DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **28**, Number **2**, October **2010**

doi:10.3934/dcds.2010.28.789

pp. **789–807**

A STRUCTURAL CONDITION FOR MICROSCOPIC CONVEXITY PRINCIPLE

Baojun Bian

Department of mathematics
Tongji University
Shanghai 200092, China

Pengfei Guan

Department of Mathematics and Statistics
McGill University
Montreal, Quebec, H3A 2K6, Canada

Dedicated to Professor Nirenberg on the occasion of his 85th birthday

Abstract. The arguments in paper [2] have been refined to prove a microscopic convexity principle for fully nonlinear elliptic equation under a more natural structure condition. We also consider the corresponding result for the partial convexity case.

1. **Introduction.**

Consider fully nonlinear elliptic equation in the form

$$F(\nabla^2 u, \nabla u, u, x) = 0, \quad x \in \Omega \subset \mathbb{R}^n \text{ is a domain.} \tag{1}$$

Assume F is elliptic at some $u \in C^2(\Omega)$ in the sense that

$$\left(\frac{\partial F}{\partial r_{\alpha\beta}}(\nabla^2 u(x), \nabla u(x), u(x), x)\right) > 0, \quad \forall x \in \Omega. \tag{2}$$

The following microscopic convexity principle was proved in [2].

Theorem 1.1. *([2]) Let $F = F(r, p, u, x) \in C^{2,1}(\mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega)$ and let $u \in C^{2,1}(\Omega)$ be a convex solution of (1). If F is elliptic and*

$$F(A^{-1}, p, u, x) \quad \text{is locally convex in } (A, u, x) \text{ for each } p \text{ fixed,} \tag{3}$$

then the rank of Hessian $(\nabla^2 u(x))$ is constant in Ω.

This type of *constant rank theorem* was first established by Caffarelli-Friedman [3] for convex solutions of semilinear elliptic equation

$$\Delta u = f(\nabla u, u, x), \quad \Omega \subset \mathbb{R}^2, \tag{4}$$

under the condition that

$$\tfrac{1}{f(\nabla u, u, x)} \text{ is convex in } (u, x) \text{ for each fixed } \nabla u. \tag{5}$$

2000 *Mathematics Subject Classification.* Primary: 35K10, 35B99.

Key words and phrases. Convexity, partial convexity, fully nonlinear equations, constant rank principle.

Research of the first author was supported in part by NSFC No.10671144 and NBRP of China (2007CB814903). Research of the second author was supported in part by an NSERC Discovery Grant.

A similar result was also discovered by Yau [13] at about the same time and the result for equation (4) in [3] was generalized to $\Omega \subset \mathbb{R}^n$ by Korevaar-Lewis [11]. The microscopic convexity principle is a powerful tool in the study of geometric properties of solutions of nonlinear differential equations and is very useful for the existence of convex solutions of differential equations [8, 7, 9, 4, 2]. The constant rank theorem shares the same spirit with the results of Hartman-Nirenberg in [10] for the gradient mapping.

Theorem 1.1 is general in the sense it covers a wide class of fully nonlinear elliptic differential equations, including Hessian equations whose elliptic structure was studied in the pioneer work of Caffarelli-Nirenberg-Spruck in [5]. But condition (3) in Theorem 1.1 is tricky to apply. For example, for equation (4), if apply Theorem 1.1 directly, one needs that

$$f(\nabla u, u, x) \text{ is concave in } (u, x) \text{ for each fixed } \nabla u. \tag{6}$$

This condition is obviously stronger than (5). On the other hand, rewrite equation (4) as

$$-\frac{1}{\Delta u} = -\frac{1}{f(\nabla u, u, x)}, \tag{7}$$

then condition (5) fits Theorem 1.1. This disparity indicates that there should be a more natural structural condition for microscopic convexity principle.

Denote $\mathcal{S}^n_+$ the space of positive definite real symmetric $n \times n$ matrices, for each fixed $p \in \mathbb{R}^n$, define the zero sub-level set

$$\Gamma_F = \{(A, u, x) \in \mathcal{S}^n_+ \times \mathbb{R} \times \Omega | F(A^{-1}, p, u, x) \leq 0\}. \tag{8}$$

In the rest of the paper, we assume

$$F(\mathbf{0}, \nabla u(x), u(x), x) \neq 0, \quad \forall x \in \Omega. \tag{9}$$

Theorem 1.2. *Let $F = F(r, p, u, x) \in C^{2,1}(\mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega)$ and let $u \in C^{2,1}(\Omega)$ be a convex solution of equation (1). Suppose F satisfies condition (2) and (9) at $(\nabla^2 u(x), \nabla u(x), u(x), x)$ for each $x \in \Omega$. If for each $x \in \Omega$ and $p = \nabla u(x)$,*

$$\Gamma_F \text{ is locally convex at } (A, u(x), x), \tag{10}$$

then the rank of the hessian $(\nabla^2 u(x))$ is constant in Ω. If l is the rank of $\nabla^2 u$, then $\forall x_0 \in \Omega$, there exist a neighborhood U of x_0 and $(n-l)$ fixed directions $V_1, \ldots, V_{n-l}$ such that $\nabla^2 u(x) V_j = 0$ for all $1 \leq j \leq n - l$ and $x \in U$.

In other words, the point-wise convexity condition on F in Theorem 1.1 can be replaced by the convexity of the zero sub-level set of F in Theorem 1.2. This resolves the problem regarding equation (4) we just discussed since the zero sub-sets $\{\Delta u - f \leq 0\}$ and $\{\frac{-1}{\Delta u} + \frac{1}{f} \leq 0\}$ are the same.

In fact, condition (10) can be weakened further. Denote $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. For each $\theta \in \mathbb{S}^{n-1}$, define

$$\theta\mathbb{R} = \{t\theta | t \in \mathbb{R}\}, \quad (\theta\mathbb{R})^\perp = \{\eta \in \mathbb{R}^n | <\eta, \theta>= 0\}$$

and

$$\mathcal{S}_\theta = \{A \in \mathcal{S}^n | A\theta = 0\}, \quad \mathcal{S}^+_\theta = \{A \in \mathcal{S}^n | A\theta = 0, A > 0 \text{ on } (\theta\mathbb{R})^\perp\}$$

Let $(A,p,u,x) \in \mathcal{S}_\theta^+ \times \mathbb{R}^n \times \mathbb{R} \times \Omega$ and $B \in \mathcal{S}_\theta^+$ with $B = A^{-1}$ on $(\theta\mathbb{R})^\perp$. For each fixed $p \in \mathbb{R}^n$ and $x_0 \in \Omega$, set

$$\begin{aligned}\Gamma_F^\theta =&\{(B,u,x) \in \mathcal{S}_\theta^+ \times \mathbb{R} \times \theta\mathbb{R} \\ &\qquad |F(A,p,u,x+x_0) \le 0, A \in \mathcal{S}_\theta^+,\ A = B^{-1} \text{ on } (\theta\mathbb{R})^\perp\}.\end{aligned}$$

Theorem 1.3. *The same conclusion in Theorem 1.2 is true if condition (10) is replaced by the following structural condition: for any fixed $x_0 \in \Omega$, $p = \nabla u(x_0)$ and $\theta \in \mathbb{S}^{n-1}$*

$$\Gamma_F^\theta \text{ is locally convex at } (B, u(x_0), 0) \text{ with } B = A^{-1} \in \mathcal{S}_\theta^+. \tag{11}$$

There are corresponding theorems for the partially convex solutions of equation (1). They appear in the last section of the paper.

2. **Convexity.**

We follow the same noation as in [2]. For each function $F(r,p,u,x)$, denote

$$\begin{aligned}&F^{\alpha\beta} = \frac{\partial F}{\partial r_{\alpha\beta}},\quad F^u = \frac{\partial F}{\partial u},\quad F^{x_i} = \frac{\partial F}{\partial x_i},\quad F^{p_i} = \frac{\partial F}{\partial p_i},\\ &F^{\alpha\beta,\gamma\eta} = \frac{\partial^2 F}{\partial r_{\alpha\beta}\partial r_{\gamma\eta}},\quad F^{\alpha\beta,u} = \frac{\partial^2 F}{\partial r_{\alpha\beta}\partial u},\quad F^{\alpha\beta,x_k} = \frac{\partial^2 F}{\partial r_{\alpha\beta}\partial x_k},\\ &F^{u,u} = \frac{\partial^2 F}{\partial^2 u},\quad F^{u,x_i} = \frac{\partial^2 F}{\partial u \partial x_i},\quad F^{x_i,x_j} = \frac{\partial^2 F}{\partial x_i \partial x_j},\end{aligned} \tag{12}$$

the partial derivatives with respect to the corresponding variables. Set

$$X_F^* = X_F^*(A,p,u,x)) = ((F^{\alpha\beta}), F^u, (F^{x_1}, \cdots, F^{x_n})), \tag{13}$$

$$\Gamma^\perp_{X_F^*} = \Gamma^\perp_{X_F^*}(A,p,u,x) = \{\tilde X \in \mathcal{S}_\theta \times \mathbb{R} \times \theta\mathbb{R} |< \tilde X, X_F^*(A,p,u,x) >= 0\}. \tag{14}$$

X_F^* is a vector in $\mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n$, where functions $F^{\alpha\beta}, F^u, F^{x_1}, \cdots, F^{x_n}$ are evaluated at (A,p,u,x).

For $\tilde X = \big((X_{ij}), Y, (Z_k)\big) \in \mathcal{S}_\theta \times \mathbb{R} \times \theta\mathbb{R}$, define a quadratic form

$$\begin{aligned}Q^*(\tilde X, \tilde X) = &\sum_{i,j,k,l=1}^n F^{ij,kl} X_{ij} X_{kl} + 2 \sum_{i,j,k,l=1}^n F^{ij} B_{kl} X_{ik} X_{jl}\\ &+ 2\sum_{i,j=1}^n F^{ij,u} X_{ij} Y + 2 \sum_{i,j,k=1}^n F^{ij,x_k} X_{ij} Z_k\\ &+ F^{u,u} Y^2 + 2\sum_{i=1}^n F^{u,x_i} Y Z_i + \sum_{i,j=1}^n F^{x_i,x_j} Z_i Z_j,\end{aligned} \tag{15}$$

again functions $F^{ij,kl}, F^{ij}, F^{u,u}, F^{ij,u}, F^{ij,x_k}, F^{u,x_i}, F^{x_i,x_j}$ are evaluated at (A,p,u,x), and $B \in \mathcal{S}_\theta^+$ with $B = A^{-1}$ on $(\theta\mathbb{R})^\perp$.

Lemma 2.1. *If $(A,p,u,x) \in \mathcal{S}_+^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega$ such that $F(A,p,u,x) = 0$, then Γ_F is locally convex at (A^{-1}, u, x) if and only if*

$$Q^*(\tilde X, \tilde X) \ge 0 \tag{16}$$

for every $\tilde X = \big((X_{ij}), Y, (Z_k)\big) \in \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n$ with $< \tilde X, X_F^*(A,p,u,x) >= 0$.

Proof. Fix p, let $\tilde{F}(B,u,x) = F(B^{-1},p,u,x)$ for $(B,u,x) \in \mathcal{S}^n_+ \times \mathbb{R} \times \Omega$. Then the condition (10) is equivalent to

$$
\begin{aligned}
&\sum_{\alpha,\beta,\gamma,\eta=1}^{n} \tilde{F}^{\alpha\beta,\gamma\eta}(B,u,x)\hat{X}_{\alpha\beta}\hat{X}_{\gamma\eta} + 2\sum_{\alpha,\beta=1}^{n} \tilde{F}^{\alpha\beta,u}(B,u,x)\hat{X}_{\alpha\beta}\hat{Y} \\
&+ 2\sum_{\alpha,\beta,k=1}^{n} \tilde{F}^{\alpha\beta,x_k}(B,u,x)\hat{X}_{\alpha\beta}\hat{Z}_k + \tilde{F}^{u,u}(B,u,x)\hat{Y}^2 \\
&+ 2\sum_{k=1}^{n} \tilde{F}^{u,x_k}(B,u,x)\hat{Y}\hat{Z}_k + \sum_{i,j=1}^{n} \tilde{F}^{x_i,x_j}(B,u,x)\hat{Z}_i\hat{Z}_j \geq 0
\end{aligned}
\tag{17}
$$

for every $\hat{X} \in \mathcal{S}^n$, $\hat{Y} \in \mathbb{R}$, $\hat{Z} = (\hat{Z}_i) \in \mathbb{R}^n$, with

$$\sum_{\alpha,\beta} \tilde{F}^{\alpha\beta}(B,u,x)\hat{X}_{\alpha\beta} + \tilde{F}^{u}(B,u,x)\hat{Y} + \tilde{F}^{x_i}(B,u,x)\hat{Z}_i = 0.$$

A direct computation yields

$$
\begin{aligned}
\tilde{F}^{\alpha\beta}(B,u,x) =& - F^{ij}(B^{-1},p,u,x)B^{i\alpha}B^{j\beta}, \\
\tilde{F}^{\alpha\beta,u}(B,u,x) =& - F^{ij,u}(B^{-1},p,u,x)B^{i\alpha}B^{j\beta}, \\
\tilde{F}^{\alpha\beta,\gamma\eta}(B,u,x) =& F^{ij,kl}(B^{-1},p,u,x)B^{i\alpha}B^{j\beta}B^{k\gamma}B^{l\eta} \\
&+ F^{ij}(B^{-1},p,u,x)(B^{i\gamma}B^{j\beta}B^{\eta\alpha} + B^{i\alpha}B^{j\eta}B^{\beta\gamma}).
\end{aligned}
$$

Other derivatives can be calculated in a similar way. Substituting these into (17), equation (16) follows directly. ◻

Lemma 2.2. *If $(A,p,u,x) \in \mathcal{S}^+_\theta \times \mathbb{R}^n \times \mathbb{R} \times \Omega$ such that $F(A,p,u,x) = 0$, Γ^θ_F is locally convex near (B,u,x) (where $B = A^{-1}$ on $\mathcal{S}^+_\theta$) if and only if*

$$Q^*(\tilde{X},\tilde{X}) \geq 0, \quad \forall \tilde{X} \in \Gamma^\perp_{X^*_F}, \tag{18}$$

where Q^* is evaluated at (A,p,u,x).

Proof. Note that A may not be invertible. But the same computations in the proof of Lemma 2.1 can be carried out without difficult. We may assume $\theta = (1,0,\cdots,0)$. In this case, all $X_{1j} = X_{j1} = \hat{X}_{1j} = \hat{X}_{j1} = 0$ for all $j = 1,\cdots,n$. Therefore, we can still perform corresponding inversions in the proof of Lemma 2.1. Also notice that $Z_j = \hat{Z}_j = 0$ for all $j = 2,\cdots,n$, because we restrict x variable in $\theta\mathbb{R} = \mathbb{R}^1$. ◻

It is clear that condition (10) is weaker than condition (3). The fact condition (11) is weaker than condition (10) can be seen from Lemma 2.1 and Lemma 2.2. Note that condition (11) has a dimensional deduction in symmetric matrix A. The remaind part of this paper is to refine the arguments in [2] to prove Theorem 1.3 under weaker condition (11).

With the assumptions of F and u in Theorem 1.3, u is automatically in $C^{3,1}$. This will be assumed in the rest of this paper. Let $W(x) = \nabla^2 u(x)$ and $l = \min_{x\in\Omega} \operatorname{rank}(\nabla^2 u(x))$. Since $l = n$ is of full rank, only $l \leq n-1$ is of interest. And this will be assume in the rest of the proof. Suppose $z_0 \in \Omega$ is a point where W is of minimal rank l.

Throughout this paper we use convention that $\sigma_j(W) = 0$ if $j < 0$ or $j > n$. For any symmetric function $f(W)$, denote

$$f^{ij} = \frac{\partial f(W)}{\partial u_{ij}}, \quad f^{ij,km} = \frac{\partial^2 f(W)}{\partial u_{ij} \partial u_{km}}$$

For each $z_0 \in \Omega$ where W is of minimal rank l. Pick an open neighborhood $\mathcal{O}$ of z_0, for any $x \in \mathcal{O}$, let $\lambda_1(x) \le \lambda_2(x) ... \le \lambda_n(x)$ be the eigenvalues of W at x. There is a positive constant $C > 0$ depending only on $\|u\|_{C^{3,1}}$, $W(z_0)$ and $\mathcal{O}$, such that $\lambda_n(x) \ge \lambda_{n-1}(x) ... \ge \lambda_{n-l+1}(x) \ge C$ for all $x \in \mathcal{O}$. Let $G = \{n-l+1, n-l+2, ..., n\}$ and $B = \{1, ..., n-l\}$ be the "good" and "bad" sets of indices respectively. Let $\Lambda_G = (\lambda_{n-l+1}, ..., \lambda_n)$ be the "good" eigenvalues of W at x and $\Lambda_B = (\lambda_1, ..., \lambda_{n-l})$ be the "bad" eigenvalues of W at x. For the simplicity, write $G = \Lambda_G$, $B = \Lambda_B$ if there is no confusion. Note that for any $\delta > 0$, we may choose $\mathcal{O}$ small enough such that $\lambda_i(x) < \delta$ for all $i \in B$ and $x \in \mathcal{O}$. Use notation $h = O(f)$ if $|h(x)| \le Cf(x)$ for $x \in \mathcal{O}$ with the positive constant C under control. It is clear that $\lambda_i = O(\phi)$ for all $i \in B$.

For $\epsilon > 0$ sufficient small, define

$$q_\epsilon(W) = \frac{\sigma_{l+2}(W_\epsilon)}{\sigma_{l+1}(W_\epsilon)}, \quad \phi_\epsilon(W) = \sigma_{l+1}(W_\epsilon) + q_\epsilon(W), \tag{19}$$

where $W_\epsilon = W + \epsilon I$. We will also denote

$$G_\epsilon = (\lambda_{n-l+1} + \epsilon, ..., \lambda_n + \epsilon), \quad B_\epsilon = (\lambda_1 + \epsilon, ..., \lambda_{n-l} + \epsilon).$$

To simplify the notation, we will write q for q_ϵ, W for W_ϵ, G for G_ϵ and B for B_ϵ with the understanding that all the estimates will be independent of ϵ. In this setting, with $\mathcal{O}$ is small enough, there is $C > 0$ independent of ϵ such that

$$\sigma_{l+1}(W(x)) \ge C\epsilon, \quad \text{and} \quad \sigma_1(B(x)) \ge C\epsilon, \quad \text{for all } x \in \mathcal{O}. \tag{20}$$

Similarly write $h = O(f)$ if $|h(x)| \le Cf(x)$ for $x \in \mathcal{O}$ with positive constant C under control independent of ϵ.

The importance of the function q is reflected in the following proposition.

Proposition 1. *[Proposition 2.1 and Corollary 2.2 in [2]] For each $z \in \mathcal{O}$ with $W(z)$ diagonal at z,*

$$\begin{aligned}\sum_{i,j,k,m} q^{ij,km} u_{ij\alpha} u_{km\beta} = O(\phi + \sum_{i,j \in B} |\nabla u_{ij}|) - \frac{\sum_{i,j \in B, i \ne j} u_{ij\alpha} u_{ji\beta}}{\sigma_1(B)} \\ - \frac{\sum_{i \in B} V_{i\alpha} V_{i\beta}}{\sigma_1^3(B)} - 2 \sum_{i \in B, j \in G} \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)\lambda_j} u_{ij\alpha} u_{ji\beta},\end{aligned} \tag{21}$$

where

$$V_{i\alpha} = u_{ii\alpha}\sigma_1(B) - (u_{ii} + \epsilon)\Big(\sum_{j \in B} u_{jj\alpha}\Big). \tag{22}$$

If $u \in C^{3,1}(\Omega)$ is a convex function and $l = \min_{x \in \Omega} \operatorname{rank}(W(x))$, then the function $q(x) = q(W(x))$ defined in (19) is in $C^{1,1}(\Omega)$ and its $C^{1,1}$ norm is bounded independent of ϵ.

We now prove a strong maximum principle for ϕ defined in (19) for equation (1). Theorem 1.3 is a direct consequence of Lemma 2.2 and the following proposition.

Proposition 2. *Suppose that the function F satisfies conditions (2) and (18) and let $u \in C^{3,1}(\Omega)$ be a convex solution of (1). If $\nabla^2 u$ attains its minimum rank l at certain point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O}$ of x_0 and a positive constant C independent of ϕ (defined in (19)), such that*

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta}(x) \le C(\phi(x) + |\nabla\phi(x)|), \quad \forall x \in \mathcal{O}. \tag{23}$$

In turn, $\nabla^2 u$ is of constant rank l in $\mathcal{O}$. Moreover, for each $x_0 \in \mathcal{O}$, there exist a neighborhood $\mathcal{U}$ of x_0 and $(n-l)$ fixed directions $V_1, \cdots, V_{n-l}$ such that $\nabla^2 u(x) V_j = 0$ for all $1 \le j \le n-l$ and $x \in \mathcal{U}$.

Proof. Let $u \in C^{3,1}(\Omega)$ be a convex solution of equation (1) and $W(x) = (u_{ij}(x))$. Let $z_0 \in \Omega$ be a point where $W = (\nabla^2 u)$ attains minimal rank l. We may assume $l \le n-1$, otherwise there is nothing to prove. Pick an open neighborhood $\mathcal{O}$ of z_0, for any $x \in \mathcal{O}$, let $G = \{n-l+1, n-l+2, ..., n\}$ and $B = \{1, ..., n-l\}$ be the "good" and "bad" sets of indices for eigenvalues of $\nabla^2 u(x)$ respectively.

Setting ϕ as (19), $\phi \in C^{1,1}(\mathcal{O})$ by Proposition 1. There is a constant $C > 0$ such that for all $x \in \mathcal{O}$,

$$\frac{1}{C}\sigma_1(B)(x) \le \phi(x) \le C\sigma_1(B)(x), \ \frac{1}{C}\sigma_1(B)(x) \le \sigma_{l+1}(W(x)) \le C\sigma_1(B)(x).$$

For each $z \in \mathcal{O}$ fixed, letting $\lambda_1 \le \lambda_2 ... \le \lambda_n$ be the eigenvalues of $(u_{ij}(z))$ at z, one may assume $(u_{ij}(z))$ is diagonal with proper choice of orthonormal coordinates, and $u_{ii}(z) = \lambda_i, i = 1, \cdots, n$. We will work on equation (1) to obtain the differential inequality (23) for ϕ_ϵ defined in (19) with constant C_1, C_2 independent of ϵ.

Note that (20) implies

$$\epsilon \le C\phi(x), \quad \text{for all } x \in \mathcal{O}. \tag{24}$$

And

$$\phi_\alpha = \frac{\partial\phi}{\partial x_\alpha} = \phi^{ij}u_{ij\alpha}, \quad \phi_{\alpha\beta} = \frac{\partial^2\phi}{\partial x_\alpha \partial x_\beta} = \phi^{ij}u_{ij\alpha\beta} + \phi^{ij,km}u_{ij\alpha}u_{km\beta}.$$

Differentiate equation (1) in x_i and then x_j to obtain

$$\sum_{\alpha,\beta} F^{\alpha\beta}u_{\alpha\beta i} + \sum_k F^{p_k}u_{ki} + F^u u_i + F^{x_i} = 0, \tag{25}$$

$$\begin{aligned}
&\sum_{\alpha,\beta} F^{\alpha\beta}u_{\alpha\beta ij} + \sum_k F^{p_k}u_{kij} + F^u u_{ij} \\
&+ \sum_{\alpha,\beta}\Big(\sum_{\gamma,\eta} F^{\alpha\beta,\gamma\eta}u_{\gamma\eta j} + \sum_k F^{\alpha\beta,p_k}u_{kj} + F^{\alpha\beta,u}u_j + F^{\alpha\beta,x_j}\Big)u_{\alpha\beta i} \\
&+ \sum_k\Big(\sum_{\alpha,\beta} F^{p_k,\alpha\beta}u_{\alpha\beta j} + \sum_l F^{p_k,p_l}u_{lj} + F^{p_k,u}u_j + F^{p_k,x_j}\Big)u_{ki} \\
&+ \Big(\sum_{\alpha,\beta} F^{u,\alpha\beta}u_{\alpha\beta j} + \sum_l F^{u,p_l}u_{lj} + F^{u,u}u_j + F^{u,x_j}\Big)u_i \\
&+ \sum_{\alpha,\beta} F^{x_i,\alpha\beta}u_{\alpha\beta j} + \sum_k F^{x_i,p_k}u_{kj} + F^{x_i,u}u_j + F^{x_i,x_j} = 0.
\end{aligned} \tag{26}$$

As $u_{\alpha\beta ij} = u_{ij\alpha\beta}$, we get

$$
\begin{aligned}
&\sum F^{\alpha\beta}\phi_{\alpha\beta} = \sum F^{\alpha\beta}\phi^{ij}u_{ij\alpha\beta} + \sum F^{\alpha\beta}\phi^{ij,km}u_{ij\alpha}u_{km\beta} \\
=&\sum F^{\alpha\beta}\phi^{ij,km}u_{ij\alpha}u_{km\beta} - \sum \phi^{ij}F^{p_k}u_{kij} - \sum \phi^{ij}[2\sum F^{\alpha\beta,p_k}u_{\alpha\beta i}u_{kj} \\
&+ F^u u_{ij} + \sum F^{p_k,p_l}u_{ki}u_{lj} + 2\sum F^{p_k,u}u_{ki}u_j + 2\sum F^{p_k,x_j}u_{ki}] \\
&- \sum \phi^{ij}[F^{\alpha\beta,\gamma\eta}u_{\alpha\beta i}u_{\gamma\eta j} + 2\sum F^{\alpha\beta,u}u_{\alpha\beta i}u_j + 2\sum F^{\alpha\beta,x_j}u_{\alpha\beta i} \\
&+ \sum F^{u,u}u_iu_j + 2\sum F^{u,x_i}u_j + \sum F^{x_ix_j}]
\end{aligned}
\tag{27}
$$

We estimate the terms in the right hand side of (27). The analysis those terms with third order derivatives which have with at least two indices in B is completely same as in [2], with the help of the concavity properties of the function q in (19). The remaining terms in (27) will be sorted out in such way so that condition (18) can be used to obtain appropriate control.

Since $W = (v_{ij}) = (u_{ij} + \varepsilon\delta_{ij})$ is diagonal at z, by Lemma 2.4 in [2],

$$
\phi^{ij}(z) = \begin{cases} \sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)} + O(\phi), & \text{if } i = j \in B \\ O(\phi), & \text{otherwise.} \end{cases}
\tag{28}
$$

Hence at z

$$
\begin{aligned}
&\sum_{i,j}\phi^{ij}[F^u u_{ij} + 2\sum F^{\alpha\beta,p_k}u_{\alpha\beta i}u_{kj} + \sum F^{p_k,p_l}u_{ki}u_{lj} \\
&\qquad + 2\sum(F^{p_k,u}u_{ki}u_j + F^{p_k,x_j}u_{ki})] \\
=&O(\phi) + \sum_{i=1}^{n}\phi^{ii}[F^u u_{ii} + 2\sum F^{\alpha\beta,p_i}u_{\alpha\beta i}u_{ii} + F^{p_i,p_i}u_{ii}u_{ii} \\
&\qquad + 2F^{p_i,u}u_{ii}u_i + 2F^{p_i,x_i}u_{ii}] \\
=&O(\phi) + \sum_{i\in B}\phi^{ii}[F^u + 2\sum F^{\alpha\beta,p_i}u_{\alpha\beta i} + F^{p_i,p_i}u_{ii} + 2F^{p_i,u}u_i + 2F^{p_i,x_i}]u_{ii} \\
\le&O(\phi) + C\sum_{i\in B}(\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)})u_{ii} = O(\phi),
\end{aligned}
\tag{29}
$$

since $\lambda_i = O(\phi), i \in B$ and $\sigma_{l+1}(W) \ge \sigma_l(G)\sigma_1(B)$. This takes care of the third term on the right hand side of (27). For the second term we have

$$
\sum \phi^{ij}F^{p_k}u_{kij} = O(\phi) + \sum_{i\in B}\phi^{ii}F^{p_k}u_{kii} = O(\phi + \sum_{i,j\in B}|\nabla u_{ij}|)
\tag{30}
$$

For the fourth term in (27), by (28) we have,

$$
\begin{aligned}
&\sum \phi^{ij}[F^{\alpha\beta,\gamma\eta}u_{\alpha\beta i}u_{\gamma\eta j}+2F^{\alpha\beta,u}u_{\alpha\beta i}u_j+2F^{\alpha\beta,x_j}u_{\alpha\beta i}\\
&\qquad\qquad +F^{u,u}u_iu_j+2F^{u,x_i}u_j+F^{x_ix_j}]\\
=&O(\phi)+\sum_{i\in B}\phi^{ii}[\sum F^{\alpha\beta,\gamma\eta}u_{\alpha\beta i}u_{\gamma\eta i}+2\sum F^{\alpha\beta,u}u_{\alpha\beta i}u_i\\
&\qquad\qquad +2\sum F^{\alpha\beta,x_i}u_{\alpha\beta i}+F^{u,u}u_i^2+2F^{u,x_i}u_i+F^{x_ix_i}]\\
=&O(\phi+\sum_{i,j\in B}|\nabla u_{ij}|)+\sum_{i\in B}(\sigma_l(G)+\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)})\\
&\qquad [\sum_{\alpha,\beta,\gamma,\eta\in G}F^{\alpha\beta,\gamma\eta}u_{i\alpha\beta}u_{i\gamma\eta}+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,u}u_{i\alpha\beta}u_i\\
&\qquad +2\sum_{\alpha,\beta\in G}F^{\alpha\beta,x_i}u_{i\alpha\beta}+F^{u,u}u_i^2+2F^{u,x_i}u_i+F^{x_ix_i}].
\end{aligned}
\tag{31}
$$

Now deal with the first term $\sum F^{\alpha\beta}\phi^{ij,km}u_{ij\alpha}u_{km\beta}$ in (27). Note that

$$\phi^{ij,km}=\sigma_{l+1}^{ij,km}+q^{ij,km}.$$

Since $\sigma_{l-1}(W|ij)=O(\phi)$ for $i,j\in G, i\neq j$, for α,β fixed, by Lemma 2.3 in [2],

$$
\begin{aligned}
\sum\sigma_{l+1}^{ij,km}u_{ij\alpha}u_{km\beta}=&\sum_{i\neq k}\sigma_{l+1}^{ii,kk}u_{ii\alpha}u_{kk\beta}+\sum_{i\neq j}\sigma_{l+1}^{ij,ji}u_{ij\alpha}u_{ji\beta}\\
=&\sum_{i\neq k}\sigma_{l-1}(W|ik)v_{ii\alpha}u_{kk\beta}-\sum_{i\neq j}\sigma_{l-1}(W|ij)u_{ij\alpha}u_{ji\beta}\\
=&O(\phi+\sum_{i,j\in B}|\nabla u_{ij}|)-2\sum_{i\in B,j\in G}\sigma_{l-1}(G|j)u_{ij\alpha}u_{ij\beta}.
\end{aligned}
$$

As $\sigma_{l-1}(G|j)=\frac{\sigma_l(G)}{\lambda_j}, j\in G$, we have

$$\sigma_{l+1}^{ij,km}u_{ij\alpha}u_{km\beta}=O(\phi+\sum_{i,j\in B}|\nabla u_{ij}|)-2\sigma_l(G)\sum_{i\in B,j\in G}\frac{1}{\lambda_j}u_{ij\alpha}u_{ij\beta}.$$

By Proposition 1,

$$
\begin{aligned}
\sum_{i,j,k,m}q^{ij,km}u_{ij\alpha}u_{km\beta}=&O(\phi+\sum_{i,j\in B}|\nabla u_{ij}|)-\frac{1}{\sigma_1(B)}\sum_{i,j\in B,i\neq j}u_{ij\alpha}u_{ji\beta}\\
&-\frac{\sum_{i\in B}V_{i\alpha}V_{i\beta}}{\sigma_1^3(B)}-2\sum_{i\in B,j\in G}\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)\lambda_j}u_{ij\alpha}u_{ji\beta},
\end{aligned}
$$

where $V_{i\alpha}$ is defined in (22). We conclude that

$$
\begin{aligned}
\sum F^{\alpha\beta}\phi^{ij,km}u_{ij\alpha}u_{km\beta}=&-\sum_{\alpha,\beta}F^{\alpha\beta}[\frac{\sum_{i\in B}V_{i\alpha}V_{i\beta}}{\sigma_1^3(B)}+\frac{\sum_{i,j\in B,i\neq j}u_{ij\alpha}u_{ji\beta}}{\sigma_1(B)}\\
&+2\sum_{i\in B}(\sigma_l(G)+\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)})\frac{u_{ij\alpha}u_{ji\beta}}{\lambda_j}]\\
&+O(\phi+\sum_{i,j\in B}|\nabla u_{ij}|).
\end{aligned}
\tag{32}
$$

Combining (29)-(32), one reduces (27) to

$$
\begin{aligned}
&\sum F^{\alpha\beta}\phi_{\alpha\beta}\\
=&O(\phi+\sum_{i,j\in B}|\nabla u_{ij}|)-\sum_{\alpha,\beta}F^{\alpha\beta}[\frac{\sum_{i\in B}V_{i\alpha}V_{i\beta}}{\sigma_1^3(B)}+\frac{\sum_{i,j\in B,i\neq j}u_{ij\alpha}u_{ji\beta}}{\sigma_1(B)}]\\
&-\sum_{i\in B}[\sigma_l(G)+\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)}][\sum_{\alpha,\beta,\gamma,\eta\in G}F^{\alpha\beta,\gamma\eta}u_{i\alpha\beta}u_{i\gamma\eta}\\
&+2\sum_{\alpha\beta\in G}F^{\alpha\beta}\sum_{j\in G}\frac{1}{\lambda_j}u_{ij\alpha}u_{ij\beta}+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,u}u_{i\alpha\beta}u_i\\
&+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,x_i}u_{i\alpha\beta}+F^{u,u}u_i^2+2F^{u,x_i}u_i+F^{x_i,x_i}].
\end{aligned}\tag{33}
$$

For each $i\in B$, set

$$
\begin{aligned}
J_i=&\sum_{\alpha,\beta,\gamma,\eta\in G}F^{\alpha\beta,\gamma\eta}u_{i\alpha\beta}u_{i\gamma\eta}+2\sum_{\alpha,\beta\in G}F^{\alpha\beta}\sum_{j\in G}\frac{1}{\lambda_j}u_{ij\alpha}u_{ij\beta}\\
&+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,u}u_{i\alpha\beta}u_i+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,x_i}u_{i\alpha\beta}\\
&+F^{u,u}u_i^2+2F^{u,x_i}u_i+F^{x_i,x_i},
\end{aligned}\tag{34}
$$

where functions $F^{\alpha\beta,\gamma\eta},\cdots$ are evaluated at $(\nabla^2u(z),\nabla u(z),u(z),z)$. So far, we have followed same lines of arguments in [2]. We now modify the arguments in [2] to use of new structural condition (18) to control J_i in (34).

Condition (9) implies that $G\neq\emptyset$, we may assume

$$
u_{nn}(z)>0,\quad i=1\in B,\quad \text{and}\quad \theta=(1,0\cdots,0)=e_1.
$$

By condition (2), and the assumption of F and u (since $\bar{\mathcal{O}}\subset\Omega$), there exists a constant $\delta_0>0$ independent of the lower bound of ∇^2u and ϵ, such that

$$
(F^{\alpha\beta})\geq\delta_0 I,\ \forall y\in\mathcal{O}.\tag{35}
$$

In particular, $F^{nn}\geq\delta_0$. For any $\delta>0$ small enough, set

$$
\tilde{A}_\delta=\begin{pmatrix}0 & 0\\ 0 & (u_{ij}(z)+\delta\delta_{ij})_{i,j=2}^n\end{pmatrix}.
$$

So, $\tilde{A}_\delta\in\mathcal{S}^+_{e_1}$ and

$$
\tilde{A}_\delta-\nabla^2u(z)=O(\delta+\phi),\quad F(\tilde{A}_\delta,\nabla u(z),u(z),z)=O(\delta+\phi).
$$

If $\mathcal{O}$ is sufficient small around the minimal rank point x_0 and choose $0<\delta<<u_{nn}(z)$, (35) and the mean value theorem imply that there is $|\lambda_\delta|\leq C(\delta+\phi)$,

$$
A_\delta=\tilde{A}_\delta+\begin{pmatrix}\mathbf{0} & 0\\ 0 & \lambda_\delta\end{pmatrix}\in\mathcal{S}^+_{e_1},\quad F(A_\delta,\nabla u(z),u(z),z)=0.\tag{36}
$$

where $\mathbf{0}$ is the $(n-1)\times(n-1)$ zero matrix.

(25) and (36) yield

$$
\sum_{\alpha,\beta\in G}F^{\alpha\beta}u_{\alpha\beta1}+F^uu_1+F^{x_1}=g;
$$

where $F^{\alpha\beta}, F^u, etc.$ are evaluated at $(A_\delta, \nabla u(z), u(z), z)$, and

$$g = O(\delta + \phi + \sum_{i,j\in B} |\nabla u_{ij}|).$$

Set

$$\begin{aligned}&X_{nn} = u_{1nn} - \frac{g}{F^{nn}};\\ &X_{\alpha\beta} = 0, \forall \alpha \in B; \quad X_{\alpha\beta} = u_{\alpha\beta 1}, \quad \text{otherwise};\\ &Y = u_1, \quad Z_k = \delta_{k1}, \forall k,\end{aligned}$$

again, F^{nn} is evaluated at $(A_\delta, \nabla u(z), u(z), z)$.

Thus $\tilde{X} = ((X_{\alpha\beta}), Y, Z_1, \cdots, Z_n) \in \Gamma^{\perp}_{X_F^*}(A_\delta, \nabla u(z), u(z), z)$. Conditions (18) and (36) imply (by letting $\delta \to 0$)

$$J_i \geq -C(\phi + \sum_{i,j\in B} |\nabla u_{ij}|). \tag{37}$$

Since $C \geq \sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)} \geq 0$,

$$\begin{aligned}\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \leq& C(\phi + \sum_{i,j\in B} |\nabla u_{ij}|)\\ &- \sum_{\alpha,\beta} F^{\alpha\beta}(\frac{\sum_{i\in B} V_{i\alpha}V_{i\beta}}{\sigma_1^3(B)} + \frac{\sum_{i,j\in B, i\neq j} u_{ij\alpha}u_{ji\beta}}{\sigma_1(B)}).\end{aligned} \tag{38}$$

The term $\sum_{i,j\in B} |\nabla u_{ij}|$ in (38) can be controlled in the same way as in [2] using the remaind terms on the right hand side. Here is a sketch.

By (35),

$$\sum_{\alpha,\beta} F^{\alpha\beta} V_{i\alpha}V_{i\beta} \geq \delta_0 \sum_{\alpha=1}^n V_{i\alpha}^2, \quad \sum_{\alpha,\beta} F^{\alpha\beta} u_{ij\alpha}u_{ij\beta} \geq \delta_0 \sum_{\alpha=1}^n u_{ij\alpha}^2.$$

Inserting above inequalities into (38),

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \leq C(\phi + \sum_{i,j\in B} |\nabla u_{ij}|) - \delta_0 \sum_{\alpha=1}^n [\frac{\sum_{i\in B} V_{i\alpha}^2}{\sigma_1^3(B)} + \frac{\sum_{i,j\in B\ i\neq j} |u_{ij\alpha}|^2}{\sigma_1(B)}]. \tag{39}$$

From Proposition 1, it follows that

$$\phi_\alpha = O(\phi) + \sum_{i\in B} (\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}) u_{ii\alpha}. \tag{40}$$

By Lemma 3.3 in [2] and (39), there exist positive constants C_1, C_2 independent of ϵ and the lower bound of $tr(\nabla^2 u(x))$, such that

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \leq C_1(\phi + |\nabla\phi|) - C_2 \sum_{i,j\in B} |\nabla u_{ij}|. \tag{41}$$

Taking $\epsilon \to 0$, (23) is verified for all $z \in \mathcal{O}$.

The Strong Maximum Principle implies $\phi \equiv 0$ in $\mathcal{O}$. Since Ω is flat, following the arguments in [3, 11], for any $x_0 \in \Omega$, there is a neighborhood $\mathcal{U}$ and $(n-l)$ fixed directions $V_1, \cdots, V_{n-l}$ such that $\nabla^2 u(x) V_j = 0$ for all $1 \leq j \leq n-l$ and $x \in \mathcal{U}$. $\square$

There is also a parabolic version of Theorem 1.3 for the equation in the form

$$u_t = F(\nabla^2 u, \nabla u, u, x, t). \tag{42}$$

Here one needs stronger structural condition for (42): for each p, x_0 and $t > 0$,

$$F(A, p, u, x + x_0, t) \text{ is locally convex in}(B, u, x), \tag{43}$$

for $A \in \mathcal{S}^+_\theta$, $B = A^{-1}$ on $(\theta\mathbb{R})^\perp$. Condition (43) was discussed in [12] for the preservation of convexity of equation (42) in whole space $\mathbb{R}^n$. The same lines of proof in [2] with modifications in this paper, we can prove

Theorem 2.3. *Suppose that the function F satisfies conditions (2) and (43) for each $t \in [0, T]$, let $u \in C^{3,1}(\Omega \times [0, T]$ is a convex solution of (42). For each $0 < t_0 \le T$, if $\nabla^2 u$ attains minimum rank l at certain point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O}$ of x_0 and two positive constant C_1, C_2 independent of ϕ (defined in (19)), such that for t close to t_0, $\sigma_l(u_{ij}(x, t)) > 0$ for $x \in \mathcal{O}$, and*

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta}(x) - \phi_t \le C_1\phi(x) + C_2|\nabla\phi(x)|, \quad \forall x \in \mathcal{O}. \tag{44}$$

Remark 1. Condition (9) forces $G \neq \emptyset$, that was used in the proof to create appropriate $\tilde{X}$ to get (37). When $G = \emptyset$, this trick can not apply if $|F^u| + |F^{x_i}|$ does not have lower bound. On the other hand, in this case,

$$J_i = F^{u,u}u_i^2 + 2F^{u,x_i}u_i + F^{x_i,x_i}, \tag{45}$$

where $F^{u,u}, F^{u,x_i}, F^{x_i,x_i}$ are evaluated at $(\nabla^2 u(z), \nabla u(z), u(z), z)$. For each p, set

$$\Gamma_0 = \{F(\mathbf{0}, p, u, x) = 0\}.$$

$G = \emptyset$ implies $|\nabla^2 u(z)| \le C\phi(z)$, therefore the following condition will guarantee (37) in this case: for each $p = \nabla u(x)$ fixed, $F(\mathbf{0}, p, u, x)$ is locally convex in (u, x) near Γ_0.

3. **Partial convexity.**

We treat the the constant rank theorem for partially convex solutions of fully nonlinear elliptic equation (1) with ellipticity assumption (2).

Let $N = N' + N''$ with N' and N'' are two positive integers. Write $x = (x', x'')$ with $x' \in \mathbb{R}^{N'}$ and $x'' \in \mathbb{R}^{N''}$ respectively. As in the case of the study of the full convexity, homotopic deformation argument (provided if there is a homotopic path) would reduce this problem to a constant rank theorem. The question we want to address is, *when $\nabla^2_{x'} u(x', x'')$ has constant rank?*

Let us write $x = (x', x'') \in \Omega$ and $p = (p', p'') \in R^N$ with $p' \in R^{N'}$, $p'' \in R^{N''}$ and split a matrix $A \in \mathcal{S}^N$ into $\begin{pmatrix} a & b \\ b^T & c \end{pmatrix}$ with $a \in \mathcal{S}^{N'}$, $b \in \mathbb{R}^{N' \times N''}$ and $c \in \mathcal{S}^{N''}$. Let

$$\mathcal{S}^{N,\oplus} = \{A \in \mathcal{S}^N | A = \begin{pmatrix} a & b \\ b^T & c \end{pmatrix}, a \in \mathcal{S}^{N'}_+\}$$

Define for $(A, p, u, x) \in \mathcal{S}^{N,\oplus} \times \mathbb{R}^N \times \mathbb{R} \times \Omega$

$$\tilde{F}(A, p'', u, x') = F(\begin{pmatrix} a^{-1} & a^{-1}b \\ (a^{-1}b)^T & c + b^T a^{-1} b \end{pmatrix}, p, u, x)$$

For each fixed x'' and $p' \in \mathbb{R}^{N'}$, define the zero sub-level set

$$\Gamma_F = \{(A, p'', u, x') \in \mathcal{S}^{N,\oplus} \times \mathbb{R}^{N''} \times \mathbb{R} \times \mathbb{R}^{N'} | \tilde{F}(A, p'', u, x') \le 0\}. \tag{46}$$

We say u is partially convex if $u(x^{'}, x^{''})$ is convex in the first variable $x^{'}$ for each fixed $x^{''}$.

Theorem 3.1. *Let $F = F(r, p, u, x) \in C^{2,1}(\mathcal{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega)$ and let $u \in C^{2,1}(\Omega)$ be a partially convex solution of (1). Suppose F satisfies condition (2) at $(\nabla^2 u(x), \nabla u(x), u(x), x)$ for each $x \in \Omega$. If for each $x \in \Omega$ and $p = \nabla u(x)$,*

$$\Gamma_F \text{ is locally convex in } (A, p^{''}, u, x^{'}), \tag{47}$$

then the rank of the hessian $(\nabla^2_{x^{'}} u(x^{'}, x^{''}))$ is constant in Ω.

Under the stronger structural condition

$$\tilde{F}(A, p^{''}, u, x^{'}) \text{ is convex}, \tag{48}$$

the above theorem was proved by C. Chen [6] following the arguments in [2]. The partial convexity of solutions of equation (1) under (48) with state constraint boundary condition on convex domains was studied in [1]. The proof of Theorem 3.1 will make use of the refined arguments in section 2.

Set

$$X_F^* = X_F^*(A, p, u, x)) = ((F^{\alpha\beta}), (F^{p_{N'+1}}, \cdots, F^{p_N}), F^u, (F^{x_1}, \cdots, F^{x_{N'}})), \tag{49}$$

$$\begin{aligned} \Gamma^{\perp}_{X_F^*} =& \Gamma^{\perp}_{X_F^*}(A, p, u, x) \\ =& \{\tilde{X} \in \mathcal{S}^N \times \mathbb{R}^{N^{''}} \times \mathbb{R} \times \mathbb{R}^{N^{'}} \mid < \tilde{X}, X_F^*(A, p, u, x) >= 0\}. \end{aligned} \tag{50}$$

X_F^* is a vector in $\mathcal{S}^N \times \mathbb{R}^{N^{''}} \times \mathbb{R} \times \mathbb{R}^{N^{'}}$, where functions $F^{\alpha\beta}, F^{p_{N'+1}}, \cdots, F^{p_N}, F^u, F^{x_i}$ are evaluated at (A, p, u, x).

For $\tilde{X} = \big((X_{ij}), (P_i), Y, (Z_k)\big) \in \mathcal{S}^N \times \mathbb{R}^{N^{''}} \times \mathbb{R} \times \mathbb{R}^{N^{'}}$, define a quadratic form

$$\begin{aligned} Q^*(\tilde{X}, \tilde{X}) =& \sum_{i,j,k,l=1}^{N} F^{ij,kl} X_{ij} X_{kl} + 2 \sum_{i,j=1}^{N} \sum_{k,l=1}^{N^{'}} F^{ij} a^{kl} X_{ik} X_{jl} \\ &+ 2 \sum_{i,j=1}^{N} F^{ij,u} X_{ij} Y + 2 \sum_{i,j=1}^{N} \sum_{k=1}^{N^{'}} F^{ij,x_k} X_{ij} Z_k + \sum_{i,j=N'+1}^{N} F^{p_i,p_j} P_i P_j \\ &+ 2 \sum_{i=N'+1}^{N} F^{p_i,u} P_i Y + 2 \sum_{i=N'+1}^{N} \sum_{j=1}^{N^{'}} F^{p_i,x_j} P_i Z_j + 2 \sum_{i=1}^{N^{'}} F^{u,x_i} Y Z_i \\ &+ F^{u,u} Y^2 + \sum_{i,j=1}^{N^{'}} F^{x_i,x_j} Z_i Z_j + 2 \sum_{i,j=1}^{N} \sum_{k=N'+1}^{N} F^{ij,p_k} X_{ij} P_k, \end{aligned} \tag{51}$$

again functions $F^{ij,kl}, F^{ij}, F^{u,u}, F^{ij,u}, F^{ij,x_k}, F^{u,x_i}, F^{x_i,x_j}$ are evaluated at (A, p, u, x) with $A = \begin{pmatrix} a & b \\ b^T & c \end{pmatrix} \in \mathcal{S}^{N,\oplus}$ and $a^{-1} = (a^{kl})$. Theorem 3.1 is based on the following lemma.

Lemma 3.2. *If $(A, p, u, x) \in \mathcal{S}^{N,\oplus} \times \mathbb{R}^N \times \mathbb{R} \times \Omega$, $A = \begin{pmatrix} a & b \\ b^T & c \end{pmatrix}$, such that $F(A, p, u, x) = 0$, then Γ_F is locally convex at $(\begin{pmatrix} a^{-1} & a^{-1}b \\ (a^{-1}b)^T & c - b^T a^{-1} b \end{pmatrix}, p^{''}, u, x^{'})$*

if and only if

$$Q^*(\tilde{X},\tilde{X}) \geq 0 \tag{52}$$

for every $\tilde{X} = \big((X_{ij}),(P_i),Y,(Z_k)\big) \in \Gamma^{\perp}_{X^*_F}(A,p,u,x)$.

Proof of Lemma 3.2. Fix $p^{'}$ and $x^{''}$, let

$$\tilde{F}(\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b}^T & \tilde{c} \end{pmatrix}, p^{''}, u, x^{'}) = F(\begin{pmatrix} \tilde{a}^{-1} & \tilde{a}^{-1}\tilde{b} \\ (\tilde{a}^{-1}\tilde{b})^T & \tilde{c}+\tilde{b}^T\tilde{a}^{-1}\tilde{b} \end{pmatrix}, p, u, x).$$

Then the condition (47) is equivalent to

$$\begin{aligned} &\sum_{\alpha,\beta,\gamma,\eta=1}^{N} \tilde{F}^{\alpha\beta,\gamma\eta}\hat{X}_{\alpha\beta}\hat{X}_{\gamma\eta} + 2\sum_{\alpha,\beta=1}^{N}(\sum_{k=N'+1}^{N}\tilde{F}^{\alpha\beta,p_k}\hat{X}_{\alpha\beta}\hat{P}_k \\ &+ \tilde{F}^{\alpha\beta,u}\hat{X}_{\alpha\beta}\hat{Y} + \sum_{k=1}^{N^{'}}\tilde{F}^{\alpha\beta,x_k}\hat{X}_{\alpha\beta}\hat{Z}_k) + \sum_{i,j=N'+1}^{N}\tilde{F}^{p_i,p_j}\hat{P}_i\hat{P}_j \\ &+ 2\sum_{i=N'+1}^{N}\tilde{F}^{p_i,u}\hat{P}_i\hat{Y} + 2\sum_{i=N'+1}^{N}\sum_{j=1}^{N^{'}}\tilde{F}^{p_i,x_j}\hat{P}_i\hat{Z}_j \\ &+ \tilde{F}^{u,u}\hat{Y}^2 + 2\sum_{k=1}^{N^{'}}\tilde{F}^{u,x_k}\hat{Y}\hat{Z}_k + \sum_{i,j=1}^{N^{'}}\tilde{F}^{x_i,x_j}\hat{Z}_i\hat{Z}_j \geq 0 \end{aligned} \tag{53}$$

for every $\hat{X} \in \mathcal{S}^N$, $\hat{P} = (\hat{P}_i) \in \mathbb{R}^{N^{''}}$, $\hat{Y} \in \mathbb{R}$, $\hat{Z} = (\hat{Z}_i) \in \mathbb{R}^{N^{'}}$, with

$$\sum_{\alpha,\beta=1}^{N}\tilde{F}^{\alpha\beta}\hat{X}_{\alpha\beta} + \sum_{i=N'+1}^{N}\tilde{F}^{P_i}\hat{P}_i + \tilde{F}^u\hat{Y} + \sum_{i=1}^{N^{'}}\tilde{F}^{x_i}\hat{Z}_i = 0.$$

where functions $\tilde{F}^{\alpha\beta,\gamma\eta},\cdots$ are evaluated at

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b}^T & \tilde{c} \end{pmatrix} = (\begin{pmatrix} a^{-1} & a^{-1}b \\ (a^{-1}b)^T & c-b^Ta^{-1}b \end{pmatrix}.$$

Decompose $\begin{pmatrix} \tilde{a}^{-1} & \tilde{a}^{-1}\tilde{b} \\ (\tilde{a}^{-1}\tilde{b})^T & \tilde{c}+\tilde{b}^T\tilde{a}^{-1}\tilde{b} \end{pmatrix} = E^TD^{-1}E + C$ with $D = \begin{pmatrix} \tilde{a} & 0 \\ 0 & I \end{pmatrix}$, $E = \begin{pmatrix} I & \tilde{b} \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{c} \end{pmatrix}$. A direct computation yields

$$\tilde{F}^{\alpha\beta} = \begin{cases} -\sum_{i,j=1}^{N}F^{ij}\sum_{k,l=1}^{N^{'}}E_{ki}E_{lj}\tilde{a}^{k\alpha}\tilde{a}^{l\beta}, & 1 \leq \alpha,\beta \leq N^{'}, \\ \sum_{i,j=1}^{N}F^{ij}\sum_{k=1}^{N^{'}}(E_{ki}\delta_{j\beta}+E_{kj}\delta_{i\beta})\tilde{a}^{k\alpha}, & 1 \leq \alpha \leq N^{'}, N^{'}+1 \leq \beta \leq N, \\ F^{\alpha\beta}, & N^{'}+1 \leq \alpha,\beta \leq N. \end{cases}$$

And we get

$$\begin{aligned} \tilde{F}^{\alpha\beta,\gamma\eta} = &\sum_{i,j,k,l=1}^{N}F^{ij,kl}(\sum_{m,n=1}^{N^{'}}E_{mi}E_{nj}\tilde{a}^{m\alpha}\tilde{a}^{n\beta})(\sum_{m,n=1}^{N^{'}}E_{mk}E_{nl}\tilde{a}^{m\gamma}\tilde{a}^{n\eta}) \\ &+ \sum_{i,j=1}^{N}F^{ij}\sum_{k,l=1}^{N^{'}}E_{ki}E_{lj}(\tilde{a}^{k\gamma}\tilde{a}^{\eta\alpha}\tilde{a}^{\beta l} + \tilde{a}^{k\alpha}\tilde{a}^{\beta\eta}\tilde{a}^{l\gamma}) \end{aligned}$$

for $1 \le \alpha, \beta, \gamma, \eta \le N^{'}$,

$$\tilde{F}^{\alpha\beta,\gamma\eta} = -\sum_{i,j,k,l=1}^{N} F^{ij,kl} \sum_{m,n,h=1}^{N^{'}} E_{mi}E_{nj}\tilde{a}^{m\alpha}\tilde{a}^{n\beta}\tilde{a}^{\gamma h}(E_{hl}\delta_{k\eta} + E_{hk}\delta_{l\eta})$$

$$-\sum_{i,j=1}^{N} F^{ij}\sum_{k=1}^{N^{'}}(E_{kj}\delta_{i\eta}\tilde{a}^{\gamma\alpha}\tilde{a}^{k\beta} + E_{ki}\delta_{j\eta}\tilde{a}^{\gamma\beta}\tilde{a}^{\alpha k})$$

for $1 \le \alpha, \beta, \gamma \le N^{'}, N^{'} + 1 \le \eta \le N$,

$$\tilde{F}^{\alpha\beta,\gamma\eta} = -\sum_{i,j=1}^{N} F^{ij,\gamma\eta} \sum_{m,n=1}^{N^{'}} E_{mi}E_{nj}\tilde{a}^{m\alpha}\tilde{a}^{n\beta}$$

for $1 \le \alpha, \beta \le N^{'}$, $N^{'} + 1 \le \gamma, \eta \le N$,

$$\tilde{F}^{\alpha\beta,\gamma\eta} = \sum_{i,j,k,l=1}^{N} F^{ij,kl}[\sum_{m=1}^{N^{'}} \tilde{a}^{\alpha m}(E_{mj}\delta_{i\beta} + E_{mi}\delta_{j\beta})][\sum_{n=1}^{N^{'}} \tilde{a}^{\gamma n}(E_{nl}\delta_{k\eta} + E_{nk}\delta_{l\eta})]$$

$$+\sum_{i,j=1}^{N} F^{ij}\tilde{a}^{\alpha\gamma}(\delta_{i\beta}\delta_{j\eta} + \delta_{j\beta}\delta_{i\eta})$$

for $1 \le \alpha, \gamma \le N^{'}, N^{'} + 1 \le \beta, \eta \le N$,

$$\tilde{F}^{\alpha\beta,\gamma\eta} = \sum_{i,j=1}^{N} F^{ij,\gamma\eta} \sum_{m=1}^{N^{'}}(E_{mj}\delta_{i\beta} + E_{mi}\delta_{j\beta})\tilde{a}^{m\alpha}$$

for $1 \le \alpha \le N^{'}, N^{'} + 1 \le \gamma, \beta, \eta \le N$,

$$\tilde{F}^{\alpha\beta,\gamma\eta} = F^{\alpha\beta,\gamma\eta}$$

for $N^{'} + 1 \le \alpha, \gamma, \beta, \eta \le N$. Other derivatives can be calculated in a similar way. Where functions $F^{ij,kl}, F^{ij}, F^{u,u}, F^{ij,u}, F^{ij,x_k}, F^{u,x_i}, F^{x_i,x_j}$ are evaluated at (A, p, u, x). Set the relation between $\hat{X}$ and X is as follow

$$X_{ij} = \begin{cases} -\sum_{\alpha,\beta=1}^{N^{'}} \tilde{a}^{i\alpha}\tilde{a}^{j\beta}\hat{X}_{\alpha\beta}, & 1 \le i, j \le N^{'}, \\ 2\sum_{k=1}^{N^{'}} \tilde{a}^{ik}\hat{X}_{kj} \\ \qquad -\sum_{\alpha,\beta=1}^{N^{'}}\sum_{k=1}^{N^{'}} \tilde{a}^{i\alpha}\tilde{a}^{k\beta}\tilde{b}_{kj}\hat{X}_{\alpha\beta}, & 1 \le i \le N^{'},\ N^{'} + 1 \le j \le N, \\ 2\sum_{k=1}^{N^{'}} \tilde{a}^{kj}\hat{X}_{ki} \\ \qquad -\sum_{\alpha,\beta=1}^{N^{'}}\sum_{k=1}^{N^{'}} \tilde{a}^{j\alpha}\tilde{a}^{k\beta}\tilde{b}_{ki}\hat{X}_{\alpha\beta}, & N^{'} + 1 \le i \le N, 1 \le j \le N^{'}, \\ \hat{X}_{ij} + 2\sum_{k,l=1}^{N^{'}} \tilde{a}^{kl}(\tilde{b}_{ki}\hat{X}_{lj} + \tilde{b}_{kj}\hat{X}_{li}) \\ \qquad -\sum_{\alpha,\beta=1}^{N^{'}}\sum_{k,l=1}^{N^{'}} \tilde{a}^{k\alpha}\tilde{a}^{l\beta}\tilde{b}_{ki}\tilde{b}_{lj}\hat{X}_{\alpha\beta}, & N^{'} + 1 \le i, j \le N, \end{cases}$$

Substituting these into (53), equation (52) follows directly. ☐

The key inequality is (52). This inequality holds under even further weakened condition. Following the same notation as in section 2, denote $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ for integer n. For each $\theta^{'} \in \mathbb{S}^{N^{'}-1}$, let $\theta = (\theta^{'}, 0) \in \mathbb{S}^{N-1}$ and define

$$\theta^{'}\mathbb{R} = \{t\theta^{'} | t \in \mathbb{R}\}, \quad (\theta^{'}\mathbb{R})^{\perp} = \{\eta \in \mathbb{R}^{N^{'}} \mid <\eta, \theta >= 0\}$$

and

$$\mathcal{S}^{N'}_{\theta'} = \{a \in \mathcal{S}^{N'} | a\theta' = 0\}, \quad \mathcal{S}^{N',+}_{\theta'} = \{a \in \mathcal{S}^{N'} | a\theta' = 0, a > 0 \text{ on } (\theta'\mathbb{R})^{\perp}\}$$

$$\mathcal{S}^{N,\oplus}_{\theta'} = \{A \in \mathcal{S}^{N} | A = \begin{pmatrix} a & b \\ b^T & c \end{pmatrix}, a \in \mathcal{S}^{N',+}_{\theta'}\}, \quad \mathcal{S}^{N}_{\theta} = \{A \in \mathcal{S}^{N} | A\theta = 0\}$$

$$\mathcal{S}^{N,\oplus}_{\theta} = \{A \in \mathcal{S}^{N} | A = \begin{pmatrix} a & b \\ b^T & c \end{pmatrix}, a \in \mathcal{S}^{N',+}_{\theta'}, A\theta = 0\}$$

Suppose $A_0 = \begin{pmatrix} \hat{a}_0 & b_0 \\ b_0^T & c_0 \end{pmatrix} \in \mathcal{S}^{N,\oplus}_{\theta'}$, $a_0 \in \mathcal{S}^{N',+}_{\theta'}$ with $a_0 = \hat{a}_0^{-1}$ on $(\theta'\mathbb{R})^{\perp}$. Assume $x_0 \in \Omega$, $u_0 = u(x_0)$ and $p_0 = \nabla u(x_0)$ such that $F(A_0, p_0, u_0, x_0) = 0$. Let $(A, p'', u, x') \in \mathcal{S}^{N,\oplus}_{\theta} \times \mathbb{R}^{N''} \times \mathbb{R} \times \theta'\mathbb{R}$, $\hat{a} \in \mathcal{S}^{N',+}_{\theta'}$ with $\hat{a} = a^{-1}$ on $(\theta'\mathbb{R})^{\perp}$ and define

$$\tilde{F}_\theta(A, p'', u, x') = F(\begin{pmatrix} \hat{a} & b_0 + \hat{a}b \\ b_0^T + (\hat{a}b)^T & c_0 + c + b^T\hat{a}b \end{pmatrix}, p_0', p'', u, x' + x_0', x_0'').$$

Set

$$\Gamma^\theta_F = \{(A, p'', u, x') \in \mathcal{S}^{N,\oplus}_{\theta} \times \mathbb{R}^{N''} \times \mathbb{R} \times \theta'\mathbb{R} | \tilde{F}_\theta(A, p'', u, x') \le 0\}.$$

For $\tilde{X} = \big((X_{ij}), (P_i), Y, (Z_k)\big) \in \mathcal{S}^N_\theta \times \mathbb{R}^{N''} \times \mathbb{R} \times \theta'\mathbb{R}$, define a quadratic form

$$\begin{aligned} Q^*_\theta(\tilde{X}, \tilde{X}) = & \sum_{i,j,k,l=1}^{N} F^{ij,kl} X_{ij} X_{kl} + 2 \sum_{i,j=1}^{N} \sum_{k,l=1}^{N'} F^{ij} (a_0)_{kl} X_{ik} X_{jl} \\ & + 2 \sum_{i,j=1}^{N} F^{ij,u} X_{ij} Y + 2 \sum_{i,j=1}^{N} \sum_{k=1}^{N'} F^{ij,x_k} X_{ij} Z_k + \sum_{i,j=N'+1}^{N} F^{p_i,p_j} P_i P_j \\ & + 2 \sum_{i=N'+1}^{N} F^{p_i,u} P_i Y + 2 \sum_{i=N'+1}^{N} \sum_{j=1}^{N'} F^{p_i,x_j} P_i Z_j + 2 \sum_{i=1}^{N'} F^{u,x_i} Y Z_i \\ & + F^{u,u} Y^2 + \sum_{i,j=1}^{N'} F^{x_i,x_j} Z_i Z_j + 2 \sum_{i,j=1}^{N} \sum_{k=N'+1}^{N} F^{ij,p_k} X_{ij} P_k, \end{aligned} \tag{54}$$

where functions $F^{ij,kl}, \cdots$ are evaluated at (A_0, p_0, u_0, x_0) with $A_0 = \begin{pmatrix} \hat{a}_0 & b_0 \\ b_0^T & c_0 \end{pmatrix}$, $a_0 \in \mathcal{S}^{N',+}_{\theta'}$ and $a_0 = \hat{a}_0^{-1}$ on $(\theta'\mathbb{R})^{\perp}$.

Lemma 3.3. *For any fixed $x_0 \in \Omega$, $\theta' \in \mathbb{S}^{N'-1}$ and $A_0 = \begin{pmatrix} \hat{a}_0 & b_0 \\ b_0^T & c_0 \end{pmatrix} \in \mathcal{S}^{N,\oplus}_{\theta'}$ with $F(A_0, p_0, u_0, x_0) = 0$, Γ^θ_F is locally convex at $(\begin{pmatrix} a_0 & 0 \\ 0 & 0 \end{pmatrix}, p_0'', u(x_0), 0')$ if and only if*

$$Q^*_\theta(\tilde{X}, \tilde{X}) \ge 0 \tag{55}$$

for every $\tilde{X} = \big((X_{ij}), (P_i), Y, (Z_k)\big) \in \mathcal{S}^N_\theta \times \mathbb{R}^{N''} \times \mathbb{R} \times \theta'\mathbb{R}$ satisfying

$$< \tilde{X}, X^*_F(A_0, p_0, u_0, x_0) >= 0.$$

Condition (56) is weaker than condition (47) by Lemma 3.2 and Lemma 3.3.

Proof. Note that the matrix a may not be invertible. But the same computations in the proof of Lemma 3.2 can be carried out without difficult. We may assume $\theta^{'} = e_1^{'} = (1, 0, \cdots, 0)$. In this case, all $X_{1j} = X_{j1} = \hat{X}_{1j} = \hat{X}_{j1} = 0$ for all $j = 1, \cdots, N^{'}$. Therefore, we can still perform corresponding inversions in the proof of Lemma 3.2. Also notice that $Z_j = \hat{Z}_j = 0$ for all $j = 2, \cdots, N^{'}$, because we restrict $x^{'}$ variable in $\theta^{'}\mathbb{R} = \mathbb{R}^1$. □

Theorem 3.4. *The same conclusion in Theorem 3.1 is true if condition (47) is replaced by the following structural condition: for any fixed $x_0 \in \Omega$, $\theta^{'} \in \mathbb{S}^{N^{'}-1}$ and $A_0 = \begin{pmatrix} \hat{a}_0 & b_0 \\ b_0^T & c_0 \end{pmatrix} \in \mathcal{S}_{\theta^{'}}^{N,\oplus}$ with $a_0 \in \mathcal{S}_{\theta^{'}}^{N^{'},+}$, $a_0 = \hat{a}_0^{-1}$ on $(\theta^{'}\mathbb{R})^{\perp}$,*

$$\text{if } F(A_0, p_0, u_0, x_0) = 0 \text{ then } \Gamma_F^{\theta} \text{ is locally convex at } (\begin{pmatrix} a_0 & 0 \\ 0 & 0 \end{pmatrix}, p_0^{''}, u_0, 0^{'}). \tag{56}$$

Notice that u is automatically in $C^{3,1}$ by the assumptions of F and u in Theorem 3.4. As in section 2, let $W(x) = (\nabla^2 u(x))_{N^{'}\times N^{'}}$ and $l = \min_{x\in\Omega} \operatorname{rank} W(x)$. Since $l = N^{'}$ is of full rank, only $l \leq N^{'} - 1$ is of interest. And this will be assume in the rest of the proof. Suppose $z_0 \in \Omega$ is a point where W is of minimal rank l.

For each $z_0 \in \Omega$ where W is of minimal rank l. Pick an open neighborhood $\mathcal{O}$ of z_0, for any $x \in \mathcal{O}$, let $\lambda_1(x) \leq \lambda_2(x) ... \leq \lambda_{N^{'}}(x)$ be the eigenvalues of W at x. There is a positive constant $C > 0$ depending only on $\|u\|_{C^{3,1}}$, $W(z_0)$ and $\mathcal{O}$, such that $\lambda_{N^{'}}(x) \geq \lambda_{N^{'}-1}(x) ... \geq \lambda_{N^{'}-l+1}(x) \geq C$ for all $x \in \mathcal{O}$. Let $G = \{N^{'} - l + 1, N^{'} - l + 2, ..., N^{'}\}$ and $B = \{1, ..., N^{'} - l\}$ be the "good" and "bad" sets of indices respectively. Let $\Lambda_G = (\lambda_{N^{'}-l+1}, ..., \lambda_N^{'})$ be the "good" eigenvalues of W at x and $\Lambda_B = (\lambda_1, ..., \lambda_{N^{'}-l})$ be the "bad" eigenvalues of W at x. For the simplicity, write $G = \Lambda_G$, $B = \Lambda_B$ if there is no confusion. Note that for any $\delta > 0$, we may choose $\mathcal{O}$ small enough such that $\lambda_i(x) < \delta$ for all $i \in B$ and $x \in \mathcal{O}$. Use notation $h = O(f)$ if $|h(x)| \leq Cf(x)$ for $x \in \mathcal{O}$ with the positive constant C under control. It is clear that $\lambda_i = O(\phi)$ for all $i \in B$.

For $\epsilon > 0$ sufficient small, define

$$q_\epsilon(W) = \frac{\sigma_{l+2}(W_\epsilon)}{\sigma_{l+1}(W_\epsilon)}, \quad \phi_\epsilon(W) = \sigma_{l+1}(W_\epsilon) + q_\epsilon(W), \tag{57}$$

where $W_\epsilon = W + \epsilon I$. We will also denote

$$G_\epsilon = (\lambda_{N^{'}-l+1} + \epsilon, ..., \lambda_{N^{'}} + \epsilon), \quad B_\epsilon = (\lambda_1 + \epsilon, ..., \lambda_{N^{'}-l} + \epsilon).$$

As in section 2, we will drop subindex ϵ with the understanding that the estimates we carry on will be independent of ϵ. In this setting, with $\mathcal{O}$ is small enough, there is $C > 0$ independent of ϵ such that

$$\sigma_{l+1}(W(x)) \geq C\epsilon, \quad \text{and} \quad \sigma_1(B(x)) \geq C\epsilon, \quad \text{for all } x \in \mathcal{O}. \tag{58}$$

Similarly write $h = O(f)$ if $|h(x)| \leq Cf(x)$ for $x \in \mathcal{O}$ with positive constant C under control independent of ϵ.

Theorem 3.4 is a direct consequence of Lemma 3.3 and the following proposition.

Proposition 3. *Suppose that the function F satisfies conditions (2) and (55) and let $u \in C^{3,1}(\Omega)$ be a partially convex solution of (1). If $(\nabla^2 u)_{N'\times N'}$ attains its minimum rank l at certain point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O}$ of x_0 and a positive constant C independent of ϕ (defined in (57)), such that*

$$\sum_{\alpha,\beta=1}^{N} F^{\alpha\beta}\phi_{\alpha\beta}(x) \le C(\phi(x) + |\nabla\phi(x)|), \quad \forall x \in \mathcal{O}. \tag{59}$$

In turn, $(\nabla^2 u)_{N'\times N'}$ is of constant rank l in $\mathcal{O}$. Moreover, for each $x_0 \in \mathcal{O}$, there exist a neighborhood $\mathcal{U}$ of x_0 and $(N^{'} - l)$ fixed directions $V_1, \cdots, V_{N'-l} \in \mathbb{R}^{N^{'}}$ such that $(\nabla^2 u)_{N'\times N'} V_j = 0$ for all $1 \le j \le N^{'} - l$ and $x \in \mathcal{U}$.

Proof. The proof is similar to the proof of Proposition 2 in section 2. Following the same arguments as in section 2, one deduces

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^{N} F^{\alpha\beta}\phi_{\alpha\beta} \\
=&O(\phi + \sum_{i,j\in B}|\nabla u_{ij}|) - \sum_{\alpha,\beta=1}^{N} F^{\alpha\beta}[\frac{\sum_{i\in B} V_{i\alpha}V_{i\beta}}{\sigma_1^3(B)} + \frac{\sum_{i,j\in B, i\neq j} u_{ij\alpha}u_{ji\beta}}{\sigma_1(B)}] \\
&- \sum_{i\in B}[\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}][\sum_{\alpha,\beta,\gamma,\eta\notin B} F^{\alpha\beta,\gamma\eta}u_{i\alpha\beta}u_{i\gamma\eta} \\
&+ 2\sum_{\alpha,\beta\notin B} F^{\alpha\beta}\sum_{j\in G}\frac{1}{\lambda_j}u_{ij\alpha}u_{ij\beta} + 2\sum_{\alpha,\beta\notin B}(\sum_{k=N'+1}^{N} F^{\alpha\beta,p_k}u_{\alpha\beta i}u_{ik} \\
&+ F^{\alpha\beta,u}u_{i\alpha\beta}u_i + F^{\alpha\beta,x_i}u_{i\alpha\beta}) + \sum_{k,l=N'+1}^{N} F^{p_k,p_l}u_{ik}u_{il} \\
&\quad + 2\sum_{k=N'+1}^{N}(F^{p_k,u}u_{ik}u_i + F^{p_k,x_i}u_{ik}) + F^{u,u}u_i^2 + 2F^{u,x_i}u_i + F^{x_i,x_i}].
\end{aligned} \tag{60}$$

For each $i \in B$, set

$$\begin{aligned}
J_i =& \sum_{\alpha,\beta,\gamma,\eta\notin B} F^{\alpha\beta,\gamma\eta}u_{i\alpha\beta}u_{i\gamma\eta} + 2\sum_{\alpha,\beta\notin B} F^{\alpha\beta}\sum_{j\in G}\frac{1}{\lambda_j}u_{ij\alpha}u_{ij\beta} \\
&+ 2\sum_{\alpha,\beta\notin B}(\sum_{k=N'+1}^{N} F^{\alpha\beta,p_k}u_{\alpha\beta i}u_{ik} + F^{\alpha\beta,u}u_{i\alpha\beta}u_i + F^{\alpha\beta,x_i}u_{i\alpha\beta}) \\
&+ \sum_{k,l=N'+1}^{N} F^{p_k,p_l}u_{ik}u_{il} + 2\sum_{k=N'+1}^{N}(F^{p_k,u}u_{ik}u_i + F^{p_k,x_i}u_{ik}) \\
&+ F^{u,u}u_i^2 + 2F^{u,x_i}u_i + F^{x_i,x_i},
\end{aligned} \tag{61}$$

where functions $F^{\alpha\beta,\gamma\eta}, \cdots$ are evaluated at $(\nabla^2 u(z), \nabla u(z), u(z), z)$. We now want to make use of new structural condition (55) to control J_i in (61).

We may assume

$$i = 1 \in B, \quad \theta^{'} = (1, 0\cdots, 0) = e_1^{'} \quad \text{and} \quad \theta = (1, 0\cdots, 0) = e_1.$$

By condition (2), and the assumption of F and u (since $\bar{\mathcal{O}} \subset \Omega$), there exists a constant $\delta_0 > 0$ independent of the lower bound of $\nabla^2 u$ and ϵ, such that

$$(F^{\alpha\beta}) \geq \delta_0 I, \ \forall y \in \mathcal{O}. \tag{62}$$

In particular, $F^{NN} \geq \delta_0$. For any $\delta > 0$ small enough, set

$$\tilde{A}_\delta = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & (u_{ij}(z) + \delta\delta_{ij})_{i,j=2}^{N'} \end{pmatrix} & (u_{ij}(z)) \\ (u_{ji}(z)) & (u_{ij}(z))_{i,j=N'+1}^{N} \end{pmatrix}.$$

So, $\tilde{A}_\delta \in \mathcal{S}_{e_1'}^{N,\oplus}$ and

$$\tilde{A}_\delta - \nabla^2 u(z) = O(\delta + \phi), \quad F(\tilde{A}_\delta, \nabla u(z), u(z), z) = O(\delta + \phi).$$

If $\mathcal{O}$ is sufficient small around the minimal rank point x_0 and choose $\delta > 0$ small enough, (62) and the mean value theorem imply that there is $|\lambda_\delta| \leq C(\delta + \phi)$,

$$A_\delta = \tilde{A}_\delta + \begin{pmatrix} \mathbf{0} & 0 \\ 0 & \lambda_\delta \end{pmatrix} \in \mathcal{S}_{e_1'}^{N,\oplus}, \quad F(A_\delta, \nabla u(z), u(z), z) = 0. \tag{63}$$

where $\mathbf{0}$ is the $(N-1) \times (N-1)$ zero matrix.

Differentiate (1) in x_1, together with (63),

$$\sum_{\alpha,\beta \notin B} F^{\alpha\beta} u_{\alpha\beta 1} + \sum_{k=N'+1}^{N} F^{p_k,u} u_{1k} + F^u u_1 + F^{x_1} = g;$$

where $F^{\alpha\beta}, F^u, etc.$ are evaluated at $(A_\delta, \nabla u(z), u(z), z)$, and

$$g = O(\delta + \phi + \sum_{i,j \in B} |\nabla u_{ij}|).$$

Set

$$\begin{aligned} &X_{NN} = u_{1NN} - \frac{g}{F^{NN}}; \\ &X_{\alpha\beta} = 0, \forall \alpha \in B; \quad X_{\alpha\beta} = u_{\alpha\beta 1}, \quad \text{otherwise}; \\ &P_k = u_{k1}, N' + 1 \leq k \leq N, \quad Y = u_1, \quad Z_k = \delta_{k1}, \forall k, \end{aligned}$$

again, F^{NN} is evaluated at $(A_\delta, \nabla u(z), u(z), z)$.

Thus $\tilde{X} = ((X_{\alpha\beta}), (P_i), Y, Z_1, \cdots, Z_n) \in \Gamma^{\perp}_{X_F^*}(A_\delta, \nabla u(z), u(z), z)$. Conditions (55) and (63) imply (by letting $\delta \to 0$)

$$J_i \geq -C(\phi + \sum_{i,j \in B} |\nabla u_{ij}|). \tag{64}$$

Since $C \geq \sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)} \geq 0$,

$$\begin{aligned} \sum_{\alpha,\beta} F^{\alpha\beta} \phi_{\alpha\beta} \leq & C(\phi + \sum_{i,j \in B} |\nabla u_{ij}|) \\ & - \sum_{\alpha,\beta} F^{\alpha\beta} \left(\frac{\sum_{i \in B} V_{i\alpha} V_{i\beta}}{\sigma_1^3(B)} + \frac{\sum_{i,j \in B, i \neq j} u_{ij\alpha} u_{ji\beta}}{\sigma_1(B)} \right). \end{aligned} \tag{65}$$

The term $\sum_{i,j \in B} |\nabla u_{ij}|$ in (65) can be controlled in the same way as in [2] and section 2, we won't repeat it here. □

Remark 2. Since $N^{''} \geq 1$, we don't need the extra assumption (9) in Theorem 3.1. On the other hand, the structural condition (47) is much stronger than (10). In the sense, Theorem 3.1 is not as useful as Theorem 1.2. The partial convexity of solutions to fully nonlinear equations in the form (1) has significant geometric implications. In particular, it is important to understand this property for solutions of Monge-Ampère type equations.

A parabolic version of Theorem 3.4 can be proved for the equation in the form

$$u_t = F(\nabla^2 u, \nabla u, u, x, t).$$

In this case, the structural condition is: *for each fixed* $p^{'}$, x_0, b_0, c_0 *and* $t > 0$,

$$F(\begin{pmatrix} \hat{a} & b_0 + \hat{a}b \\ b_0^T + (\hat{a}b)^T & c_0 + c + b^T\hat{a}b \end{pmatrix}, p, u, x^{'} + x_0^{'}, x_0^{''}, t)$$

is locally convex in $\left(\begin{pmatrix} a & b \\ b^T & c \end{pmatrix}, p^{''}, u, x^{'}\right) \in \mathcal{S}_\theta^{N,\oplus} \times \mathbb{R}^{N^{''}} \times \mathbb{R} \times \theta^{'}\mathbb{R}$, *with* $\hat{a} \in \mathcal{S}_{\theta^{'}}^{N^{'},+}$, $\hat{a} = a^{-1}$ *on* $(\theta^{'}\mathbb{R})^{\perp}$.

REFERENCES

[1] O. Alvarez, J. M. Lasry and P.-L. Lions, *Convexity viscosity solutions and state constraints*, J. Math. Pures Appl., **76** (1997), 265–288.

[2] B. Bian and P. Guan, *A microscopic convexity principle for nonlinear partial differential equations*, Inventiones Math., **177** (2009), 307–335.

[3] L. Caffarelli and A. Friedman, *Convexity of solutions of some semilinear elliptic equations*, Duke Math. J., **52** (1985), 431–455.

[4] L. Caffarelli, P. Guan and X. Ma, *A constant rank theorem for solutions of fully nonlinear elliptic equations*, Comm. on Pure and Appl. Math., **60** (2007), 1769–1791.

[5] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian*, Acta Math., **155** (1985), 261–301.

[6] C. Chen, *A constant rank theorem for partial convex solutions of partial differential equations*, preprint.

[7] P. Guan, C. S. Lin and X. N. Ma, *The Christoffel-Minkowski problem II: Weingarten curvature equations*, Chinese Annals of Mathematics, Series B, **27** (2006), 595–614.

[8] P. Guan and X. N. Ma, *The Christoffel-Minkowski problem I: convexity of solutions of a Hessian equations*, Inventiones Math., **151** (2003), 553–577.

[9] P. Guan, X. N. Ma and F. Zhou, *The Christoffel-Minkowski problem III: Existence and convexity of admissible solutions*, Comm. Pure and Appl. Math., **59** (2006), 1352–1376.

[10] P. Hartman and L. Nirenberg, *On spherical image maps whose Jacobians do not change sign*, Amer. J. Math., **81** (1959), 901–920.

[11] N. Korevaar and J. Lewis, *Convex solutions of certain elliptic equations have constant rank hessians*, Arch. Rational Mech. Anal., **91** (1987), 19–32.

[12] P. Lions and M. Musiela, *Convexity of solutions of parabolic equations*, C. R. Math. Acad. Sci. Paris, **342** (2006), 915–921.

[13] I. Singer, B. Wong, S. T. Yau and Stephen S. T. Yau, *An estimate of gap of the first two eigenvalues in the Schrodinger operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4), **12** (1985), 319–333.

Received February 2010; revised April 2010.

E-mail address: bianbj@tongji.edu.cn
E-mail address: guan@math.mcgill.ca