Problem 1 [10], Let $\mathbf{c}$ be a smooth path.

(a) Suppose $\mathbf{F}$ is perpendicular to $\mathbf{c}'(t)$ at the point $\mathbf{c}(t)$. Show that

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0.$$

(b) If $\mathbf{F}$ is parallel to $\mathbf{c}'(t)$ at the point $\mathbf{c}(t)$, show that

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \|\mathbf{F}\| ds.$$

(By parallel to $\mathbf{c}'(t)$ we mean that $\mathbf{F}(\mathbf{c}(t)) = \lambda(t)\mathbf{c}'(t)$, where $\lambda(t) > 0$.)

Problem 2 [10], Let D be a region for which Green's Theorem holds. Suppose u is harmonic; that is,

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, \quad \forall (x,y) \in D. \tag{1}$$

Prove that

$$\int_{\partial D} \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy = 0.$$

Problem 3 [10], Suppose $(0,0) \in D$ such that $\bar{B}_R(0) \subset D$, and suppose u is C^2 in D and u satisfies

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) \geq 0, \quad \forall (x,y) \in D. \tag{2}$$

1. Set $I(\rho) = \frac{1}{\rho}\int_{\partial B_\rho(p)} u ds$, for $0 < \rho \leq R$, compute $\frac{d}{d\rho}I(\rho)$, using Green's Theorem to deduce that $\frac{d}{d\rho}I(\rho) \geq 0$,
2. Prove

$$u(0) \leq \frac{1}{2\pi R}\int_{\partial B_R(p)} u ds. \tag{3}$$

[Hint: what is $\lim_{\rho \to 0} I(\rho)$? Problem 1 in assignment 4.]

Problem 4 [10] Use (3) to prove the following strong maximum principle: suppose u as in Problem 3, if $u(0,0)$ is a local maximum, then u is a constant function near $(0,0)$.

Problem 5 [10], Let $B = \{x^2 + y^2 \leq 1\}$, and $\forall \delta > 0$, denote $B_\delta = \{x^2 + y^2 \leq \delta\}$. Suppose f is a continuous function and $\|\nabla f\| \leq 1$ on B, suppose

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{x^2+y^2}, \quad in \quad B \setminus \{0\}.$$

Use the boundedness of ∇f to show

$$\lim_{\delta \to 0} \int_{\partial B_\delta} f_y dx - f_x dy = 0.$$

Use this fact and the Green Theorem to evaluate

$$\int_{\partial B} f_y dx - f_x dy.$$

Problem 6 [10], Let Φ be a regular surface at (u_0, v_0) (i.e., Φ is C^1 and $T_u \times T_v \neq 0$ at (u_0, v_0)).

(a) Use the implicit function theorem to show that the image of Φ near (u_0, v_0) is the graph of a C^1 function of two variables. If the z component of $T_u \times T_v$ is nonzero, we can write it as $z = f(x, y)$.

(b) Show that the tangent plane at $\Phi(u_0, v_0)$ defined by the plane spanned by T_u and T_v coincides with the tangent plane of the graph of $z = f(x, y)$ at this point.

Problem 7 [10], Find the area of the surface defined by $z = xy$ and $x^2 + y^2 \leq 2$.

Problem 8 [10], Find a parametrization of the surface $x^2 + 3xy + z^2 = 2, z > 0$, and use it to find the tangent plane at the point $x = 1, y = 1/3, z = 0$. Compare your answer with that using level sets.

Problem 9 [10], Evaluate the integral

$$\int\int_S (1 - z) dS,$$

where S is the graph of $z = 1 - x^2 - y^2$, with $x^2 + y^2 \leq 1$.

Problem 10 [10], Let S be a sphere of radius r and $\mathbf{p}$ be a point inside or outside the sphere (but not on it). Show that

$$\int\int_S \frac{1}{\|\mathbf{x} - \mathbf{p}\|} dS = \begin{cases} 4\pi r, & \text{if } \mathbf{p} \text{ is inside S} \\ 4\pi r^2/d, & \text{if } \mathbf{p} \text{ is outside S} \end{cases}$$

where d is the distance from $\mathbf{p}$ to the center of the sphere nd integration is over the sphere. (Hint: assume $\mathbf{p}$ is on the z-axis. Then change variables and evaluate. Why this assumption on $\mathbf{p}$ justified?)