MATHEMATICS 222 CALCULUS III ASSIGNMENT 5 ANSWERS

1. The function f is defined by $\frac{2xy}{x^2+y^2)}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$

   (a) $\dfrac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\left(\dfrac{2h(0)}{h^2+0^2} - 0\right)/h = 0$ Similarly $\frac{\partial f}{\partial y}(0,0) = 0$
   For any other point, simply apply the quotient rule to get
$$\frac{\partial f}{\partial x}(x,y) = \frac{2y(y^2-x^2)}{(x^2+y^2)^2} \text{ and } \frac{\partial f}{\partial y}(x,y) = \frac{2x(x^2-y^2)}{(x^2+y^2)^2}$$

   (b) In order to find the second partial derivatives at $(x, y) \neq (0, 0)$ apply the quotient rule again to obtain
$$f_{xx} = \frac{4xy(x^2-3y^2}{(x^2+y^2)^3} \quad f_{yy} = \frac{4xy(y^2-3x^2)}{(x^2+y^2)^3} \quad f_{xy} = \frac{2(6x^2y^2-x^4-y^4)}{(x^2+y^2)^3}$$
   Substituting in to (recall the correction $y^2 f_{yy}$)
$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = \frac{4xy}{(x^2+y^2)^3}\left[x^4 - 3x^2y^2 + y^4 - 3x^2y^2 + 6x^2y^2 - x^4 - y^4\right] = 0$$

2. (a) if $V = x^3 - 3xy^2$ then $V_{xx} = 6x$ and $V_{yy} = -6x$ which add up to zero as required.
   (b) if $V = e^{-y}\cos x$ then $V_{xx} = -e^{-y}\cos x$ and $V_{yy} = (-1)^2 e^{-y}\cos x$ which add up to zero.
   (c) If, for $(x, y) \neq (0, 0)$, $V = \tan^{-1}\frac{y}{x}$ then $V_x = \frac{-y}{x^2+y^2}$ and $V_y = \frac{x}{x^2+y^2}$.
   Hence $V_{xx} = \frac{-2xy}{(x^2+y^2)^2}$ and $V_{yy} = \frac{2xy}{(x^2+y^2)^2)}$ which add up to zero as required.

3. (a) $\nabla w = \frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$ for $(x, y, z) \neq (0, 0, 0)$
   (b) The rate of increase $D_{\mathbf{u}}w$ in the direction of a unit vector $\mathbf{u}$ is $\nabla w \cdot \mathbf{u}$. We calculate this at the point $(2, 1, 2)$ where $\nabla w = \frac{-1}{27}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$
      i. if $\mathbf{u} = \frac{1}{3}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ then $D_{\mathbf{u}}w = \frac{-8}{81}$
      ii. if $\mathbf{u} = \frac{1}{9}(\mathbf{i} + 4\mathbf{j} + 8\mathbf{k})$ then $D_{\mathbf{u}}w = \frac{-22}{243}$

4. (a) $dz = \frac{1}{2}\sqrt{\frac{y}{x}}dx + \frac{1}{2}\sqrt{\frac{x}{y}}dy$. Note that if $x = 9$ and $y = 4$ then $z = 6$. The equation of the tangent plane to the surface at this point is then
$$Z - 6 = \frac{1}{2}\sqrt{\frac{4}{9}}(X-9) + \frac{1}{2}\sqrt{\frac{9}{4}}(Y-4) = \frac{1}{3}(X-9) + \frac{3}{4}(Y-4).$$
   A normal to this plane is $\frac{1}{3}\mathbf{i} + \frac{3}{4}\mathbf{j} - \mathbf{k}$

(b) Simply note that $\sqrt{(kt)(\frac{t}{k})} = \sqrt{t^2} = t$. As a consequence of the definition of this line,

$$\frac{dz}{dt} = (k, \frac{1}{k}, 1)$$

(c) In order for the line to go through $(9, 4, 6)$ we must have $t = z = 6$ and so $k = \frac{9}{6} = \frac{3}{2}$ and the direction of the line is $\frac{3}{2}\mathbf{i} + \frac{2}{3}\mathbf{j} - \mathbf{k}$ which is perpendicular to the normal to the tangent plane–hence the line lies in the tangent plane.

5. (a) $\frac{\partial V}{\partial r} = \cos\theta \frac{\partial V}{\partial x} + \sin\theta \frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial \theta} = -r\sin\theta \frac{\partial V}{\partial x} + r\cos\theta \frac{\partial V}{\partial y}$.

(b) Substitute into the right hand side and simplify to get the left hand side.

6. (a) $\frac{\partial \phi}{\partial u} = 2x(v) + 2y(2(u+v+w)) + 2z(0)$, $\frac{\partial \phi}{\partial v} = 2x(u) + 2y(2(u+v+w)) + 2z(0)$, $\frac{\partial \phi}{\partial w} = 2x(1) + 2y(2(u+v+w)) + 2z(1)$

(b) in order to do the differential approximation, evaluate these derivatives at $u = 1, v = 2, w = 1$ that is to say, $x = 3, y = 16, z = 1$

$\phi(1.01, 2.01, 1.02) \approx (2(3)(2) + 2(16)(2)(4) + 0)(0.01) + (2(3)(1) + 2(16)(2)(4) + 0)(0.01) + (2(3)(1) + 2(16)(2)(4) + 2(1))(0.02) = 8.66$