First Assignment, due on September 18, 2017.

Problem 1, Show that the subset $D = \{(x,y)|x \neq 0 \;\; and \;\; y < -1\}$ is an open set in $\mathbb{R}^2$.

Problem 2, Compute the following limits of functions defined in $\mathbb{R}^2$ (if they exist):

a $\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{\sqrt{x^2+y^2}}$,

b $\lim_{(x,y)\to(0,0)} \frac{1-\cos xy}{y^2}$,

c $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$.

Problem 3,

a. Prove that if U and V are neighborhoods of $\mathbf{x} \in \mathbb{R}^n$, then so are $U \bigcap V$ and $U \bigcup V$.

b Prove that the boundary points of an open interval $(a,b) \subset \mathbb{R}$ are the points a and b.

c $\forall \mathbf{x} \in \mathbb{R}^n$, define $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n || \mathbf{y} - \mathbf{x}| < r\}$. Prove that for $\mathbf{x} \in \mathbb{R}^n$ and $0 < s < t, B_s(\mathbf{x}) \subset B_t(\mathbf{x})$.

Problem 4, Suppose $f(x,y)$ is a function defined in $\mathbb{R}^2$. Set $g(x) = f(x,0)$, $h(y) = f(0,y)$. If g and h are continuous at 0 as functions in one variable, does it follows that f is continuous at the origin in $\mathbb{R}^2$? If your answer is "yes", provide a proof; if your answer is "no", construct an example.

Problem 5, Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ satisfy $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq K\|\mathbf{x} - \mathbf{y}\|^\alpha$ for all $\mathbf{x}$ and $\mathbf{y}$ in A for positive constant K and α. Show that f is continuous. (Such functions are called **Hölder-continuous** or, if $\alpha = 1$, **Lipschitz-continuous**.)

Problem 6, Find the partial derivatives $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ for function $w(x,y) = \sin(y^2 e^{xy})\cos(x^2)$.

Problem 7, Show that the function $f(r,\theta) = r^2 \sin 2\theta, r > 0$ (in polar coordinates) is differentiable at each point in its domain. Decide if it is C^1.

Problem 8, Compute the matrix of partial derivatives of $f(x,y) = (e^{y^2}, \sin xy)$.

Problem 9, Evaluate the gradient of $f(x,y,z) = \log(x^2 + y^2 + z^2)$ at $(1,-1,0)$.

Problem 10, Describe all Hölder-continuous functions with $\alpha > 1$ (see problem 5). (Hint: What is the derivative of such function?)