Sixth Assignment, due by 14:00, on November 23, 2017.

Problem 1 [10], Compute the Gauss curvature of the sphere $\mathbb{S}^2_R = \{x^2 + y^2 + z^2 = R^2\}$ of radius $R > 0$.

Problem 2 [10], Evaluate the surface integral

$$\int\int_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$ and S is the surface parameterized by $\Phi(u, v) = (2 \sin u, 3 \cos u, v)$, with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$.

Problem 3 [10], Evaluate $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F} = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$ and S is the surface $x^2 + y^2 + z^2 = 16, z \geq 0$. (Let $\mathbf{n}$, the unit normal, be upward pointing.)

Problem 4 [10], Find the area of surface S, where S is the surface defined by the graph $z = \frac{y^7}{7} + x$ over $\Omega = \{x^{\frac{1}{5}} \leq y \leq 1, 0 \leq x \leq 1\}$.

Problem 5 [10], Evaluate the integral $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the portion of the surface of a sphere defined by $x^2 + y^2 + z^2 = 1$ and $x + y + z \geq 1$, and where $\mathbf{F} = \mathbf{r} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, by observing that $\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial \Sigma} \mathbf{F} \cdot d\mathbf{r}$ for any other surface Σ with the same boundary as S. By picking Σ appropriately, $\int\int_\Sigma (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ may be easy to compute. Show that this is the case if Σ is taken to be the portion of the plane $x + y + z = 1$ inside the circle ∂S.

Problem 6 [10], Let $\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$, show that $\mathbf{F}$ is irrotaional in the domain $\Omega = \{x^2 + y^2 > 0\} \subset \mathbb{R}^3$, but $\mathbf{F}$ is not a gradient vector field in Ω.

Problem 7 [10], For a surface S and a fixed vector $\mathbf{v}$, prove that

$$2 \int\int_S \mathbf{v} \cdot \mathbf{n} dS = \int_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{s},$$

where $\mathbf{r} = (x, y, z)$.

Problem 8 [10], Let $\mathbf{F} = \frac{-GmM\mathbf{r}}{\|\mathbf{r}\|^3}$ be the gravitational force field defined in $\mathbb{R}^3 \setminus \{0\}$.

(a) Show that $div\mathbf{F} = 0$.

(b) Show that $\mathbf{F} \neq curl\mathbf{G}$ for any C^1 vector field $\mathbf{G}$ on $\mathbb{R}^3 \setminus \{0\}$.

Problem 9 [10], Evaluate the surface integral $\int\int_S \mathbf{F} \cdot \mathbf{n} dA$, where $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + z(x^2 + y^2)^2\mathbf{k}$ and S is the surface of the cylinder $x^2 + y^2 \leq 1, 0 \leq z \leq 1$.

Problem 10 [10], Suppose $\mathbf{F}$ is tangent to the closed surface $S = \partial W$ of a region W. Prove that

$$\int\int\int_W (div\mathbf{F})dV = 0.$$