Contemporary Mathematics

Convex solutions of fully nonlinear elliptic equations in classical differential geometry

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1. Introduction

The convexity is an issue of interest for a long time in PDE, it is intimately related to the study of geometric properties of solutions of general elliptic partial differential equations. A beautiful result of Gabriel [**18**] states that: the level sets of the Green function in three convex domains are strictly convex. Makar-Limanov [**35**] considered equation

$$\begin{aligned} \Delta u &= -1 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

in bounded plane convex domain Ω. By an ingenious argument involving the maximum principle, he proved that $\sqrt{u}$ is concave.

In an important article by Brascamp-Lieb [**9**], they established the log-concavity of the fundamental solution of diffusion equation with convex potential. As a consequence, they proved the log-concavity of the first eigenfunction of Laplace equation in convex domains and the Brunn-Minkowski inequality for the first eigenvalues.

For the case of dimension two, another proof of Brascamp-Lieb's result was found in Acker-Payne-Philippin [**1**]. It was observed that the function $v = \log u$ satisfies the following equation,

$$\begin{aligned} \Delta v &= -(\lambda_1 + |Dv|^2) \quad \text{in} \quad \Omega, \\ v &\longrightarrow -\infty \quad \text{on} \quad \Omega. \end{aligned} \tag{1.2}$$

If we let $v = \sqrt{u}$ in (1.1), then it satisfies equation

$$\begin{aligned} v\Delta v &= -(1 + |Dv|^2) \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.3}$$

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The second author [**34**] gave a new proof the concavity of the function v when Ω is a bounded convex plane domain, moreover he obtained a sharp estimate on the lower bound of the Gauss curvature of the graph of v in terms of the curvature of $\partial\Omega$. But the methods in [**1**] and [**34**] are restricted to two dimensions.

In [**30, 31**], Korevaar studied the convexity of the capillary surface. He introduced a very useful maximum principle (now named Korevaar's concavity maximum principle) in convex domains. Under certain boundary value conditions, he established convexity results for the mean curvature type equations. New proofs of the log-concavity of the first eigenfunction of convex domains were also given by Korevaar [**31**] and Caffarelli-Spruck [**13**]. Their methods were developed further by Kawohl [**28**] (for the intermediate case) and by Kennington [**29**] to establish an improved maximum principle, which enables them to give a higher dimensional generalization of the result of Makar-Limanov [**35**]. Recently, Alvarez-Lasry-Lions [**5**] found a new approach for the convexity problem and they treated a large class fully nonlinear elliptic equations.

In a fundamental work of Singer-Wong-Yau-Yau [**43**] and Caffarelli-Friedman [**10**], they devised a new technique to deal with the convexity issue via homotopy method of deformation. Caffarelli-Friedman [**10**] establish the strictly convexity of level sets of solution of the following equation in two dimension:

$$\begin{aligned} \Delta u(x) &= f(u(x)), \qquad x \in \Omega, \\ u &= 0 \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.4}$$

Their result was generalized by Korevaar-Lewis [**33**] to higher dimensions. This deformation approach (see also [**32**] for the earlier contribution of Yau related to this development) is very powerful, it is the main inspiration for our discussion on the convexity problem of some nonlinear elliptic equations in classical differential geometry in the next sections.

2. The Christoffel-Minkowski problem

In this section, we consider fully nonlinear differential equations on $\mathbb{S}^n$ associated to the intermediate Christoffel-Minkowski problem. The Minkowski problem is a problem of finding a convex hypersurface with the prescribed Gauss curvature on its outer normals. The general problem of finding a convex hypersurface M with kth symmetric function S_k of principal radii prescribed as a function of φ on its outer normals is often called Christoffel-Minkowski problem, where S_k is defined as follows.

Definition 1. *For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, $S_k(\lambda)$ is defined as*

$$S_k(\lambda) = \sum \lambda_{i_1} ... \lambda_{i_k},$$

where the sum is taking over for all increasing sequences $i_1, ..., i_k$ of the indices chosen from the set $\{1, ..., n\}$. The definition can be extended to symmetric matrices.

The support function u of a convex hypersurface M satisfies the following nonlinear elliptic Hessian equation (e.g., [**21**]):

$$S_k(\{u_{ij} + u\delta_{ij}\}) = \varphi \quad \text{on} \quad \mathbb{S}^n, \tag{2.1}$$

where u_{ij} are the second order covariant derivatives with respect to any given local orthonormal frame $\{e_1, e_2, ..., e_n\}$ on $\mathbb{S}^n$. Since M is convex, u satisfies the following *convexity* condition:

$$(u_{ij}(x) + u(x)\delta_{ij}) > 0, \qquad \forall x \in \mathbb{S}^n. \tag{2.2}$$

In what follows in this section, a function $u \in C^2(\mathbb{S}^n)$ is called *convex* if u satisfies (2.2). For $1 \leq k \leq n$, define

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \quad S_1(\lambda) > 0, ..., S_k(\lambda) > 0\}.$$

A function $u \in C^2(\mathbb{S}^n)$ is called k-convex if the eigenvalues of $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\}$ is in Γ_k for each $x \in \mathbb{S}^n$. u is called an admissible solution of (2.1) if it is k-convex. u is *convex* if and only u is n-convex.

We first note that

$$\int_{\mathbb{S}^n} x_i \varphi(x)\, dx = 0, \qquad i = 1, ..., n+1, \tag{2.3}$$

is a necessary condition for (2.1) to be solvable (e.g., see [**41, 21**]).

At one end $k = n$, equation (2.1) corresponds to the Minkowski problem. By the work of Nirenberg [**36**], Pogorelov [**39, 41**] and Cheng-Yau [**14**], (2.3) is also sufficient in this case. But when $1 \leq k < n$, the natural solution class for this of type equations is much larger than the class of convex functions. Hence the major issue is to find conditions for the existence of *convex* solutions of (2.1). At the other end $k = 1$, equation (2.1) is linear and it corresponds to the Christoffel problem. The necessary and sufficient conditions for the existence of a convex solution can be read off from the Green function [**17**]. For the intermediate cases ($2 \leq k \leq n-1$), (2.1) is a fully nonlinear equation, Pogorelov in [**41**] found a sufficient condition for the existence of *convex* solutions. But it is rather restrictive (see remark 5.5 in [**26**]). In [**26**], we introduced a general sufficient condition for the solution of the intermediate Christoffel-Minkowski problem. We deal with the problem via continuity method as a deformation process together with strong minimum principle to enforce the *convexity*.

Definition 2. *Let f be a positive $C^{1,1}$ function on $\mathbb{S}^n$ satisfies (2.3), $\forall s \in \mathbb{R}$, we say f is in $\mathcal{C}_s$ if $(f^s_{ij} + \delta_{ij} f^s)$ is semi-positive definite almost everywhere in $\mathbb{S}^n$.*

The following full rank theorem was proved in [**26**].

Theorem 1. *Suppose u is an admissible solution of equation (2.1) with semi-positive definite spherical Hessian $W = \{u_{ij} + u\delta_{ij}\}$ on $\mathbb{S}^n$. If $\varphi \in \mathcal{C}_{-\frac{1}{k}}$, then W is positive definite on $\mathbb{S}^n$.*

As a consequence, an existence result can be established for the Christoffel-Minkowski problem.

THEOREM 2. *Let $\varphi(x) \in \mathcal{C}_{-\frac{1}{k}}$, then Christoffel-Minkowski problem (2.1) has a unique convex solution up to translations.*

Theorem 2 was first proved in [**26**] under further assumption that φ is connected to 1 in $C_{-\frac{1}{k}}$. It turns out this extra condition is redundant as $C_{-\frac{1}{k}}$ is indeed connected. This fact was first proved in the joint work of Andrews and the second author [**7**] via curvature flow approach. More recently, this fact was also verified directly by Sheng-Trudinger-Wang [**42**].

The proof of Theorem 1 relies on a deformation lemma for Hessian equation (2.1), a fully nonlinear version of the results of Caffarelli-Friedman [**10**] and Korevaar-Lewis [**33**]. This type of deformation lemma enables us to apply the strong Maximum Principal to enforce the constant rank of $(u_{ij} + u\delta_{ij})$ on $\mathbb{S}^n$. The proof of such deformation lemma in [**26**] relies on some delicate algebraic structure of S_k. Theorem 1 was subsequently generalized to Hessian quotient equations in [**27**]. All these results point to a general phenomenon, as we observe that $-S_k^{-\frac{1}{k}}$ is concave. The following general convexity principle was established recently in [**11**].

THEOREM 3. *Let f be C^2 symmetric function of homogeneous degree -1 defined on a symmetric domain $\Psi \subset \Gamma_1$ in $\mathbb{R}^n$. Let $\tilde{\Psi} = \{A \in Sym(n) : \lambda(A) \in \Psi\}$, and define $F : \tilde{\Psi} \to R$ by $F(A) = f(\lambda(A))$. Suppose $\tilde{F}(A) = -F(A^{-1})$ is locally concave on the positive definitive matrices, $f_{\lambda_i} = \frac{\partial f}{\partial \lambda_i} > 0$. If $0 > g \in C^2(\mathbb{S}^n)$ and $g(x)$ is concave in $\mathbb{R}^{n+1}$ after being extended to $\mathbb{R}^{n+1}$ as a function of homogeneous function of degree 1, if u is an admissible solution of the equation on $\mathbb{S}^n$*

$$F(u_{ij} + u\delta_{ij}) = g, \tag{2.4}$$

with $(u_{ij} + u\delta_{ij})$ semi-positive definite on $\mathbb{S}^n$, then the Hessian $(u_{ij} + u\delta_{ij})$ positive definite on $\mathbb{S}^n$.

Theorem 3 has a counter part for domains in $\mathbb{R}^n$.

THEOREM 4. *Let f be a C^2 symmetric function defined on a symmetric domain $\Psi \subset \Gamma_1$ in $\mathbb{R}^n$. Let $\tilde{\Psi} = \{A \in Sym(n) : \lambda(A) \in \Psi\}$, and define $F : \tilde{\Psi} \to R$ by $F(A) = f(\lambda(A))$. If $\tilde{F} = -F(A^{-1})$ is a concave function on the positive definitive matrices, $f_{\lambda_i} = \frac{\partial f}{\partial \lambda_i} > 0$. Then if u is a C^4 convex solution of the following equation in a domain Ω in $\mathbb{R}^n$*

$$F(u_{ij}) = g(x) \tag{2.5}$$

and $g(x)$ is concave function in Ω. Then the Hessian u_{ij} is constant rank in Ω.

REMARK 1. The concavity condition on $G(W) = -F(W^{-1})$ was introduced by Alvarez-Lasry-Lions [**5**]. It was used by Andrews [**6**] in a similar spirit to obtain a pinching estimate of curvature flow for convex hypersurfaces. Theorem 1 is a special case of Theorem 3 with the setting of $F(A) = -S_k^{-\frac{1}{k}}(A)$. Both of them implies that there is a priori upper bound of principal curvatures of the convex hypersurface M satisfying (2.1). The existence of such estimate has been known for sometime if a stronger condition $((\varphi^{-\frac{1}{k}})_{ij} + \varphi^{-\frac{1}{k}}\delta_{ij}) > 0$ is imposed. Under this condition, the upper bound of the principal curvatures can be deduced simply from the equation (2.1) combining the ellipticity and concavity of the fully nonlinear operators $F(W) = -S_k^{-\frac{1}{k}}(A)$ and $G(W) = -F(A^{-1})$.

Proposition 1. *Suppose there exists a positive constant c_0 such that $((\varphi^{-\frac{1}{k}})_{ij} + \varphi^{-\frac{1}{k}}\delta_{ij}) \geq c_0\delta_{ij}$ on S^n, then there is a positive constant C depends only on $n, k, c_0, \inf_{\mathbb{S}^n}\varphi$ and $\|\varphi\|_{C^{1,1}}(S^n)$ such that for the admissible solution u of (2.1) we have the following estimate on S^n*

$$\frac{1}{C}\delta_{ij} \geq \{u_{ij} + u\delta_{ij}\} \geq C\delta_{ij}.$$

Proof. An upper bound of $\{u_{ij} + u\delta_{ij}\}$ has been established in [**26**]. Let $W_{ij} = u_{ij} + u\delta_{ij}$, we rewrite (2.1) as

$$F(W_{ij}) = -S_k^{-\frac{1}{k}}(W_{ij}) = -\varphi^{-\frac{1}{k}}. \tag{2.6}$$

Let W^{ij} be inverse matrix of W_{ij}, and $P(x) = W^{kl}\xi_k\xi_l(x)$, where ξ is a unit vector in R^n. Assume $P(x)$ attains its maximum on $x_o \in S^n$. We can also assume ξ is e_1 and take other directions $e_2, ..., e_n$ such that $(e_1, e_2, ..., e_n)$ is a local orthonormal frame near x_o and $W_{ij}(x_o)$ is diagonal. Then the function

$$P(x) = W^{11}$$

attains its maximum at $x_o \in S^n$. Define

$$F^{ij} = \frac{\partial F}{\partial W_{ij}}, \quad F^{ij,st} = \frac{\partial^2 F}{\partial W_{ij}\partial W_{st}}, \quad g(x) = \varphi^{-\frac{1}{k}}.$$

A straightforward computation yields that, at x_0,

$$\begin{aligned} 0 \quad \geq \quad & 2(W^{11})^2[\sum_{ik} F^{ii}W^{kk}W_{1ki}^2 + \sum_{ijkl} F^{ij,kl}W_{ij1}W_{kl1}] - W^{11}\sum_i F^{ii} \\ & +(W^{11})^2\sum_i F^{ii}W_{ii} + (W^{11})^2 g_{11}. \end{aligned}$$

As

$$\sum_i F^{ii}W_{ii} = S_k^{-\frac{1}{k}} = g, \qquad \sum_i F^{ii} = \frac{n-k+1}{k}S_k^{-\frac{1}{k}-1}S_{k-1} > 0.$$

By the upper bound of $\{u_{ij} + u\delta_{ij}\}$, there exists a positive constant C such that

$$\sum_i F^{ii} \leq C.$$

It follows from the concavity of functions $\tilde{F}(A) = -F(A^{-1})$ and $F(A)$ as in [**46**] that

$$0 \leq \sum_{ijkl} F^{ij,st}W_{ij1}W_{st1} + 2\sum_{ik} F^{ii}W^{kk}W_{1ki}^2.$$

Combing the above facts, we obtain

$$W^{11}(g_{11}+g) \le C. \tag{2.7}$$

The proof is complete. □

This type direct estimate on the upper bound of the principal curvatures blows up when some of the eigenvalues of $(\varphi_{ij}^{-\frac{1}{k}} + \varphi^{-\frac{1}{k}}\delta_{ij})$ vanish. On the other hand, Theorem 1 or Theorem 4 implies the following stronger result, which even allows the eigenvalues of $(\varphi_{ij}^{-\frac{1}{k}} + \varphi^{-\frac{1}{k}}\delta_{ij})$ to go negative.

THEOREM 5. *For any constant $1 > \beta > 0$, there is a positive constant $\gamma > 0$ such that if $\varphi(x) \in C^{1,1}(\mathbb{S}^n)$ is a positive function with $\frac{\inf_{\mathbb{S}^n}\varphi}{\sup_{\mathbb{S}^n}\varphi} \ge \beta$, $\frac{\sup_{\mathbb{S}^n}\varphi}{\|\varphi\|_{C^{1,1}(\mathbb{S}^n)}} \ge \beta$, and φ satisfies the necessary condition (2.3) and*

$$(\varphi_{ij}^{-\frac{1}{k}} + \varphi^{-\frac{1}{k}}\delta_{ij}) \ge -\gamma\varphi^{-\frac{1}{k}}\delta_{ij} \quad on \quad \mathbb{S}^n, \tag{2.8}$$

then Christoffel-Minkowski problem (2.1) has a unique $C^{3,\alpha}$ ($\forall 0<\alpha<1$) convex solution upto translations.

Proof. We argue by contradiction. If the result is not true, for some $0<\beta<1$, there is a sequence of positive functions $\varphi_l \in C^{1,1}(\mathbb{S}^n)$ such that $\sup_{\mathbb{S}^n}\varphi_l = 1$, $\inf_{\mathbb{S}^n}\varphi_l \ge \beta$, $\|\varphi_l\|_{C^{1,1}(\mathbb{S}^n)} \le \frac{1}{\beta}$, $(({\varphi_l}^{-\frac{1}{k}})_{ij} + {\varphi_l}^{-\frac{1}{k}}\delta_{ij}) \ge -\frac{1}{l}{\varphi_l}^{-\frac{1}{k}}\delta_{ij}$, φ_l satisfies (2.3), and equation (2.1) has no *convex* solution. By [**27**], equation (2.3) has an admissible solution u_l with

$$\|u_l\|_{C^{3,\alpha}(\mathbb{S}^n)} \le C,$$

independent of l. Therefore, there exist subsequences, we still denote φ_l and u_l,

$$\varphi_l \to \varphi \quad \text{in} \quad C^{1,\alpha}(\mathbb{S}^n), \quad u_l \to u \quad \text{in} \quad C^{3,\alpha}(\mathbb{S}^n),$$

for some positive $\varphi \in C^{1,1}(\mathbb{S}^n)$ with $(\varphi_{ij}^{-\frac{1}{k}} + \varphi^{-\frac{1}{k}}\delta_{ij}) \ge 0$, and u satisfies equation (2.1) and $(u_{ij}(x)+u(x)\delta_{ij}) \le 0$ at some point x. On the other hand, Theorem 2 yields a *convex* solution $\tilde{u} \in C^{3,\alpha}(\mathbb{S}^n)$ for such φ. By the uniqueness theorem in [**27**], $u-\tilde{u} = \sum_i^{n+1} a_i x_i$. In turn, $(u_{ij}+u\delta_{ij}) = (\tilde{u}_{ij}+\tilde{u}\delta_{ij}) > 0$ everywhere. This is a contradiction. □

The same argument also produces a similar a priori estimate for a lower bound of eigenvalues of $(u_{ij}+u\delta_{ij})$ of solution u of equation (2.4) in Theorem 3 with a weaker condition that $g-\beta|x|$ concave in $\mathbb{R}^{n+1}$ for some $\beta>0$.

3. Weingarten curvature equations

The Christoffel-Minkowski problem was deduced to a convexity problem of a spherical Hessian equation on $\mathbb{S}^n$ in the last section. It can also be considered as a curvature equation on the hypersurface via inverse Gauss map. In this section, we discuss some curvature equations related to problems in the classical differential geometry. For a compact hypersurface M in $\mathbb{R}^{n+1}$, the kth Weingarten curvature at $x \in M$ is defined as

$$\mathcal{W}_k(x) = S_k(\kappa_1(x), \kappa_2(x), \cdots, \kappa_n(x))$$

where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n)$ the principal curvatures of M. In particular, $\mathcal{W}_1$ is the mean curvature, $\mathcal{W}_2$ is the scalar curvature, and $\mathcal{W}_n$ is the Gauss-Kronecker curvature. If the surface is starshaped about the origin, it follows that the surface can be parameterized as a graph over $\mathbb{S}^n$:

$$X = \rho(x)x, \qquad x \in \mathbb{S}^n, \tag{3.1}$$

where ρ is the radial function. In this correspondence, the Weingarten curvature can be considered as a function on $\mathbb{S}^n$ or in $\mathbb{R}^{n+1}$. The problem of prescribing curvature function has attracted much attention. For example, given a positive function F in $\mathbb{R}^{n+1} \setminus \{0\}$, one would like to find a starshaped hypersurface M about the origin such that its kth Weingarten curvature is F. The problem is equivalent to solve the following equation

$$S_k(\kappa_1, \kappa_2, ..., \kappa_n)(X) = F(X) \quad \text{for any} \quad X \in M. \tag{3.2}$$

The uniqueness question of starshaped hypersurfaces with prescribed curvature was studied by Alexandrov [**4**] and Aeppli [**2**]. The problem of prescribing Weingarten curvature and similar problems have been studied by various authors, we refer to [**8, 47, 45, 38, 12, 15, 48, 19, 20, 21**] and references there.

We will use notions of admissible solutions as in last section

Definition 3. *A C^2 surface M is called k-admissible if at every point $X \in M$, $\kappa \in \Gamma_k$.*

Under some barrier conditions, an existence result for equation (3.2) was obtained by Bakelman-Kantor [**8**], Treibergs-Wei [**45**] for $k = 1$, and by Caffarelli-Nirenberg-Spruck in [**12**] for general $1 \le k \le n$. The solution of the problem [**12**] in general is not *convex* if $k < n$. The question of convexity of solution in [**12**] was treated by Chou [**15**] (see also [**48**]) for the mean curvature case under concavity assumption on F, and by Gerhardt [**19**] for general Weingarten curvature case under concavity assumption on $\log F$, see also [**20**] for the work on general Riemannian manifolds.

The following is proved in [**24**].

Theorem 6. *Suppose M is a k-admissible surface of equation (3.2) in $\mathbb{R}^{n+1}$ with semi-positive definite second fundamental form $\mathcal{W} = \{h_{ij}\}$ and $F(X) : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^+$ is a given smooth positive function. If $F(X)^{-\frac{1}{k}}$ is a convex function in a neighborhood of M, then $\{h_{ij}\}$ is positive definite. that is, M is strictly convex.*

As a consequence, we deduce the existence of convex hypersurface with prescribed Weingarten curvature in (3.2) (in [**24**]): if in addition to the barrier condition in [**12**], $F(X)^{-\frac{1}{k}}$ is a convex function in the region $r_1 < |X| < r_2$, then the k-admissible solution in Theorem [**12**] is strictly convex.

In the literature, the homogeneous Weingarten curvature problem

$$S_k(k_1, k_2, ..., k_n)(X) = \gamma f(\frac{X}{|X|})|X|^{-k}, \quad \forall X \in M, \tag{3.3}$$

also draws some attention. If M is a starshaped hypersurface about the origin in $\mathbb{R}^{n+1}$, by dilation property of the curvature function, the kth Weingarten curvature can be considered as a function of homogeneous degree $-k$ in $\mathbb{R}^{n+1} \setminus \{0\}$. If F is of homogeneous degree $-k$, then the barrier condition in [**12**] can not be valid unless the function is constant. Therefore equation (3.3) needs a different treatment. In fact, this problem is a nonlinear eigenvalue problem for the curvature equation. When $k = n$, then equation (3.3) can be expressed as a Monge-Ampère equation of radial function ρ on $\mathbb{S}^n$, the problem was studied by Delanoë [**16**]. The other special case $k = 1$ was considered by Treibergs in [**44**]. The difficulty for equation (3.3) is the lack of gradient estimate, such kind of estimate does not hold in general (see [**44, 24**]). Therefore, some conditions have to be in place for f in (3.3). In [**24**], a uniform treatment for $1 \le k \le n$ was given, and together with some discussion on the existence of convex solutions.

Theorem 7. *Suppose $n \ge 2$, $\quad 1 \le k \le n$ and f is a positive smooth function on $\mathbb{S}^n$. If $k < n$, assume further that f satisfies*

$$\sup_{\mathbb{S}^n} \frac{|\nabla f|}{f} < 2k, \tag{3.4}$$

Then there exist a unique constant $\gamma > 0$ with

$$\frac{C_n^k}{\max_{\mathbb{S}^n} f} \le \gamma \le \frac{C_n^k}{\min_{\mathbb{S}^n} f} \tag{3.5}$$

and a smooth k-admissible hypersurface M satisfying (3.3) and solution is unique up to homothetic dilations. Furthermore, for $1 \le k < n$, if in addition $|X| f(\frac{X}{|X|})^{-\frac{1}{k}}$ is convex in $\mathbb{R}^{n+1} \setminus \{0\}$, then M is strictly convex.

Remark 2. Condition (3.4) in Theorem 7 can be weakened, we refer to [**24**] for the precise statement. When $k = n$, the above result was proved by Delanoë [**16**]. In this case, the solution is convex automatically. The treatment in [**24**] is different from [**16**]. When $k = 1$, the existence part of Theorem 7 was proved in [**44**], along with a sufficient condition for the convexity of solutions.

We now switch to a similar curvature equation arising from the problem of prescribing curvature measures in the theory of convex bodies. For a bounded convex body Ω in $\mathbb{R}^{n+1}$ with C^2 boundary M, the corresponding curvature measures of Ω

can be defined according to some geometric quantities of M. The k-th curvature measure of Ω is defined as

$$\mathcal{C}_k(\Omega,\beta) := \int_{\beta\cap M} \mathcal{W}_{n-k} dF_n,$$

for every Borel measurable set β in $\mathbb{R}^{n+1}$, where dF_n is the volume element of the induced metric of $\mathbb{R}^{n+1}$ on M. Since M is convex, M is star-shaped about some point. We may assume that the origin is inside of Ω. Since M and $\mathbb{S}^n$ is diffemorphism through radial correspondence R_M. Then the k-th curvature measure can also be defined as a measure on each Borel set β in $\mathbb{S}^n$:

$$\mathcal{C}_k(M,\beta) = \int_{R_M(\beta)} \mathcal{W}_{n-k} dF_n.$$

Note that $\mathcal{C}_k(M,\mathbb{S}^n)$ is the k-th quermassintegral of Ω.

The problem of prescribing curvature measures is dual to the Christoffel- Minkowski problem in the previous section. The case $k=0$ is named as the Alexandrov problem, which can be considered as a counterpart to Minkowski problem. The existence and uniqueness results were obtained by Alexandrov [**3**]. The regularity of the Alexandrov problem in elliptic case was proved by Pogorelov [**40**] for $n=2$ and by Oliker [**37**] for higher dimension case. A general regularity result (degenerate case) of the problem was obtained in [**22**]. Yet, very little is known for the existence problem of prescribing curvature measures $\mathcal{C}_{n-k}$ for $k<n$.

The problem is equivalent to solve the following curvature equation

$$S_k(\kappa_1,\kappa_2,...,\kappa_n) = \frac{f(x)}{g(x)}, \quad 1\le k\le n \quad \text{on} \quad \mathbb{S}^n \tag{3.6}$$

where f is the given function on $\mathbb{S}^n$ and $g(x)$ is a function involves the gradient of solution. The major difficulty around equation (3.6) is the lack of C^2 a priori estimates for admissible solutions. Though equation (3.6) is similar to the equation of prescribing Weingarten curvature equation (3.2), the function g (depending on the gradient of solution) makes the matter very delicate. Equation (3.6) was studied in an unpublished notes [**23**] by Yanyan Li and the first author. The uniqueness and C^1 estimates were established for admissible solutions there. In [**25**], we make use of some ideas in the convexity estimate for curvature equations to overcome the difficulty on C^2 estimate.

Theorem 8. *Suppose $f(x)\in C^2(\mathbb{S}^n)$, $\quad f>0$, $n\ge 2$, $1\le k\le n-1$. If f satisfies the condition*

$$|X|^{\frac{n+1}{k}} f(\frac{X}{|X|})^{-\frac{1}{k}} \quad \textit{is a strictly convex function in} \quad \mathbb{R}^{n+1}\setminus\{0\}, \tag{3.7}$$

then there exists a unique strictly convex hypersurface $M\in C^{3,\alpha}, \alpha\in(0,1)$ such that it satisfies (3.6).

For the C^2 estimates for admissible solutions of (3.6), it is equivalent to estimate the upper bounds of principal curvatures. If the hypersurface is strictly convex, it is simple to observe that a positive lower bound on the principal curvatures implies an upper bound of the principal curvatures. To achieve such a lower bound, we shall use the inverse Gauss map and consider the equation for the support function of the hypersurface. The role of the Gauss map here should be compared with the role of the Legendre transformation on the graph of convex surface in a domain in $\mathbb{R}^n$. We note that a lower bound on the principal curvature is an upper bound on the principal radii which are exactly the eigenvalues of the spherical Hessian of the support function. We give a brief illustration how the idea of convexity estimates can help us to obtain C^2 estimate for solutions of equation (3.6).

Let $X : M \to \mathbb{R}^{n+1}$ be a closed strictly convex smooth hypersurface in $\mathbb{R}^{n+1}$. We may assume the X is parameterized by the inverse Gauss map. The support function of X is defined by

$$u(x) =< x, X(x) >, \quad \forall x \in \mathbb{S}^n.$$

Let $e_1, e_2, ..., e_n$ be a smooth local orthonormal frame field on $\mathbb{S}^n$. The inverse second fundamental form of X is

$$h_{ij} = u_{ij} + u\delta_{ij},$$

and the metric of X is

$$g_{ij} = \sum_{l=1}^{n} h_{il}h_{jl}.$$

The principal radii of curvature are the eigenvalues of matrix

$$W_{ij} = u_{ij} + u\delta_{ij}.$$

Equation (3.6) can be written as an equation on support function u.

$$F(W_{ij}) = [\frac{\det W_{ij}}{S_{n-k}(W_{ij})}]^{\frac{1}{k}}(x) = G(X)u^{-\frac{1}{k}} \quad \text{on} \quad \mathbb{S}^n, \tag{3.8}$$

where X is position vector of hypersurface, and

$$G(X) = |X|^{\frac{n+1}{k}} f^{-\frac{1}{k}}(\frac{X}{|X|}).$$

Lemma 1. *Suppose f satisfies condition (3.7). If M is a convex hypersurface in $\mathbb{R}^{n+1}$ respect to the origin satisfying (3.6), then the following estimates hold for its radial function $\rho = |X|$,*

$$|\nabla^2 \rho| \le C. \tag{3.9}$$

Proof: C^1 estimate for general admissible solutions follows simply from the equation by the maximum principle. We only need to obtain an upper bound of $H = \sum_{l=1}^{n} = \Delta u + nu$. Assume the maximum of H attains at some point $x_o \in \mathbb{S}^n$, choose an orthonormal frame $e_1, e_2, ..., e_n$ near x_o such that $u_{ij}(x_o)$ is diagonal.

Set $F^{ij} = \frac{\partial F(W)}{\partial W_{ij}}$. At x_0, we compute

$$0 \ge \sum_{i=1}^{n} F^{ii}\Delta(W_{ii}) - nF + (C_n^{n-k})^{-\frac{1}{k}}H. \tag{3.10}$$

By equation (3.8) and the concavity of F , we get

$$\sum_{l=1}^{n}[G(X)u^{-\frac{1}{k}}]_{ll}-nF+(C_n^{n-k})^{-\frac{1}{k}}H\leq 0. \tag{3.11}$$

The main observation is that $[G(X)u^{-\frac{1}{k}}]_{ll}$ is the dominating term. We use the standard convention that repeated indices on α,β denote summation over the indices from $1,2,\cdots,n+1$. Denote $G_\alpha=\frac{\partial G}{\partial X^\alpha}$, $G_{\alpha\beta}=\frac{\partial^2 G}{\partial X^\alpha\partial X^\beta}$.

We calculate that, at x_o

$$\begin{aligned}\sum_{l=1}^{n}[G(X)u^{-\frac{1}{k}}]_{ll} \quad &= \quad G_{\alpha,\beta}e_l^\alpha e_l^\beta W_{ll}^2u^{-\frac{1}{k}}-[G_\alpha x^\alpha u^{-\frac{1}{k}}+\frac{1}{k}G(X)u^{-\frac{1}{k}-1}]H\\ &\quad -\frac{2}{k}(G_\alpha e_l^\alpha u_lW_{ll})u^{-\frac{1}{k}-1}+\frac{1}{k}(\frac{1}{k}+1)G(X)u^{-\frac{1}{k}-2}|Du|^2+\frac{n}{k}G(X)u^{-\frac{1}{k}}.\end{aligned} \tag{3.12}$$

Now (3.11) becomes

$$\begin{aligned}&G_{\alpha,\beta}e_l^\alpha e_l^\beta W_{ll}^2u^{-\frac{1}{k}}-[G_\alpha x^\alpha u^{-\frac{1}{k}}+\frac{1}{k}G(X)u^{-\frac{1}{k}-1}]H-nF+(C_n^{n-k})^{-\frac{1}{k}}H\\ &-\frac{2}{k}(G_\alpha e_l^\alpha u_lW_{ll})u^{-\frac{1}{k}-1}+\frac{1}{k}(\frac{1}{k}+1)G(X)u^{-\frac{1}{k}-2}|Du|^2++\frac{n}{k}G(X)u^{-\frac{1}{k}}\leq 0.\end{aligned} \tag{3.13}$$

Since $G(X)$ is strictly convex and $\sum_{l=1}^n W_{ll}^2\geq\frac{H^2}{n}$, we obtain $H(x_o)\leq C$. □

When $k=1$ or 2, the strict convexity condition (3.7) can be weakened.

THEOREM 9. *Suppose $k=1$, or 2 and $k<n$, and suppose $f(x)\in C^2(\mathbb{S}^n)$ is a positive function. If f satisfies*

$$|X|^{\frac{n+1}{k}}f(\frac{X}{|X|})^{-\frac{1}{k}} \quad \textit{is a convex function in} \quad \mathbb{R}^{n+1}\setminus\{0\}, \tag{3.14}$$

then there exists unique strictly convex hypersurface $M\in C^{3,\alpha},\alpha\in(0,1)$ such that it satisfies equation (3.6).

LEMMA 2. *If M is a convex hypersurface in $\mathbb{R}^{n+1}$ respect to the origin satisfying (3.6) for $k=2$, then the following estimates hold for its radial function $\rho=|X|$,*

$$\|\rho\|_{C^2}\leq C. \tag{3.15}$$

Proof: We only need to get an upper bound of the mean curvature H.

Equation (3.6) can be expressed as

$$S_2(\kappa_1,\kappa_2,...,\kappa_n)(X)=\phi(X)<X,e_{n+1}>, \quad \text{on} \quad M, \tag{3.16}$$

where $\phi(X)=|X|^{-(n+1)}f(\frac{X}{|X|})$. This time, we perform calculation at a maximum point $X_o\in M$ of the function $P=H+\frac{a}{2}|X|^2$, where a is a constant to be chosen later.

As before, we may assume $\{h_{ij}\}$ and $F^{ij}=\frac{\partial S_2(\lambda\{h_{ij}\})}{\partial h_{ij}}$ are diagonal at X_o. At this point,

$$\sum_{ij=1}^{n}F^{ij}P_{ij}=\sum_{i=1}^{n}F^{ii}H_{ii}+a\sum_{i=1}^{n}F^{ii}-a<X,e_{n+1}>\sum_{i=1}^{n}F^{ii}h_{ii}\leq 0, \tag{3.17}$$

Set $|A|^2=\sum_{i=1}^n h_{ii}^2$, we compute that

$$\sum_{i=1}^n F^{ii}H_{ii}=\sum_{ij=1}^n F^{ii}h_{iijj}+|A|^2\sum_{i=1}^n F^{ii}h_{ii}-H\sum_{i=1}^n F^{ii}h_{ii}^2. \tag{3.18}$$

By equation (3.16),

$$\begin{aligned}\sum_{ij=1}^n F^{ii}h_{iijj}= \quad & \Delta\phi<X,e_{n+1}>+2\sum_{j=1}^n \phi_j h_{jj}<X,e_j>+\phi\sum_{j=1}^n<X,e_{n+1}>_{jj}\\ &+\sum_{j,k\neq l}h_{jkl}^2-\sum_{j,k,l}h_{jkk}h_{jll}+\sum_{j,k}h_{jkk}^2.\end{aligned}$$

At X_0,

$$\sum_{i=1}^n<X,e_{n+1}>_{ii}=-a\sum_{i=1}^n<x,e_i>^2+H-|A|^2<X,e_{n+1}>,$$

and

$$\begin{aligned}\sum_{ij=1}^n F^{ii}h_{iijj} \quad \geq \quad & -|A|^2S_2(h_{ij})+\phi H+\Delta\phi<X,e_{n+1}>+2\sum_{j=1}^n\phi_j h_{jj}<X,e_j>\\ & -a\phi\sum_{i=1}^n<x,e_i>^2-a^2\sum_{i=1}^n<x,e_i>^2 .\end{aligned} \tag{3.19}$$

In turn,

$$\begin{aligned}&a(n-1)H+\phi H+2\sum_{i=1}^n\phi_i h_{ii}<X,e_i>+\Delta\phi<X,e_{n+1}>+3HS_3(h_{ij})\\ &\leq 2S_2(h_{ij})^2+2a<X,e_{n+1}>S_2(h_{ij})+[a\phi+a^2]\sum_{i=1}^n<X,e_i>^2 .\end{aligned} \tag{3.20}$$

Since M is convex, if a is suitable large, we obtain an upper bound of H at X_0. □

To ensure the convexity of solutions in the process of applying the method of continuity in the proof of Theorem 9, we establish a corresponding deformation lemma for equation (3.6) as in Theorem 1 and Theorem 6 under the condition that $|X|^{\frac{n+1}{k}}f(\frac{X}{|X|})^{-\frac{1}{k}}$ is a convex in $\mathbb{R}^{n+1}\setminus\{0\}$. We refer [**25**] for the detail of proof.

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