $\varphi_1$ is arbitrary.

Let $\varphi_0$ and $\varphi_1$ be two constant scalar curvature transverse Kähler metrics in the same basic Kähler class $\mathcal{K}$. By the identity between $\mathcal{K}$ and $\mathcal{H}_0 \subset \mathcal{H}$, we can consider $\varphi_0$ and $\varphi_1$ as two functions in $\mathcal{H}$. Let $\{\varphi(t)|t \in [0,1]\}$ be a $\epsilon$-approximate geodesic in $\mathcal{H}$ and satisfies (1.12). Integrating (6.28) from $t = 0$ to $t = 1$,

\begin{equation}
(\frac{d}{dt} \mu(\varphi(t)))|_{t=0}^1 = \int_0^1 \int_M \frac{1}{2} |\overline{\partial}_B V_{\varphi'}(\varphi')|^2 Q^{-1} (d\eta_B)^n \wedge \eta_B dt + \epsilon \dot{S} + \int_0^1 \int_M |\overline{\partial}_B |Q|_{\varphi'}^2 Q^{-1} (\eta_{\varphi})^n \wedge \eta_{\varphi} dt - \int_0^1 \int_M \text{tr}_{g_{\varphi}}(\text{Ric}^T)(\eta_{\varphi})^n \wedge \eta_{\varphi} dt.
\end{equation}

Since $\varphi_0$ and $\varphi_1$ are two metrics with transverse constant scalar curvature, by (7.5),

\begin{equation}
(\frac{d}{dt} \mu(\varphi(t)))|_{t=0}^1 = 0. \quad (1.12) \text{and (6.33)} \implies
\end{equation}

\begin{equation}
\int_0^1 \int_M \left\{\frac{1}{2} |\overline{\partial}_B V_{\varphi'}(\varphi')|^2 Q^{-1} + |\overline{\partial}_B \log Q|_{\varphi'}^2 \right\} (d\eta_B)^n \wedge \eta_B dt = \int_0^1 \int_M \text{tr}_{g_{\varphi}}(\text{Ric}^T)(d\eta)^n \wedge \eta dt - \dot{S}.
\end{equation}

If $C^B(M) = 0$, then the constant $\dot{S} = 0$, by the initial assumption, $\text{Ric}^T = 0$. Consequently

\begin{equation}
\int_0^1 \int_M \left\{\frac{1}{2} |\overline{\partial}_B V_{\varphi'}(\varphi')|^2 Q^{-1} + |\overline{\partial}_B \log Q|_{\varphi'}^2 \right\} (d\eta_B)^n \wedge \eta_B dt = 0
\end{equation}

This implies the Hamiltonian vector field $V_{\varphi'}(\varphi')$ is transversely holomorphic. By Lemma 10, $\varphi'(t)$ is constant for each $t$. Therefore $\varphi_0$ and $\varphi_1$ represent the same transverse Kähler metric. That is, there exists at most one constant scalar curvature transverse Kähler metric in each basic Kähler class when $C^B(M) = 0$.

If $C^B(M) < 0$, then $\text{Ric}^T < -cg^T$ for some positive constant $c$. By (6.34), we have

\begin{equation}
\int_0^1 \int_M \left\{\frac{1}{2} |\overline{\partial}_B V_{\varphi'}(\varphi')|^2 Q^{-1} + |\overline{\partial}_B \log Q|_{\varphi'}^2 \right\} (d\eta_B)^n \wedge \eta_B dt \
\leq \int_0^1 \int_M \text{tr}_{g_{\varphi}}(g^T)(d\eta)^n \wedge \eta dt - \dot{S}.
\end{equation}

where $\dot{S}$ is a negative constant depending only the basic Kähler class. Following the same discussion in [6] (section 6.2), we may argue $\overline{\partial}_B V_{\varphi'}(\varphi') = 0$ in some weak sense. The following is a sketch of proof.
From the estimates in Theorem 1 and (5.10), there exist an uniform positive constant $C$ which independent on $\epsilon$, such that $Q \leq \varphi' \leq C$. In what follows, we will denote $C$ as an uniform constant under control, and set
\begin{equation}
(6.37) \quad d\bar{v} = (d\eta)^n \wedge \eta, \quad X = V_{\eta}\varphi' - \eta(V_{\eta'}\varphi')\xi.
\end{equation}

First we have an integral estimate on $Q^{2\bar{v}}$ ($1 < q < 2$) with respect to the measure $dvdt$:
\begin{equation}
(6.38) \quad \int_{M \times [1, \frac{1}{2}]} Q^{\frac{2\bar{v}}{q}} dvdt \leq C \int_{M \times [1, \frac{1}{2}]} Q^{\frac{2\bar{v}}{q}} dvdt
\end{equation}
\begin{equation}
= C \int_{M \times [1, \frac{1}{2}]} (Q^{\frac{\det g}{\det g}})^{\frac{2\bar{v}}{q}} \cdot (\frac{\det g}{\det g})^{\frac{2\bar{v}}{q}} dvdt
\end{equation}
\begin{equation}
\leq C \epsilon \int_{M \times [1, \frac{1}{2}]} \text{tr}_{g_\epsilon}(g^T) dvdt \to 0.
\end{equation}

The following inequality shows that vector field $X$ is uniformly bounded in $L^2$ with respect to the measure $dvdt$.
\begin{equation}
(6.39) \quad \int_{M \times [1, \frac{1}{2}]} |X|^2 q dvdt = \int_{M \times [1, \frac{1}{2}]} g_{\alpha\beta} g_{\gamma\delta} (\varphi')^\gamma (\varphi')^\delta dvdt
\end{equation}
\begin{equation}
\leq \int_{M \times [1, \frac{1}{2}]} \text{tr}_{g_\epsilon}(g^T)|d_B \varphi|^2_q dvdt \leq C.
\end{equation}

A direct calculation yields
\begin{equation}
(6.40) \quad \int_{M \times [1, \frac{1}{2}]} |\partial_B V_\varphi(\varphi')|^2_q dvdt = \int_{M \times [1, \frac{1}{2}]} |\partial_B V_\varphi(\varphi')|^2_q Q^{-\frac{2\bar{v}}{q}} dvdt
\end{equation}
\begin{equation}
\leq \{ \int_{M \times [1, \frac{1}{2}]} |\partial_B V_\varphi(\varphi')|^2_q Q^{-1} dvdt \}^{\frac{2\bar{v}}{q}} \{ \int_{M \times [1, \frac{1}{2}]} Q^{\frac{2\bar{v}}{q}} dvdt \}^{\frac{2\bar{v}}{q}} \to 0.
\end{equation}

Therefore, $|\partial_B V_\varphi(\varphi')|$ can be viewed as a function in $L^2(M \times [1, \frac{1}{2}])$. It has a weak limit in $L^2$ and it’s $L^q$ ($1 < q < 2$) norm tends to 0 as $\epsilon \to 0$.

As above, let $\mathcal{D} = ker\{g\}$ be the contact sub-bundle with respect to the Sasakian structure $(\xi, \eta, \Phi, g)$. Let $Y \in \Gamma(\mathcal{D}^{1,0})$. Choose a local coordinate $(x, z^1, \cdots, z^2)$ on the Sasakian manifold $M$. For $Y = Y^i (\frac{\partial}{\partial z^i} - \eta(\frac{\partial}{\partial z^i}))\xi$, define $\partial_B Y \in \Gamma(\wedge^1_B \mathcal{M}) \otimes \mathcal{D}^{1,0}$ by $\frac{\partial Y^i}{\partial z^j} d\tilde{z}^j \otimes (\frac{\partial}{\partial z^i} - \eta(\frac{\partial}{\partial z^i}))\xi$. One may check that
\begin{equation}
(6.41) \quad |\partial_B X|_g \leq C \sqrt{\text{tr}_{g_\epsilon} g |\partial_B V_\varphi(\varphi')|_{g_\epsilon}},
\end{equation}

\begin{equation}
(6.42) \int_{M \times [1, \frac{1}{2}]} |d_B \log \frac{\det g}{\det g_\epsilon}|^2_q dvdt = \int_{M \times [1, \frac{1}{2}]} |d_B \log Q|^2_q dvdt
\end{equation}
\begin{equation}
\leq C \int_{M \times [1, \frac{1}{2}]} |d_B \log Q|^2_{g_\epsilon} dvdt \leq C,
\end{equation}

and
\begin{equation}
(6.43) \quad \int_{M \times [1, \frac{1}{2}]} \left( \frac{\det g}{\det g_\epsilon} \right)^{\frac{2\bar{v}}{q}} dvdt \leq C \int_{M \times [1, \frac{1}{2}]} \text{tr}_{g_\epsilon} g^T dvdt \leq C.
\end{equation}

By the $C^0_u$ estimate theorem 1, there is $c > 0$ such that $e^{-c |\frac{\det g}{\det g_\epsilon}|} \leq 1$. Now define a vector field $Y$ by
\begin{equation}
(6.44) \quad Y = X e^{-c |\frac{\det g}{\det g_\epsilon}|}.
\end{equation}

We have
\begin{equation}
(6.45) \quad |Y|_g = e^{-c |X|_g} \frac{\det g}{\det g_\epsilon} \leq C.
\end{equation}
and

\begin{equation}
\int_{M \times [1, \frac{3}{2}]} \left| \partial_B Y - \partial_B (\log \frac{\det g^2}{\det g^0}) \right| Y^q \, dv \, dt
\end{equation}

\begin{align*}
&= \int_{M \times [1, \frac{3}{2}]} (\partial_B X) \frac{\det g^2}{\det g^0} \, dv \, dt \\
&= \int_{M \times [1, \frac{3}{2}]} \left( \sqrt{\frac{\det g^2}{\det g^0}} \right)^q \partial_B V_{e^i} \varphi'^q \, dv \, dt \\
&= C \int_{M \times [1, \frac{3}{2}]} \left| \partial_B V_{e^i} \varphi' \right|^q \, dv \, dt \to 0
\end{align*}

for any $0 < q < 2$.

Note that $X, Y, \partial_B Y$ and $\frac{\det g^2}{\det g^0}$ are geometric quantities which depend on $c$, and their respect Sobolev norms are uniformly bounded. We have $X(\epsilon) \to X$ weakly in $L^2(M \times [1, \frac{3}{2}])$, $Y(\epsilon) \to Y$ weakly in $L^\infty(M \times [1, \frac{3}{2}])$ and $\frac{\det g^2}{\det g^0}(\epsilon) \to u$ weakly in $L^\infty(M \times [1, \frac{3}{2}])$, as $\epsilon \to 0$. Furthermore, in local coordinate $(x, z^1, \ldots, z^n)$, since $\nabla^2_i X(\epsilon) \equiv 0$ and $\xi(\det g^2/\det g^0)(\epsilon) \equiv 0$ for any $\epsilon$, then functions $v, X^i$ and $Y^i$ are all independent of $x$, where $X = X^i(\partial / \partial x^i - \eta(\partial / \partial y^i))$ and $Y = Y^i(\partial / \partial x^i - \eta(\partial / \partial y^i))$.

Let $v = -\log u + c$. With the choice of $c$ in the definition of $Y$ in (6.44), $v \geq 0$ and it satisfies the following two equations

\begin{align}
\partial_B Y + \partial_B v \otimes Y &= 0, \quad \text{and} \quad Y = X e^{-v}
\end{align}

in the sense of $L^q$ for any $1 < q < 2$. From (6.39), (6.42) and (6.43), we have the following estimates

\begin{equation}
\int_{M \times [1, \frac{3}{2}]} |X|^2 + e^v + |\partial_B v|^2 \, dv \, dt \leq C.
\end{equation}

Define a new sequence of vector fields $X_k = Y \sum_{i=0}^k \frac{\partial}{\partial t^i}$, where $Y$ is defined by (6.44). This is well defined since $v \in L^p(M \times [1, \frac{3}{2}])$ for any $p > 1$. It’s easy to check that:

\begin{equation}
\|X_k\|^2 \leq \|X_m\|^2 - \|X_m - X_k\|^2 \leq \|X_m\|^2 - \|X_k\|^2 < \infty
\end{equation}

where $k < m$. Thus, $X_k$ is a Cauchy sequence in $L^2(M \times [1, \frac{3}{2}])$ and there exists a strong limit $X_\infty$ in $L^2(M \times [1, \frac{3}{2}])$. By definition, one may check that $X_\infty = X$ in the sense of $L^q$ for any $1 < q < 2$. In local coordinate ($x, z^1, \ldots, z^n$) as in (2,3) in an open set $U$, the functions $X_k^i$ are all invariant in $x$ direction, where $X_\infty = X^i(\partial / \partial x^i - \eta(\partial / \partial y^i))$. For any vector valued smooth function $\theta = (\theta^1, \ldots, \theta^n)$ supported in $U \times [0, 1]$, and any $1 \leq j \leq n$, we have

\begin{align*}
&\left| \int_U \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} (\theta^j) \right| \\
&= \lim_{k \to \infty} \left| \int_{U \times [0, 1]} \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} (\theta^j) \right| \\
&= \lim_{k \to \infty} \left| \int_{U \times [0, 1]} \sum_{i=1}^n (X^i \to X^j - X_{k-1}^i) \frac{\partial}{\partial x^j} \theta^j \right| \\
&\leq \lim_{k \to \infty} C \|X_k - X_{k-1}\|_{L^2} = 0.
\end{align*}

The above implies that component functions $X_k^i$ are weak holomorphic and $x$-invariant. That is $X_\infty$ is a weak transverse holomorphic vector field for almost all $t \in [0, 1]$. Recall that $\|X_\infty\|_{L^2(M \times [1, \frac{3}{2}])} \leq C$. This implies that $X_\infty$ is in $L^2(M)$ for almost all $t \in [0, 1]$. Therefore $X_\infty$ must be transverse holomorphic for almost all $t \in [0, 1]$. Since $\text{Ric}^T < -ce^T$ for some positive constant $c$, by (7.2) in lemma 10, $X_\infty(t) \equiv 0$ for those $t$ where $X_\infty(t)$ is transverse holomorphic. Thus $X \equiv 0$. We
conclude that \( \varphi' \) is constant for each \( t \) fixed. Therefore, \( \varphi_0 \) and \( \varphi_1 \) differ only by a constant, and they represent the same transverse Kähler metric. \( \square \)

7. Appendix

We now provide proof of results listed in the previous section following the same argument as in Chen [6], here we make use of our Theorems 1.

**Proof of Lemma 10.** By the transverse Calabi-Yau theorem in [16], there is a function \( \varphi_0 \in \mathcal{H} \), such that

\[
\rho_{\varphi}^T = -\sqrt{-1} \partial_B \bar{\partial}_B \log \det(g_{0}^T) \leq 0,
\]

where \( \rho_{\varphi}^T \) is the transverse Ricci form corresponding to the new Sasakian structure \((\xi, \eta_{\varphi_0}, \Phi_{\varphi_0}, g_{\varphi_0})\). Let \( \Delta_0 \) be the Laplacian corresponding to the metric \( g_{\varphi_0} \), and choosing a local coordinate \((x, z^1, \cdots, z^n)\) as in (2.3). Since \( V_{n_{\varphi}}(\psi) \) is transverse holomorphic,

\[
\begin{aligned}
\Delta_{0}\left[V_{n_{\varphi}}(\psi)^{\perp}\right]_{g_{0}}^{2} &= \nabla^{2} d V_{n_{\varphi}}(\psi)^{\perp}_{g_{0}}(\xi, \xi) \\
&= 2 \nabla^{2} \nabla d V_{n_{\varphi}}(\psi)^{\perp}_{g_{0}}(Y, \bar{Y}) \\
&= \left(\nabla^{T}(V_{n_{\varphi}}(\psi)^{\perp})\right)_{g_{0}}^{2} - 2 \text{Ric}_{0}^{T}(V_{n_{\varphi}}(\psi)^{\perp}, V_{n_{\varphi}}(\psi)^{\perp})
\end{aligned}
\]

where \( V_{n_{\varphi}}(\psi)^{\perp} = V_{n_{\varphi}}(\psi) - \eta_{\varphi_{0}}(V_{n_{\varphi}}(\psi))\xi \) is the projection of \( V_{n_{\varphi}}(\psi) \) to \( \ker \eta_{\varphi_{0}} \), and \( Y = \frac{\partial}{\partial x} - \eta_{\varphi_{0}}(\frac{\partial}{\partial x}) \frac{\partial}{\partial x} \) is a basis of \( \ker \eta_{\varphi_{0}} \). From above inequality, we have \( |V_{n_{\varphi}}(\psi)^{\perp}|_{g_{0}}^{2} \equiv \text{constant} \). On the other hand, by Remark 3,

\[
V_{n_{\varphi}}(\psi)^{\perp} = h_{\varphi}^{ij} \frac{\partial \psi}{\partial z^{i}} Y_{j}.
\]

If \( \psi \) achieve the maximum value at some point \( P \), then \( |V_{n_{\varphi}}(\psi)^{\perp}|_{g_{0}}^{2} = 0 \) at \( P \). Therefore \( d \psi \equiv 0 \), that is, \( \psi \equiv \text{constant} \). \( \square \)

**Proof of Lemma 11.** Let \( \varphi : [a, b] \to \mathcal{H} \) be a smooth path connecting \( \varphi' \) and \( \varphi'' \). Define \( \psi(s, t) = s \varphi_t \in \mathcal{H}, (s, t) \in [0, 1] \times [a, b] \). Consider

\[
\theta = \left(\frac{\partial \psi}{\partial s}, S_{\psi}^{T} - \bar{S}\right) \delta s + \left(\frac{\partial \psi}{\partial t}, S_{\psi}^{T} - \bar{S}\right) \delta t,
\]

where \( S_{\psi}^{T} \) is the transverse scalar curvature respect to the Sasakian structure \((\xi, \eta_{\psi}, \Phi_{\psi}, g_{\psi})\). A direct calculation yields

\[
\left(\frac{\partial \psi}{\partial s}, D_{\partial \psi} S_{\psi}^{T}\right)_{\psi} = \left(\frac{\partial \psi}{\partial t}, D_{\partial \psi} S_{\psi}^{T}\right)_{\psi},
\]

and

\[
\begin{aligned}
\frac{\partial}{\partial s} \left(\frac{\partial \psi}{\partial s}, S_{\psi}^{T} - \bar{S}\right)_{\psi} &= (D_{\frac{\partial \psi}{\partial s}} \frac{\partial \psi}{\partial s}, S_{\psi}^{T} - \bar{S})_{\psi} + (\frac{\partial \psi}{\partial s}, D_{\frac{\partial \psi}{\partial s}} S_{\psi}^{T})_{\psi} \\
&= (D_{\frac{\partial \psi}{\partial s}} \frac{\partial \psi}{\partial s}, S_{\psi}^{T} - \bar{S})_{\psi} + (\frac{\partial \psi}{\partial s}, D_{\frac{\partial \psi}{\partial s}} S_{\psi}^{T})_{\psi} \\
&= \frac{\partial}{\partial s} \left(\frac{\partial \psi}{\partial s}, S_{\psi}^{T} - \bar{S}\right)_{\psi}.
\end{aligned}
\]

Therefore, \( \theta \) is a closed one form on \([0, 1] \times [a, b]\). Thus, following the same discussion as in [27], we have:

\[
\int_{a}^{b} (\varphi, S_{\psi}^{T} - \bar{S})_{\psi} dt = \int_{0}^{1} (\varphi, S_{\psi_{\varphi}}^{T} - \bar{S})_{\psi_{\varphi}} ds|_{\varphi_{\varphi}} = \int_{0}^{1} (\varphi, S_{\psi_{\varphi}}^{T} - \bar{S})_{\psi_{\varphi}} ds|_{\varphi_{\varphi}}.
\]
that is, $\mathcal{M}(\varphi', \varphi'')$ is independent of the path $\varphi_t$, and $\mathcal{M}$ satisfies 1-cocycle condition, and it satisfies:

$$\mathcal{M}(\varphi_0, \varphi_1) + \mathcal{M}(\varphi_1, \varphi_0) = 0,$$

and

$$\mathcal{M}(\varphi_0, \varphi_1) + \mathcal{M}(\varphi_1, \varphi_2) + \mathcal{M}(\varphi_2, \varphi_0) = 0.$$

On the other hand, it’s easy to check that

$$\mathcal{M}(\varphi, \varphi + C) = 0, \quad \forall \varphi \in \mathcal{H}, C \in \mathbb{R}.$$

From the above 1-cocycle condition,

$$\mathcal{M}(\varphi' + C', \varphi'' + C'') - \mathcal{M}(\varphi', \varphi'') = \mathcal{M}(\varphi'', \varphi' + C') - \mathcal{M}(\varphi', \varphi' + C) = 0.$$

The lemma is proved. □

**Proof of Lemma 12.** Choose a local normal coordinates $(x, z^1, \cdots, z^2)$ as in (2.3) around the point considered. We have

$$D_{\varphi_t}S_t^T = \frac{d}{ds}

\left(\begin{array}{c}
\frac{1}{2} < d_B \dot{\varphi}_t, d_B S_t^T >_{\varphi_t} \\
-\frac{1}{4} \left\langle \left\langle \left[\left| h^{ij}_t(\varphi_t) \right| k(\dot{h}_t, h_t) h_t^{km} \right] q h_t^{sq} \right) \right. \\
-\frac{1}{4} \left\langle \left\langle \left[\left| h^{ij}_t(\varphi_t) \right| k(\dot{h}_t, h_t) h_t^{km} \right] q h_t^{sq} \right) \right. \\
\end{array} \right)$$

where $(h_t)_{ij} = h_{ij} + \frac{1}{2}(\varphi_t)_{ij}$ and $\Box_{\varphi_t} = \frac{d}{ds} \frac{\partial^2}{\partial z^i \partial z^j}.$ From the definition of $K$-energy,

$$\frac{d}{dt} \mu(\varphi_t) = -(\dot{\varphi}_t, S_t^T - S)_{\varphi_t}.$$

Hence

$$\frac{d^2}{dt^2} \mu(\varphi_t) = -(D_{\dot{\varphi}_t} \dot{\varphi}_t, S_t^T - S)_{\varphi_t} - (\dot{\varphi}_t, D_{\dot{\varphi}_t} S_t^T)_{\varphi_t}$$

and

$$\left(\begin{array}{c}
\end{array} \right) = 0.$$

For any $\varphi_0 \in \mathcal{H}$ and $\psi \in C^\infty_0(M)$, choose a smooth path $\{\varphi_t\} - \epsilon \leq t \leq \epsilon$ in $\mathcal{H}$ such that $\dot{\varphi}_t|_{t=0} = \psi$. The above identity yields

$$\frac{d^2}{dt^2} \mu(\varphi_t) = -(D_{\dot{\varphi}_t} \dot{\varphi}_t, S_t^T - S)_{\varphi_t}$$

and

$$\left(\begin{array}{c}
\end{array} \right) = 0.$$

□

**Proof of Lemma 13.** If $\varphi_1 - \varphi_0 \equiv \check{C} \neq 0$, where $\check{C}$ is a constant. Then, by the definition, $\varphi_t = \varphi_0 + t \check{C}$ is the smooth geodesic connecting $\varphi_0$ and $\varphi_1$. The length of the geodesic is $|\check{C}|$, i.e. $d(\varphi_0, \varphi_1) = \check{C} > 0$. Therefore, we may assume that $\varphi_1 - \varphi_0 - (\mathcal{I}(\varphi_1) - \mathcal{I}(\varphi_0))$ is not identically zero. Let $t(\varphi) = (\varphi_0 + t(\varphi_1 - \varphi_0)) - (\varphi_0)$, $t \in [0, 1]$. We compute that

$$t'(t) = \int_M \mathcal{I}(\varphi_0 - \varphi_0) d\nu(\varphi_0 + t(\varphi_1 - \varphi_0)).$$
and
\[ e''(t) = - \int_M \frac{1}{4} |d_B(\varphi_1 - \varphi_0)|^2_{g_0} \, dv(\varphi_0 + t(\varphi_1 - \varphi_0)) \leq 0, \]
where \( dv(\varphi) = \eta_\varphi \wedge (d\eta_\varphi)^n \). In turn, \( e'(1) \leq e(1) - e(0) \leq e'(0) \). That is
\[
\int_M \varphi_1 - \varphi_0 \, dv(\varphi_1) \leq I(\varphi_1) - I(\varphi_0) \leq \int_M \varphi_1 - \varphi_0 \, dv(\varphi_0).
\]
This means that the function \( \varphi_1 - \varphi_0 - (I(\varphi_1) - I(\varphi_0)) \) must take both positive and negative values.

Let \( \tilde{\varphi}_t \) be an \( \epsilon \)-approximate geodesic between \( \varphi_0 \) and \( \varphi_1 \). From the estimates in the previous sections, we can suppose that \( \max_{M \times [0,1]} |\varphi'(t)| \) have a uniform bound independent on \( \epsilon \). Since \( \tilde{\varphi}_t'' > 0 \),
\[
\varphi'(0) \leq \varphi_1 - \varphi_0 \leq \varphi'(1).
\]
Let \( E_\epsilon(t) = \int_M (\tilde{\varphi}')^2 \, dv_{\tilde{\varphi}_t} \) for any \( t \in [0,1] \). If \( I(\varphi_1) - I(\varphi_0) > 0 \), set \( t = 1 \), by (7.9),
\[
\sqrt{E_\epsilon(1)} \geq \int_M (\tilde{\varphi}'(1)) \, dv_{\tilde{\varphi}_1} \geq \int_{\varphi_1 - \varphi_0 > I(\varphi_1) - I(\varphi_0)} (\varphi_1 - \varphi_0) \, dv_{\varphi_1} \geq \int_{\varphi_1 - \varphi_0 > I(\varphi_1) - I(\varphi_0)} \varphi_1 - \varphi_0 - (I(\varphi_1) - I(\varphi_0)) \, dv_{\varphi_1} > 0.
\]
If \( I(\varphi_1) - I(\varphi_0) \leq 0 \), the similar argument yields,
\[
(7.11) \sqrt{E_\epsilon(0)} \leq - \int_{\varphi_1 - \varphi_0 < I(\varphi_1) - I(\varphi_0)} \varphi_1 - \varphi_0 - (I(\varphi_1) - I(\varphi_0)) \, dv_{\varphi_0} > 0.
\]
On the other hand, since \( \tilde{\varphi}_t \) is an \( \epsilon \) approximate geodesic, it’s easy to check that
\[
|\frac{d}{dt}E_\epsilon(t)| \leq C \epsilon,
\]
where \( C \) is a uniform constant. This implies
\[
|E_\epsilon(t_1) - E_\epsilon(t_2)| \leq C \epsilon
\]
for any \( t_1, t_2 \in [0,1] \). Thus
\[
\sqrt{E_\epsilon(t)} \geq e - C \epsilon,
\]
where \( e = \min\{0, -\int_{\varphi_1 - \varphi_0 > I(\varphi_1) - I(\varphi_0)} \, dv_{\varphi_0}, \int_{\varphi_1 - \varphi_0 < I(\varphi_1) - I(\varphi_0)} \, dv_{\varphi_1} \} > 0 \), and
\[
\pi = \varphi_1 - \varphi_0 - (I(\varphi_1) - I(\varphi_0)).
\]
Therefore,
\[
(7.14) \quad d(\varphi_0, \varphi_1) = \lim_{\epsilon \to 0} \int_0^1 \sqrt{E_\epsilon(t)} \, dt \geq e - C \epsilon.
\]
\[
\square
\]
**Proof of Lemma 14.** Everything follows from Theorem 1, except \( \frac{\partial^2}{\partial s^2} \leq C \) and \( \frac{\partial^2}{\partial s^2} \leq C \). The inequalities above follow from the maximum principle directly since
\[
(7.15) \quad \tilde{g}^{\alpha \beta} (\frac{\partial \psi}{\partial s})_{\alpha \beta} - r_{\alpha \beta} \frac{\partial}{\partial s} (\frac{\partial \psi}{\partial s}) = 0,
\]
and
\[
(7.16) \quad \tilde{g}^{\alpha \beta} (\frac{\partial^2 \psi}{\partial s^2})_{\alpha \beta} - r_{\alpha \beta} \frac{\partial}{\partial s} (\frac{\partial^2 \psi}{\partial s^2}) \geq 0,
\]
where \( \tilde{g} \) is the Hermitian metric induced by the positive (1,1)-form \( \Omega_\psi \). \( \square \)
Acknowledgement. It is our pleasure to thank Professor Wilbur Jusson for proof-reading the paper. The paper was written while the second author was visiting McGill University. He would like to thank ZheJiang University for the financial support and to thank McGill University for the hospitality.

References


Department of Mathematics and Statistics, McGill University, Canada

E-mail address: guan@math.mcgill.ca

School of Mathematical Science, University of Science and Technology of China, P. R. China

E-mail address: mathzx@ustc.edu.cn