SOLUTIONS OF ASSIGNMENT #6

1. (a) For $F(x,y,z) = x^2 + y^2 + 4z^2 + z^4 - 64$, we have

$$z_x = -\frac{F_x}{F_z} = \frac{x}{4z + 2z^3}, \quad z_y = -\frac{F_y}{F_z} = \frac{y}{4z + 2z^3}$$

At $P = (4,4,2)$, $z_x = z_y = 1/4$. Now,

$$z_{xx} = \frac{\partial}{\partial x}(z_x) = \frac{\partial}{\partial x}(-\frac{x}{4z + 2z^3}) = -\frac{4z + z^3 - x(4 + 3z^2)z_x}{(4z + z^3)^2} = -1/8.$$

(b) The normal is proportional to $< -1/4, -1/4, -1 >$, the equation of the tangent plane is

$$(x - 4) + (y - 4) + 4(z - 2) = 0.$$

(c) $\nabla z =< -1/4, -1/4 >$, the direction of the maximum increase is $<1,1>$.

2. $z_x = ye^{-\frac{x^2+4y^2}{2}}(1 - x^2), z_y = xe^{-\frac{x^2+4y^2}{2}}(1 - 4y^2)$. Solution for $z_x = 0$ is either $y = 0$ or $x = 1, -1$. $y = 0$ and $z_y = 0$ will force $x = 0$. If $x = 1$ or -1, $z_y = 0$ will give $y = 1/2$ or $-1/2$. We have five critical points $(0,0), (1,1/2), (1,-1/2), (-1,1/2), (-1,-1/2)$. We now calculate $\nabla^2 z$:

$z_{xx} = xye^{-\frac{x^2+4y^2}{2}}(-2 - (1 - x^2)),$

$z_{yy} = xye^{-\frac{x^2+4y^2}{2}}(-8 - 4(1 - 4y^2)),$

$z_{xy} = e^{-\frac{x^2+4y^2}{2}}(1 - x^2)(1 - 4y^2).$

At $(0,0)$, $z_{xx}z_{yy} - z_{xy}^2 = -1$, it's a saddle point.

At $(1,1/2)$ and $(-1,-1/2)$, $z_{xx} = -1, z_{xy} = 0, z_{yy} = -4$, $z_{xx}z_{yy} - z_{xy}^2 = 4$, local maximum points.

At $(1,-1/2)$ and $(-1,1/2)$, $z_{xx} = 1, z_{xy} = 0, z_{yy} = 4$, $z_{xx}z_{yy} - z_{xy}^2 = 4$, local minimum points.

3. Set $F = z^3 + 4z - 2x^2y - 12$. At point $(1,2)$ with $z = 2$,

$z_x = -\frac{F_x}{F_z} = \frac{4xy}{3z^2+4} = 1/2,$

$z_y = -\frac{F_y}{F_z} = \frac{2x^2}{3z^2+4} = 1/8,$

$z_{xx} = (\frac{4xy}{3z^2+4})_x = \frac{4y}{3z^2+4} - \frac{24xyzz_x}{(3z^2+4)^2} = \frac{5}{16},$

$z_{yy} = (\frac{2x^2}{3z^2+4})_y = -\frac{2x^2zz_x}{(3z^2+4)^2} = -\frac{2}{16^2},$

$z_{xy} = (\frac{4xy}{3z^2+4})_y = \frac{4x}{3z^2+4} - \frac{24xyzz_y}{(3z^2+4)^2} = \frac{13}{64}.$

4(a). $f_x = x - 3y^2$, $f_y = 6y^2 - 6xy - 24$. $f_x = 0$ implies $x = 3y^2$, put this to $f_y = 0$, we obtain $6y^2 - 18y^3 - 24 = 0$, it is easy to see $y = -1$ is the only real solution of this equations. Therefore, $(3,-1)$ is the only critical point of f. Now at $(3, -1)$, $f_{xx} = 1, f_{xy} = -6y = 6, f_{yy} = 12y - 6x = -30$, $f_{xx}f_{yy} - f_{xy}^2 = -66$. It a saddle point.

4(b). Since f has no critical point in the region, we only need to calculate extremal values of f on the boundary. On the edge $x = 0, 0 \le y \le 1$, $f = 2y^3 - 24y + 40$, $f^{'} < 0$ for $0 \le y \le 1$, the extremal values are 40 and 18. On the edge $x = 2, 0 \le y \le 1$, $f = 2y^3 - 6y^2 - 24y + 56$, $f^{'} = 0$ has roots $1 + \sqrt{5}, 1 - \sqrt{5}$, none of them in the interval $0 \le y \le 1$, so the extremal values are 56 and 28. On the edge $y = 0, 0 \le x \le 2$, $f = 4x^2 + 40$, it is an increasing function in $0 \le x \le 2$, the extremal values are $40, 56$. On the edge $y = 1, 0 \le x \le 2$, $f = 4x^2 - 3x + 18$, $f^{'} = 0$ has solution $x = 3/8$. $f = 18 - \frac{9}{16}$. The value at $(0, 1)$ and $(2, 1)$ have been computed. In summary, the absolute minimum is $18 - \frac{9}{16}$ and the absolute maximum is 56.

5, Let x, y and z be dimensions of the open rectangle, so the surface area is $f(x, y, z) = xy + 2xz + 2yz$ and volume is $g = xyz = 128$. We use Lagrange multipliers.

$$\nabla f = \lambda \nabla g.$$

The equations are (1). $y + 2z = \lambda yz$, (2). $x + 2z = \lambda xz$ and (3). $2(x + y) = \lambda xy$. Try $x(1) + y(2) - z(3)$, we get (since $xyz = 128$, so $x \neq 0, y \neq 0, z \neq 0$)

$$2xy = \lambda xyz <=> 2 = \lambda z.$$

Now, (1)-(2) yields

$$x - y = \lambda z(x - y) <=> 0 = (x - y)(\lambda z - 1) = x - y = 0.$$

So, we must have $x = y$. Put this to (3), we have $\lambda y = 4 = \lambda x$. From $\lambda z = 2$, we obtain $y = x = 2z$. Again, by $xyz = 128$, we get $x = y = 4(4)^{\frac{1}{3}}, z = 2(4)^{\frac{1}{3}}$. The minimum surface area is $xy + 2xz + 2yz = 3x^2 = 48(16)^{1/3}$

One may also replace z by $\frac{128}{xy}$ in $f(x, y, z)$ and reduce the problem in (x, y) directly.

6. Set $f(x, y, z) = 12x^2 + 8xy + 12y^2 + z^2$ and $g(x, y, z) = x^2 + y^2 + z^2 - 8$.

$$\nabla f = \lambda \nabla g$$

gives (1). $12x + 4y = \lambda x$, (2). $12y + 4x = \lambda y$, (3). $z = \lambda z$. From (3), either $z = 0$ or $\lambda = 1$. Try (1)-(2), we get $(8 - \lambda)(x - y) = 0$.

Now, if $\lambda = 1$, we must have $x = y$. By (1) again, must have $x = 0$, so $x = y = 0$. since $g = 0$, $z^2 = 8$, we have $f = 8$.

If $z = 0$, from $(8 - \lambda)(x - y) = 0$, either $x = y$ or $8 = \lambda$. $x = y, z = 0$ and $g = 0$ gives $x^2 = y^2 = xy = 4$, so $f = 128$. If $\lambda = 8$, (1)+(2) yields $16(x + y) = 8(x + y)$. So $x = -y$. As $g = 0$, we have $x^2 = y^2 = 4$ and $xy = -4$. With $z = 0$, we get $f = 64$.

In conclusion, the maximum and minimum values of f are 128 and 8 respectively.