NON-HOMOGENEOUS FULLY NONLINEAR CONTRACTING FLOWS OF CONVEX HYPERSURFACES

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Dedicated to Professor Joel Spruck on the occasion of his retirement

Abstract. We consider a general class of non-homogeneous contracting flows of convex hypersurfaces in $\mathbb{R}^{n+1}$, and prove the existence and regularity of the flow before extincting to a point in finite time.

1. Introduction

We consider the contraction of convex hypersurfaces in $\mathbb{R}^{n+1}$ by general fully nonlinear flows. Suppose $M \subset \mathbb{R}^{n+1}$ is a compact convex hypersurface, we are interested in the following shrinking type hypersurface flow

$$X_t = -\tilde{\psi}(\overrightarrow{\nu}, X) f(\kappa) \overrightarrow{\nu}, \quad X(0) = M, \tag{1.1}$$

where $\overrightarrow{\nu}$ is the outer normal, $\kappa = (\kappa_1, \ldots, \kappa_n)$ is the principal curvature vector of $M(t)$, $\tilde{\psi}$ is a smooth positive function defined on $\mathbb{S}^n \times \mathbb{R}^{n+1}$, and f is a positive smooth function defined in the positive cone

$$\Gamma^+ = \{(\kappa_1, \cdots, \kappa_n) \subset \mathbb{R}^n | \ \ \kappa_i > 0, \ \forall i = 1, \cdots, n.\}.$$

When $\tilde{\psi} \equiv 1$, flow (1.1) is an isotropic flow of the form

$$X_t = -f(\kappa) \overrightarrow{\nu}, \quad X(0) = M. \tag{1.2}$$

The Gauss curvature flow [11] is an example of flow (1.2). Chou [16] established existence and regularity of the Gauss flow before it contracting to a point [16]. In the case of homogeneity one speed function, flow (1.2) contracts to a point in finite time and becomes spherical in shape in various structural settings [14, 10, 1, 3].

One basic question is that under what conditions flow (1.1) will contract to a point. Sufficient conditions were discussed by Han [13] for contraction to a point of flow (1.2) when the speed functions are homogeneous. Further study was carried out by Andrews-McCoy-Zheng in [4]. Our focus here is on speed functions without homogeneity assumption. We extend result in [13] for flow (1.2) to non-homogeneous case. The results of this paper has been used in recent works on convergence of Gauss curvature type flows in space forms [7] and inhomogeneous Gauss curvature type flows [8].

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We specify the conditions on function f. The following conditions (1.3)-(1.5) were introduced in [13].

$$f \text{ is a positive symmetric function defined on } \Gamma^+, \tag{1.3}$$

and

$$\frac{\partial f(\lambda)}{\partial \lambda_i} > 0,\ \forall \lambda \in \Gamma^+,\ \forall i = 1, \cdots, n. \tag{1.4}$$

Set $F(\lambda_1, \ldots, \lambda_n) := -f(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n})$, we further assume that

$$\tfrac{1}{F(\lambda)} \text{ is continuous on } \bar{\Gamma}^+, \text{ and } \frac{1}{F(\lambda)} = 0,\ \forall \lambda \in \partial\Gamma^+. \tag{1.5}$$

We replace the inverse concavity condition in [13] by the following

$$F \text{ is a concave function of } \lambda \in \Gamma^+. \tag{1.6}$$

In rest of this paper, we mainly work on the evolution equation of support function of flow (1.1). For a strictly convex hypersurface M, $\lambda_i = \frac{1}{\kappa_i}$, $i = 1, \cdots, n$ are the principal radii of M. They are the eigenvalues of

$$W = (\nabla_g^2 u + ug),$$

where u is the support function of M and g the standard metric on $\mathbb{S}^n$. We also use lower index $i, j, k, \cdots$ to denote covariant differentiation with respect to the connection on $\mathbb{S}^n$. From the work of Caffarelli-Nirenberg-Spruck [6], F can be extended as function in W. The corresponding flow for u is in the form

$$u_t = -\psi(x, u, \nabla u) F(W).$$

The first result is for flow (1.2). The following theorem is an extension of the result in [13] to the non-homogeneity case.

Theorem 1.1. *Suppose f satisfies conditions (1.3), (1.4), (1.5) and(1.6), and suppose $X(0) = M$ is strictly convex, then there is a finite time $T^* > 0$ such that flow (1.2) exists for $0 < t < T^*$, and solution $X(t)$ remains strictly convex and $X(t)$ converges to a point as $t \to T^*$.*

We switch to anisotropic flow of the form

$$X_t = -\tilde{\psi}(\nu) f(\kappa) \overrightarrow{\nu}, \quad X(0) = M, \tag{1.7}$$

where $\tilde{\psi}$ is a positive smooth function on $\mathbb{S}^n$. This type of flow was treated in [2] when f is homogeneous, in particular for power of Gauss curvature $f(\kappa) = K^\alpha$.

Denote

$$F^{ij} = \frac{\partial F(W)}{\partial W_{ij}}, \quad F^{ij,kl} = \frac{\partial^2 F(W)}{\partial W_{kl} \partial W_{ij}}, \quad \mathcal{L} := \partial_t - \psi F^{ij}(W) \nabla_i \nabla_j. \tag{1.8}$$

Theorem 1.2. *Suppose f satisfies conditions (1.3), (1.4), (1.5) and $f(0)=0$,. Suppose $\exists\delta_0>0$ such that*

$$F^{\alpha\beta,\gamma\eta}(W)\xi_{\alpha\beta}\xi_{\gamma\eta}\leq\delta_0\frac{(F^{\alpha\beta}(W)\xi_{\alpha\beta})^2}{F(W)},\ \forall\xi_{\alpha\beta}. \tag{1.9}$$

If $X(0)=M$ is strictly convex, then there is a finite time $T^*>0$ such that flow (1.7) exists for $0<t<T^*$, and solution $X(t)$ remains strictly convex and $X(t)$ converges to a point as $t\to T^*$.

Condition (1.9) is a stronger concavity condition than (1.6), but it is weaker than the inverse concavity condition for f.

For general form of flow (1.1), we need some additional conditions: assume $\exists\delta_0>0$ such that

$$F^{\alpha\beta,\gamma\eta}(W)\xi_{\alpha\beta}\xi_{\gamma\eta}+W^{\beta\gamma}F^{\alpha\eta}(W)\xi_{\alpha\beta}\xi_{\gamma\eta}\leq\delta_0\frac{(F^{\alpha\beta}\xi_{\alpha\beta})^2}{F(W)},\ \forall W\in\Gamma^+,\ \forall\xi_{\alpha\beta}. \tag{1.10}$$

and

$$F^{ij}(W)W_{ik}W_{kj}\geq-\delta_0\sigma_1(W)F(W),\quad\forall W\in\Gamma^+. \tag{1.11}$$

Theorem 1.3. *Suppose f satisfies conditions (1.3), (1.4), (1.5), (1.10) and (1.11). Then for any initial strictly convex $X(0)=M$, there is a finite time $T^*>0$ such that flow (1.1) exists for $0<t<T^*$, and solution $X(t)$ remains strictly convex and $X(t)$ converges to a point as $t\to T^*$.*

The paper is organized as follows. Section 2 is devoted to evolution equations of corresponding geometric quantities. The lower bound of the speed function and principal curvatures along the flow will be proved in section 3. In section 4, we show the flow (1.1) converges to a point at a finite time $T^*>0$ under various conditions specified in Theorem 1.1, Theorem 1.2 and Theorem 1.3. In the last section we discuss examples of the non-homogeneous flow (1.1) .

2. Preliminaries

Let u be the support function of solution $M(t):=X(t)$ to flow (1.1) with $M(0)=M$ a strictly convex, closed smooth hypersurface in $\mathbb{R}^{n+1}$, then it satisfies the following evolution equation

$$u_t=-\tilde{\psi}(\nu,X)f(\frac{1}{\lambda_1},\ldots,\frac{1}{\lambda_n})=:\psi(x,u,\nabla u)F(W) \tag{2.1}$$

with $u(0)=u_0$. Here $(\lambda_1,\ldots,\lambda_n)$ are the eigenvalues of matrix $(W(x,t)_{ij}):=(u(x,t)_{ij}+u(x,t)\delta_{ij})$ in a local orthonormal frame of $\mathbb{S}^n$ and $\psi(x,z,p)$ is a smooth positive function defined on $(x,(z,p))\in\mathbb{S}^n\times T\mathbb{S}^n$ such that $\psi(x,z,p)=\tilde{\psi}(x,xz+p)$.

Since $M(0)=M$ is strictly convex, the standard theory for parabolic equation implies that (2.1) has a smooth solution $t\in(0,T)$ for some $T>0$ if f satisfies (1.3), (1.4).

$$X(x,t):=u(x,t)x+\nabla_{\mathbb{S}^n}u(x,t)$$

corresponds to smooth solution for (1.1) with $X(0)=M$ for $0<t<T$. The goal is to show there is maximal time $T^*>0$ such that flow converges to a point when $t\to T^*$.

Lemma 2.1. *For solution u of flow (2.1), the following equations hold*

(2.2)

$$
\begin{aligned}
\mathcal{L}u =&\psi(F - F^{ij}W_{ij} + u\sum_i F^{ii}).\\
\mathcal{L}(\psi F) =&\psi F(\psi\sum_i F^{ii} + \psi_u F) + F\psi_{u_i}(\psi F)_i.\\
\mathcal{L}W_{ij} =&\psi[F^{pq,rs}W_{pqi}W_{rsj} + (F^{pq}W_{pq} + F)\delta_{ij} - W_{ij}\sum_p F^{pp}]\\
&+ (\psi_{x_i} + \psi_u u_i + \psi_{u_k}u_{ki})F^{pq}W_{pqj} + +(\psi_{x_j} + \psi_u u_j + \psi_{u_k}u_{kj})F^{pq}W_{pqi} + F(\psi_{x_ix_j}\\
&+ \psi_{x_iu}u_j + \psi_{x_ju}u_i + \psi_{uu}u_iu_j + \psi_{x_iu_k}u_{kj} + \psi_{x_ju_k}u_{ki} + \psi_{uu_k}(u_iu_{kj} + u_ju_{ki})\\
&+ \psi_{u_ku_l}u_{ki}u_{lj} + \psi_{u_k}u_{kij} + \psi_u u_{ij}).\\
\mathcal{L}r^2 =&2F[(\frac{F^{pq}W_{pq}}{F} + 1)u\psi + \psi_{x_i}u_i + \psi_u|\nabla u|^2 + \psi_{u_k}u_{ki}u_i - \psi\frac{F^{pq}W_{pi}W_{qi}}{F}],
\end{aligned}
$$

where $r^2 := u^2 + |\nabla u|^2$.

Proof. Choose a local orthonormal frame on $\mathbb{S}^n$, a direct computation yields

$$
\begin{aligned}
(1)\quad u_t =&\psi F(W) = \psi F(W) + \psi F^{ij}u_{ij} - \psi F^{ij}(W_{ij} - u\delta_{ij}),\\
(2)\quad (\psi F)_t =&\psi F^{ij}W_{ijt} + (\psi_u u_t + \psi_{u_i}u_{it})F\\
=&\psi F^{ij}(u_{ijt} + u_t\delta_{ij}) + (\psi_u(\psi F) + \psi_{u_i}(\psi F)_i)F\\
=&\psi F^{ij}(\psi F)_{ij} + \psi^2 F\sum_i F^{ii} + \psi_u F(\psi F) + F\psi_{u_i}(\psi F)_i,\\
(3)\quad W_{ijt} =&u_{ijt} + u_t\delta_{ij} = (\psi F)_{ij} + \psi F\delta_{ij}\\
=&\psi(F^{pq,rs}W_{pqi}W_{rsj} + F^{pq}W_{pqij} + F\delta_{ij}) + (\psi_{x_i} + \psi_u u_i + \psi_{u_k}u_{ki})F^{pq}W_{pqj}\\
&+ (\psi_{x_j} + \psi_u u_j + \psi_{u_k}u_{kj})F^{pq}W_{pqi} + F(\psi_{x_ix_j} + \psi_{x_iu}u_j + \psi_{x_ju}u_i + \psi_{uu}u_iu_j\\
&+ \psi_{x_iu_k}u_{kj} + \psi_{x_ju_k}u_{ki} + \psi_{uu_k}(u_iu_{kj} + u_ju_{ki}) + \psi_{u_ku_l}u_{ki}u_{lj} + \psi_{u_k}u_{kij} + \psi_u u_{ij})\\
=&\psi[F^{pq,rs}W_{pqi}W_{rsj} + F^{pq}(W_{ijpq} + W_{pq}\delta_{ij} - W_{ij}\delta_{pq} + W_{iq}\delta_{jp} - W_{jp}\delta_{iq}) + F\delta_{ij}]\\
&+ (\psi_{x_i} + \psi_u u_i + \psi_{u_k}u_{ki})F^{pq}W_{pqj} + +(\psi_{x_j} + \psi_u u_j + \psi_{u_k}u_{kj})F^{pq}W_{pqi} + F(\psi_{x_ix_j}\\
&+ \psi_{x_iu}u_j + \psi_{x_ju}u_i + \psi_{uu}u_iu_j + \psi_{x_iu_k}u_{kj} + \psi_{x_ju_k}u_{ki} + \psi_{uu_k}(u_iu_{kj} + u_ju_{ki})\\
&+ \psi_{u_ku_l}u_{ki}u_{lj} + \psi_{u_k}u_{kij} + \psi_u u_{ij}).
\end{aligned}
$$

For the last equation, we have

$$
\begin{aligned}
&r_t^2 = 2uu_t + 2u_iu_{it} = 2F[u\psi + \psi_{x_i}u_i + \psi_u|\nabla u|^2 + \psi_{u_k}u_{ki}u_i + \psi\frac{F^{pq}}{F}W_{pqi}u_i],\\
&(r^2)_{pq} = 2(W_{pi}u_i)_q = 2W_{pqi}u_i + 2W_{pi}W_{qi} - 2uW_{pq}.
\end{aligned}
$$

□

The following simple inequality (2.3) will be used extensively in the rest of the paper.

Lemma 2.2. *Suppose F is a concave function in Γ^+, then*

$$s\sum_i F^{ii}(W) \geq -F(W) + F^{ij}(W)W_{ij} + F(sI), \quad \forall s \in \mathbb{R}^+, \ \forall W \in \Gamma^+. \tag{2.3}$$

Proof. By concavity, $\forall s > 0, \ W \in \Gamma^+$,

$$F(sI) \leq F(W) + \sum_i F^{ij}(W)(s\delta_{ij} - W_{ij}).$$

□

3. Lower bound of principal curvatures

We first estimate the lower bound of speed function in flow (2.1).

Lemma 3.1. *Suppose f satisfies (1.3) , (1.4), ψ is a smooth positive function defined on $\mathbb{S}^n \times T\mathbb{S}^n$, and $X(t)$ is a smooth convex solution of (2.1) for $0 \leq t \leq T$. Then there is $C > 0$ depending only on initial data such that*

$$\min_{(x,t)\in\mathbb{S}^n\times[0,T]} -\psi(x,u,\nabla u)F(W(x,t)) \geq \frac{1}{C(T+1)}. \tag{3.1}$$

Moreover, if ψ doesn't depend on u, then

$$\min_{(x,t)\in\mathbb{S}^n\times[0,T]} -\psi(x,\nabla u)F(W(x,t)) \geq \min_{x\in\mathbb{S}^n} -\psi(x,\nabla u(x,0))F(W(x,0)). \tag{3.2}$$

Proof. Note that (2.1) is a contracting flow, u is bounded from above. As $W > 0$, $\max|\nabla u|^2 \leq \max u^2$ is also bounded from above. Thus ψ is bounded from below and above, and ψ_u is bounded from above. By the second equation in (2.2),

$$\mathcal{L}(-u_t) = -u_t\psi\sum F^{ii} - \frac{\psi_u u_t^2}{\psi} + F\psi_{u_i}(-u_t)_i \geq -C(-u_t)^2. \tag{3.3}$$

From comparison,

$$-u_t \geq \eta,$$

where η is the solution to

$$\eta_t = -C\eta^2, \quad \eta(0) = \min_{t=0} -\psi F = c_0 > 0.$$

Then $\eta = \frac{1}{Ct+c_0} \geq \frac{1}{C(T+1)}$ if we pick $C \geq c_0$. It follows (3.1).

Note that, if ψ is independent of u (e.g., (1.7)), by (3.3)

$$\mathcal{L}(-u_t) \geq -u_t\psi\sum F^{ii} + F\psi_{u_i}(-u_t)_i,$$

which implies that

$$\min_{(x,t)\in\mathbb{S}^n\times[0,T]} -\psi(x,\nabla u)F(W(x,t)) \geq \min_{x\in\mathbb{S}^n} -\psi(x,\nabla u(x,0))F(W(x,0)).$$

□

Corollary 3.1. *With the same assumptions in Lemma 3.1, we have*

$$-F \geq \frac{\min_{(x,t)\in\mathbb{S}^n\times[0,T]} -\psi(x,u,\nabla u)F(W(x,t))}{\max_{(x,t)\in\mathbb{S}^n\times[0,T]}\psi(x,u,\nabla u)} =: \varepsilon_0 > 0. \tag{3.4}$$

Proof. Note that as above, $(x, u, \nabla u)$ stays in a compact subset of $\mathbb{S}^n \times T\mathbb{S}^n$ as long as (2.1) exists. Thus $\max_{(x,t)\in\mathbb{S}^n\times[0,T]}\psi(x,u,\nabla u) \leq C(M_0,\psi) < \infty$. The corollary follows from Lemma 3.1. □

The next lemma is the lower bound of the principal curvatures of $X(t)$ along the flow (2.1).

Lemma 3.2. *Assume that one of the following holds,*

(1) conditions in Theorem 1.1 are satisfied, and $X(t)$ is a solution to (1.2),
(2) conditions in Theorem 1.2 are satisfied, $X(t)$ is a solution to (1.7),
(3) conditions in Theorem 1.3 are satisfied, $X(t)$ is a solution to (1.1),

then there exists some constant $\varepsilon > 0$ depending only on the initial data and the constants in the conditions specified above such that

$$\min_{i=1,\cdots,n,(x,t)\in\mathbb{S}^n\times[0,T]}\kappa_i(x,t) \geq \varepsilon. \tag{3.5}$$

Proof. It suffices to prove an upper bound for $\max_{i=1,\cdots,n,(x,t)\in\mathbb{S}^n\times[0,T]}\lambda_i(W(x,t))$.

(1) If $\tilde{\psi} \equiv 1$. Suppose $\max_{i=1,\cdots,n,(x,t)\in\mathbb{S}^n\times[0,T]}\lambda_i(W(x,t))$ is achieved at (x_0,t_0). If $t_0 = 0$, we are done. Otherwise, take a local orthonormal frame on $\mathbb{S}^n$, such that W is diagonal at (x_0,t_0) and $W_{11}(x_0,t_0) = \max_{i=1,\cdots,n,(x,t)\in\mathbb{S}^n\times[0,T]}\lambda_i(W(x,t))$. It follows from the evolution equation of W_{11} in Lemma 2.1 that, at (x_0,t_0),

$$\mathcal{L}W_{11} = F^{ii}(W_{ii} - W_{11}) + F^{ij,kl}W_{ij1}W_{kl1} + F \leq 0 \tag{3.6}$$

since F is concave and W_{11} is the largest eigenvalue at (x_0,t_0). By maximum principle,

$$\max_{i,=1,\cdots,n,(x,t)\in\mathbb{S}^n\times[0,T]}\lambda_i(W(x,t)) = W_{11}(x_0,t_0) \leq \max_{(x,t)\in\mathbb{S}^n\times[0,T]}W_{11}(x,t) = \max_{x\in\mathbb{S}^n}W_{11}(x,0). \tag{3.7}$$

Thus

$$W(x,t) \leq CI, \qquad (x,t)\in\mathbb{S}^n\times[0,T], \tag{3.8}$$

where $C = \max_{i=1,\cdots,n,x\in\mathbb{S}^n}\lambda_i(W(x,0))$. The lemma follows since the eigenvalues of W are the inverse of the principle curvatures.

(2) If $\tilde{\psi} = \tilde{\psi}(\nu)$ defined on $\mathbb{S}^n$. We write (1.1) as (2.1) equivalently for $\psi(x) = \tilde{\psi}(x)$, $x \in \mathbb{S}^n$. As in the first case, suppose $\max_{i=1,\cdots,n,(x,t)\in\mathbb{S}^n\times[0,T]}\lambda_i(W(x,t))$ is achieved at (x_0,t_0). If $t_0 = 0$, we are done. Otherwise, take a local orthonormal frame on $\mathbb{S}^n$, such that W is diagonal at (x_0,t_0) and $W_{11}(x_0,t_0) = \max_{i=1,\cdots,n,(x,t)\in\mathbb{S}^n\times[0,T]}\lambda_i(W(x,t))$. Again, by the evolution equation of W_{11} in Lemma 2.1 (note $\psi_u = \psi_{u_i} \equiv 0$ for this case) and (1.9), at (x_0,t_0),

$$\begin{aligned}(W_{11})_t =&\psi F + F\psi_{x_1x_1} + 2\psi_{x_1}F^{ii}W_{ii1} + \psi(F^{ij,kl}W_{ij1}W_{kl1} + F^{ii}W_{11,ii} + F^{ii}(W_{ii} - W_{11}))\\ \leq&(\psi + \psi_{x_1,x_1})F + 2\psi_{x_1}F^{ii}W_{ii1} + \delta_0\psi\frac{(F^{ii}W_{ii1})^2}{F} + \psi F^{ii}(W_{ii} - W_{11}).\end{aligned} \tag{3.9}$$

By Cauchy-Schwartz inequality and the concavity (1.6) (which was implied by (1.9))

$$
\begin{aligned}
&\psi_{x_1x_1}F + 2\psi_{x_1}F^{ii}W_{ii1} + \delta_0\psi\frac{(F^{ii}W_{ii1})^2}{F} + \psi F^{ii}(W_{ii} - W_{11}) \\
\le&\psi_{x_1x_1}F - \frac{\psi_{x_1}^2 F}{\delta_0\psi} + \psi F^{ii}(W_{ii} - W_{11}) \\
\le&C(\|\psi\|_{C^2}, \frac{1}{\delta_0})(F^{ii}W_{ii} - F) - (\inf\psi)W_{11}F^{ii} \\
\le& - C(\|\psi\|_{C^2}, \frac{1}{\delta_0})F(\frac{\inf\psi W_{11}}{2C(\|\psi\|_{C^2}, \frac{1}{\delta_0})}I) - \frac{\inf\psi}{2}W_{11}F^{ii}.
\end{aligned} \tag{3.10}
$$

By (2.3) and corollary 3.1,

$$
AF^{ii} \ge -(F - F^{ii}W_{ii}) + F(AI) \ge -F + F(AI) \ge \varepsilon_0 + F(AI),
$$

where $\varepsilon_0 = \inf_{(x,t)\in\mathbb{S}^n\times[0,T]}(-F) > 0$ is the constant in (3.4). Since $f(0) = 0$, we have $F(AI) \to 0$ as $A \to \infty$. Thus there exists $A_0(f) > 0$ large, such that

$$
F(AI) \ge -\frac{\varepsilon_0}{2} \text{ for } A \ge A_0. \tag{3.11}
$$

This implies that

$$
\sum_i F^{ii} \ge \frac{\varepsilon_0}{2A_0}. \tag{3.12}
$$

Combining (3.9), (3.10), (3.11) with (3.12), if $W_{11}(x_0, t_0) \ge \frac{2C(\|\psi\|_{C^2}, \frac{1}{\delta_0})A_0(F)}{\inf\psi}$,

$$
W_{11t} \le C(\|\psi\|_{C^2}, \frac{1}{\delta_0})\frac{\varepsilon_0}{2} - \frac{\inf\psi}{2}W_{11}\frac{\varepsilon_0}{2A_0} \le 0. \tag{3.13}
$$

(3) For the general case $\tilde{\psi} = \tilde{\psi}(\nu, X)$ defined on $\mathbb{S}^n \times \mathbb{R}^{n+1}$. We write (1.1) as (2.1) with $\psi(x, z, p)$ being a smooth positive function defined on $(x, (z, p)) \in \mathbb{S}^n \times T\mathbb{S}^n$ such that $\psi(x, z, p) = \tilde{\psi}(x, xz + p)$. Consider the function

$$
G = \log W_{11} + \frac{L}{2}r^2, \tag{3.14}
$$

where $r^2 = u^2 + |\nabla u|^2$, L is a large constant to be determined. Suppose G attains its maximum on $\mathbb{S}^n \times [0, T]$ at (x_0, t_0). Take a local orthonormal frame of $\mathbb{S}^n$ such that W_{ij} is diagonal at (x_0, t_0) and $W_{11}(x_0, t_0) = \max_{i=2,\cdots,n,(x,t)\in\mathbb{S}^n\times[0,T]} \lambda_i(W_{ij})$. Suppose $W_{11}(x_0, t_0) \ge 1$, otherwise $G(x_0, t_0) \le 1 + r^2 \le 1 + C(M_0)$ since the flow is contracting. If $t_0 = 0$, we are done. Suppose now $t_0 > 0$. First note that, at any point (x_1, t_1) ((x_1, t_1) need not to be the maximum point of G) where W_{ij} is diagonal, from Lemma 2.1, W_{11}

and r^2 satisfies

$$
\begin{aligned}
&W_{11,t} - \psi F^{pp} W_{11,pp} \\
=&F[\psi + \psi_u W_{11} - \psi_u u + \psi_{u_i} W_{11i} - \psi_{u_1} u_1 + \psi_{x_1x_1} + \psi_{uu} u_1^2 + \psi_{u_1u_1} W_{11}^2 \\
&- 2\psi_{u_1u_1} W_{11} u + \psi_{u_1u_1} u^2 + 2\psi_{x_1u} u_1 + 2\psi_{x_1u_1} W_{11} - 2\psi_{x_1u_1} u \\
&+ 2\psi_{uu_1} W_{11} u_1 - 2\psi_{uu_1} u u_1 + 2\frac{F^{ii}}{F} W_{ii1}(\psi_{x_1} + \psi_u u_1 + \psi_{u_1} W_{11} \\
&- \psi_{u_1} u) + \psi \frac{F^{ij,kl} W_{ij1} W_{kl1} + F^{ii}(W_{ii} - W_{11})}{F}],
\end{aligned} \tag{3.15}
$$

and

$$
\begin{aligned}
&r_t^2 - \psi F^{pp} (r^2)_{pp} \\
=&2F[(\frac{F^{pp} W_{pp}}{F} + 1) u\psi + \psi_{x_i} u_i + \psi_u |\nabla u|^2 + \psi_{u_i} u_{ii} u_i - \psi \frac{F^{pp} W_{pp}^2}{F}].
\end{aligned} \tag{3.16}
$$

Let $(x_1, t_1) = (x_0, t_0)$ be the maximum point of G, at (x_0, t_0),

$$
W_{11i} = -\frac{L}{2} W_{11} (r^2)_i = -L W_{11} W_{ii} u_i, \tag{3.17}
$$

and

$$
\begin{aligned}
0 \leq\ & G_t - \psi F^{pp} G_{pp} \\
=\ & \frac{W_{11t} - \psi F^{pp} W_{11,pp}}{W_{11}} + \psi F^{pp} \frac{W_{11p}^2}{W_{11}^2} + \frac{L}{2}(r_t^2 - \psi F^{pp}(r^2)_{pp}) \\
=\ & F[\psi_{u_1u_1} W_{11} + \psi_{u_i} \frac{W_{11i}}{W_{11}} + \psi_u - 2\psi_{u_1u_1} u + 2\psi_{x_1u_1} + 2\psi_{uu_1} u_1 \\
& + \frac{1}{W_{11}}(\psi - \psi_u u - \psi_{u_1} u_1 + \psi_{x_1x_1} + \psi_{uu} u_1^2 + \psi_{u_1u_1} u^2 + 2\psi_{x_1u} u_1 - 2\psi_{x_1u_1} u - 2\psi_{uu_1} u u_1) \\
& + 2\frac{F^{ii}}{F} W_{ii1}(\psi_{u_1} + \frac{\psi_{x_1} + \psi_u u_1 - \psi_{u_1} u}{W_{11}}) + \psi \frac{F^{ij,kl} \frac{W_{ij1} W_{kl1}}{W_{11}} + F^{ii}(\frac{W_{ii}}{W_{11}} - 1)}{F} \\
& + \psi \frac{F^{pp}}{F} \frac{W_{11p}^2}{W_{11}^2} + (1 + \frac{F^{pp} W_{pp}}{F}) L u\psi + L\psi_{x_i} u_i + L\psi_u |\nabla u|^2 \\
& + L\psi_{u_i} W_{ii} u_i - L\psi_{u_i} u u_i - L\psi \frac{F^{pp} W_{pp}^2}{F}] \\
\leq\ & F[-C(n, L, M_0, f, \psi) + \psi_{u_1u_1} W_{11} - L\psi_{u_i} W_{ii} u_i + 2\frac{F^{ii}}{F} W_{ii1}(\psi_{u_1} + \frac{\psi_{x_1} + \psi_u u_1 - \psi_{u_1} u}{W_{11}}) \\
& + \psi \frac{F^{ij,kl} \frac{W_{ij1} W_{kl1}}{W_{11}} + F^{ii}(\frac{W_{ii}}{W_{11}} - 1)}{F} + \psi \frac{F^{pp}}{F} \frac{W_{11p}^2}{W_{11}^2} + \frac{F^{pp} W_{pp}}{F} L u\psi \\
& + L\psi_{u_i} W_{ii} u_i - L\psi \frac{F^{pp} W_{pp}^2}{F}]
\end{aligned}
$$

$$= F[-C(n,L,M_0,f,\psi)+\psi_{u_1u_1}W_{11}-L\psi\frac{F^{pp}W_{pp}^2}{F}]+2F^{ii}W_{ii1}(\psi_{u_1}+\frac{\psi_{x_1}+\psi_u u_1-\psi_{u_1}u}{W_{11}})$$
$$+\psi F^{ij,kl}\frac{W_{ij1}W_{kl1}}{W_{11}}+F^{ii}(\frac{W_{ii}}{W_{11}}-1)+\psi F^{pp}\frac{W_{11p}^2}{W_{11}^2}+LF^{pp}W_{pp}u\psi$$

where we use (3.17) and $u^2 \le u^2+|\nabla u|^2 \le C(M_0)$ in the last inequality since the flow is contracting. By (1.10)

$$F^{ij,kl}W_{ij1}W_{kl1}+F^{ii}W^{jj}W_{ij1}^2 \le \delta_0\frac{(F^{ii}W_{ii1})^2}{F} \tag{3.18}$$

for any $W\in\Gamma^+$. This implies

$$\begin{aligned}
&2F^{ii}W_{ii1}(\psi_{u_1}+\frac{\psi_{x_1}+\psi_u u_1-\psi_{u_1}u}{W_{11}})+\psi F^{ij,kl}\frac{W_{ij1}W_{kl1}}{W_{11}}+\psi F^{pp}\frac{W_{11p}^2}{W_{11}^2}\\
\le&2F^{ii}W_{ii1}(\psi_{u_1}+\frac{\psi_{x_1}+\psi_u u_1-\psi_{u_1}u}{W_{11}})+\delta_0\frac{\psi(F^{ii}W_{ii1})^2}{FW_{11}}\\
\le&-\frac{FW_{11}}{\delta_0\psi}(\psi_{u_1}+\frac{\psi_{x_1}+\psi_u u_1-\psi_{u_1}u}{W_{11}})^2\\
\le&-C(M_0,\delta_0,\psi)FW_{11}.
\end{aligned}\tag{3.19}$$

Plugging this into the differential inequality for G and using (1.11), we get

$$\begin{aligned}
&G_t-\psi F^{pp}G_{pp}\\
\le&F[-C(n,L,M_0,f,\psi)-C(M_0,\delta_0,\psi)W_{11}-L\psi\frac{F^{pp}W_{pp}^2}{F}]+F^{ii}(\frac{W_{ii}}{W_{11}}-1)+LF^{pp}W_{pp}u\psi\\
\le&F[-C(n,L,M_0,f,\psi)-C(M_0,\delta_0,\psi)W_{11}+L\delta_0\psi\sigma_1(W)]++F^{ii}(\frac{W_{ii}}{W_{11}}-1)+LF^{pp}W_{pp}u\psi.
\end{aligned}\tag{3.20}$$

Since $M(t)$ is convex and the flow is contracting, $\sigma_1(W)\ge W_{11}\ge 1$ and $\inf_{(x,t)\in\mathbb{S}^n\times[0,T]}\psi=\varepsilon(M_0,\psi)>0$, and $(x,u,\nabla u)$ stays in a compact subset of $\mathbb{S}^n\times T\mathbb{S}^n$. Now we choose

$$L=\frac{C(M_0,\delta_0,\psi)+1}{\delta_0\inf_{(x,t)\in\mathbb{S}^n\times[0,T])}\psi}=:L(M_0,\delta_0,\psi)$$

satisfying

$$G_t-\psi F^{pp}G_{pp}\le F[-C(n,M_0,\delta_0,\psi,f)+W_{11}]+F^{ii}W_{ii}(\frac{1}{W_{11}}+Lu\psi)-\sum_i F^{ii}. \tag{3.21}$$

Let $C_1 = C_1(M_0, \delta_0, \psi) > 1$ such that $1 + L\sup_{(x,t)\in\mathbb{S}^n\times[0,T]} \psi u \le C_1$. By (2.3),

$$\begin{aligned}
G_t - \psi F^{pp} G_{pp} \le & F[-C(n, M_0, \delta_0, \psi, f) + W_{11}] + C_1(M_0, \delta_0, \psi) F^{ii} W_{ii} - \sum_i F^{ii} \\
\le & F[-C(n, M_0, \delta_0, \psi, f) + W_{11}] + C_1(M_0, \delta_0, \psi)(F - F(\frac{1}{C_1(M_0, \delta_0, \psi)} I)) \\
\le & F - \varepsilon_0 W_{11} - C_1(M_0, \delta_0, \psi) F(\frac{1}{C_1(M_0, \delta_0, \psi)} I) \\
< & 0
\end{aligned} \tag{3.22}$$

if $W_{11} \ge C(n, M_0, \delta_0, \psi, f) + \frac{C_1(M_0,\delta_0,\psi)F(\frac{1}{C_1(M_0,\delta_0,\psi)}I)}{\varepsilon_0} + 1$,where $\varepsilon_0 = \inf_{(x,t)\in\mathbb{S}^n\times[0,T]}(-F) > 0$ is the constant in (3.4). This implies an upper bound for $W_{11}(x_0, t_0)$, and hence a lower bound for $\kappa_i,\ i = 1, \cdots, n$.

□

Lemma 3.3. *(Lemma 2.2 [9]) Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a convex body with support function $u : \mathbb{S}^n \to \mathbb{R}$. Let $W = (u_{ij} + u\delta_{ij})$ be the spherical Hessian of u, $\rho_- := \rho_-(\Omega), \rho_+ := \rho_+(\Omega)$ be the inner and outer radius of Ω. Suppose $W \le C_0 I_n$ for some positive constant A. Then*

$$\frac{\rho_+^2(\Omega)}{\rho_-(\Omega)} \le C(n) C_0, \tag{3.23}$$

where $C(n)$ is a positive constant depending only on the dimension n.

4. Contraction to a point

In this section, we derive the contraction to a point of the flow (1.1) under various assumptions on f depending on different ψ (Theorem 1.1, Theorem 1.2, Theorem 1.3).

Lemma 4.1. *Suppose f satisfies (1.3), (1.4), (1.6), $\tilde{\psi}(\nu, X)$ is a positive smooth function, and $X(t)$ is a smooth convex solution of (1.1) for $0 \le t \le T$. Then*

$$-\psi F(W(x,T)) \le 3\frac{\rho_+(\Omega(T))}{\rho_-(\Omega(T))} C_1(M_0, \psi) \max\{-2F(\frac{\rho_-(\Omega(T))}{3C_2(M_0, \psi)} I), \max_{x\in\mathbb{S}^n}(-F(W(x,0)))\}. \tag{4.1}$$

where $C_2(M_0, \psi) \ge 1$, $C_1(M_0, \psi) > 0$ are positive constants depends only on M_0, ψ, $\psi(x, z, p)$ is the smooth positive function in (2.1) such that $\psi(x, z, p) = \tilde{\psi}(x, xz + p)$. Moreover, in the case when ψ doesn't depend on z, p, we can take $C_2(M_0, \psi) = 1$, and $C_1(M_0, \psi) = \max_{x\in\mathbb{S}^n} \psi(x)$.

Proof. Pick a point in $\Omega(T)$ as the origin such that $\min_{x\in\mathbb{S}^n} u(x, T) = \rho_-(\Omega(T)) = 3\epsilon$. Since (1.2) is a shrinking flow,

$$u(x, t) \ge u(x, T), \quad \forall x \in \mathbb{S}^n,\ \forall t \in [0, T].$$

Consider the function

$$\Phi := \log \frac{-\psi F(W)}{u - 2\epsilon}. \tag{4.2}$$

It follows from (2.2)

$$(-\psi F(W))_t = \psi F^{ij}(-\psi F)_{ij} + (-\psi^2 F)\sum F^{ii} + F\psi_u(-\psi F) - F\psi_{u_i}u_{it} \tag{4.3}$$

$$(u-2\epsilon)_t = \psi[F + F^{ij}(u-2\epsilon)_{ij} + (u-2\epsilon)\sum F^{ii} - F^{ij}W_{ij} + 2\epsilon\sum F^{ii}].$$

Suppose Φ attains $\max_{(x,t)\in M\times[0,T]}\Phi$ at (x_0,t_0), choose an orthonormal frame on $\mathbb{S}^n$ such that u_{ij} is diagonal at x_0, then at (x_0,t_0), $W_{ij}=u_{ij}+u\delta_{ij}$ and F^{ij} will also be diagonal. By maximum principle, at (x_0,t_0)

$$\frac{-u_{ti}}{-u_t} = \frac{u_i}{u-2\epsilon} \tag{4.4}$$

and

$$\begin{aligned} 0 \le& \Phi_t - \psi F^{ij}\Phi_{ij} \\ =&\psi F^{ii}\Phi_i(\log(-\psi F)(u-2\epsilon))_i + F\psi_u + F\psi_{u_i}\frac{u_i}{u-2\epsilon} + \psi\frac{-2\epsilon\sum F^{ii} - F + F^{ij}W_{ij}}{u-2\epsilon}. \\ =&\frac{\psi}{u-2\epsilon}[(\frac{\psi_u}{\psi}(u-2\epsilon) + \frac{\psi_{u_i}u_i}{\psi} - 1)F - 2\epsilon\sum F^{ii} + F^{ii}W_{ii}] \\ \le&\frac{\psi}{u-2\epsilon}[-C(M_0,\psi)F - 2\epsilon\sum F^{ii} + F^{ii}W_{ii}], \end{aligned} \tag{4.5}$$

for some $\infty > C(M_0,\psi) \ge 1$ as the flow is contracting and convex.

By (2.3),

$$-\epsilon\sum F^{ii} \le C(M_0,\psi)(F - F^{ij}W_{ij} - F(\frac{\epsilon}{C(M_0,\psi)}I)). \tag{4.6}$$

Plugging this into (4.5),

$$\begin{aligned} &\Phi_t - \psi F^{ij}\Phi_{ij} - \psi F^{ii}\Phi_i(\log(-F)(u-\epsilon))_i \\ \le&\psi\frac{C(M_0,\psi)F - F^{ij}W_{ij} - 2C(M_0,\psi)F(\frac{\epsilon}{C(M_0,\psi)}I)}{u-2\epsilon} \\ \le&C(M_0,\psi)\psi\frac{F - 2F(\frac{\epsilon}{C(M_0,\psi)}I)}{u-2\epsilon}. \end{aligned}$$

At the maximum point $p=(x_0,t_0)$ of Φ, we obtain

$$-\psi(x_0,u(p),\nabla u(p))F(W(p)) \le \max\{-2\psi(x_0,u(p),\nabla u(p))F(\frac{\epsilon}{C(M_0,\psi)}I), \max_{(x,t)\in\mathbb{S}^n\times\{0\}}(-\psi F)\}.$$

By the assumption, $u-2\epsilon \ge \epsilon$. That is,

$$\max_{t\le T, x\in\mathbb{S}^n}\frac{-\psi F(W)}{u-2\epsilon} \le \frac{\max\{-2\psi(x_0,u(p),\nabla u(p))F(\frac{\epsilon}{C(M_0,\psi)}I), \max_{(x,t)\in\mathbb{S}^n\times\{0\}}(-\psi F)\}}{\epsilon}.$$

Hence

$$-\psi(x)F(W(x,T)) = (u(x,T)-2\epsilon)\frac{-\psi F(W(x,T))}{u(x,T)-2\epsilon}$$

$$\leq \quad 3\frac{\rho_+(\Omega(T))}{\rho_-(\Omega(T))}C_1(\psi, M_0)\max\{-2F(\frac{\rho_-(\Omega(T))}{3C(M_0,\psi)}I), \max_{x\in\mathbb{S}^n}(-F(W(x,0)))\},$$

where $C_1(M_0,\psi) = \max\{\psi(x_0, u(p), \nabla u(p)), \max_{(x,t)\in\mathbb{S}^n\times\{0\}}\psi\} < \infty$ only depends on M_0, ψ since the flow is contracting and convex, and $(x, u, \nabla u)$ stays in a compact subset of $\mathbb{S}^n \times T\mathbb{S}^n$. It follows (4.1) with $C_2 = C$. ☐

Corollary 4.1. *With the same assumptions in Lemma (4.1), we have*

$$-F(x,t) \leq 3\frac{\rho_+(\Omega(T))}{\rho_-(\Omega(T))}\frac{C_1(M_0,\psi)}{\varepsilon_1(M_0,\psi)}\max\{-2F(\frac{\rho_-(\Omega(T))}{3C_2(M_0,\psi)}I), \max_{x\in\mathbb{S}^n}(-F(W(x,0)))\}. \tag{4.7}$$

where $\varepsilon_1(M_0,\psi) = \inf\psi > 0$ *is the minimum of* ψ *along the flow.*

Proof. Since $(x, u, \nabla u)$ stays in a compact subset of $\mathbb{S}^n \times T\mathbb{S}^n$, $\varepsilon_1 > 0$ depends only on M_0, ψ. Then the corollary follows from Lemma 4.1. ☐

Corollary 4.2. *Suppose assumptions in Lemma 4.1 are satisfied, then there is* $C > 0$ *such that for* $\rho_-(\Omega(T))$ *sufficiently small,*

$$-F(W(x,T)) \leq C\frac{-F(\frac{\rho_-(\Omega(T))}{3C_2(M_0,\psi)}I)}{\rho_-^{\frac{1}{2}}(\Omega(T))}, \quad \forall x \in \mathbb{S}^n. \tag{4.8}$$

Proof. By Lemma 3.2 and Lemma 3.3,

$$\rho_+(\Omega(T)) \leq C\rho_-^{\frac{1}{2}}(\Omega(T)).$$

We note that when $\rho_-(\Omega(T))$ is sufficiently small,

$$-F(\frac{\rho_-(\Omega(T))}{3C(M_0,\psi)}I) \geq -F(W(x,0)),\ \forall x \in \mathbb{S}^n.$$

☐

4.1. **Proof of Theorem 1.1.** In this subsection, we take $\psi = \tilde{\psi} \equiv 1$ and prove Theorem 1.1.

Lemma 4.2. *Assume* f *satisfies conditions (1.3), (1.4), (1.5), and (1.6), and suppose* $X(t)$ *is a smooth solution of (1.2) for* $0 < t \leq T$ *with initial strictly convex* $X(0) = M$, *and* $\rho_+(\Omega(T)) \geq \varepsilon_2$, *then there is* $\delta(n, M_0, f, \varepsilon_2) > 0$ *depending on* n, M_0, f, ε_2 *such that*

$$\min\{\lambda_1(W(x,t)), \cdots, \lambda_n(W(x,t))\} \geq \delta, \quad \forall 0 < t \leq T.$$

Proof. By (1) of Lemma 3.2, $W(x,t)$ is bounded from above. Thus, $W(x,t)$ is inside a compact subset of $\bar{\Gamma}^+$. By Lemma 3.3, $\rho_-(\Omega(t)) \geq C(n, M_0)\rho_+^2(\Omega(t)) \geq C(n, M_0)\rho_+^2(\Omega(T)) \geq C(n, M_0)\varepsilon_2^2$, $\forall\ 0 \leq t \leq T$. By Corollary 4.1, the speed function $-F(W(x,t))$ is bounded from above as long as outer radius of $\Omega(t)$ is positive and $W(x,t)$ is positive definite. By (1.5), $W(x,t)$ is bounded from below. ☐

Proof of Theorem 1.1. We write (1.2) as (2.1) with $\psi(x) \equiv 1$ in this case. By Lemma 3.2, the solution M_t is strictly convex if it exists. Since the flow (2.1) is contracting, the C^0 estimate of u follows. The C^1 bound follows by convexity. Moreover, W is bounded from below and above as long as outer radius is positive by Lemma 3.2 and Lemma 4.2. By Krylov's theorem, flow (1.2) exists before it converges to a point. Finally, the extinction time T^* must be finite since the speed function $\psi F \leq -C_1$ for an absolute constant $C_1 > 0$ by Lemma 3.1. □

4.2. **Proof of Theorem 1.2, 1.3.**

Proof of Theorem 1.2. We write (1.7) as (2.1) with $\psi(x) = \tilde{\psi}(x)$. The C^1 and C^0 bound of u follows from the contracting nature of (2.1) and convexity of M_t. Note $\psi > 0$ is bounded from below and above with uniform C^2 norm since $\mathbb{S}^n$ is compact. Moreover, (1.9) implies (1.6). Then we can use Lemma 3.2, Corollary 4.1, and the same argument as in the proof of Theorem 1.1. □

Proof of Theorem 1.3. We write (1.1) as (2.1) with $\psi(x, z, p) = \tilde{\psi}(x, zx + p)$. The C^1 and C^0 bound of u follows from the contracting nature of (2.1) and convexity of M_t. The uniform C^0 and C^1 bound implies that $\psi(x, u, \nabla u) > 0$ is bounded from below and above with uniform C^2 norm. We also note that (1.10) implies (1.6). Then we can use Lemma 3.2, Corollary 4.1, and the same arguments as in the proof of Theorem 1.1. In this case, the extinction time T^* is also finite since $\min_{(x,t)\in\mathbb{S}^n\times[0,T]} -\psi F \geq \frac{1}{C(T+1)}$ for an absolute constant C independent of $T > 0$ by Lemma 3.1. This implies $u(x, 0) - u(x, T^*) = \int_0^{T^*} -\psi(x, u(x, t), \nabla u(x, t))F(W(x, t))dt = \infty$ ($x \in \mathbb{S}^n$) if $T^* = \infty$, which contradicts to the fact the flow is contracting and M_0 is compact. □

5. Remarks

We discuss conditions specified in Theorem 1.1, Theorem 1.2 and 1.3. There are large classes of non-homogeneous curvature flows which evolve a strictly convex hypersurface to a point in finite time satisfying these conditions.

Remark 5.1. *(1) Concavity condition (1.6) of F is slightly weaker than concavity of f. We refer Theorem 1 and Remark 2 of* [4] *for discussion regarding condition (1.5).*

(2) There is a wide class of non-homogeneous functions satisfying conditions in Theorem 1.1 and Theorem 1.2. $f(\kappa) = \sigma_k^\alpha(\kappa)(1 \leq k \leq n, \alpha > 0)$ satisfies conditions (1.3)-(1.6) and (1.9). One may build fully non-linear flows satisfying conditions in Theorem 1.1 and Theorem 1.2 by using them as building blocks. If $f_1, \cdots, f_m$ satisfy conditions (1.3)-(1.6) and (1.9), so does $f = \sum_{i=1}^m a_i f_i^{\beta_i}$ provided $a_i > 0, \beta_i \geq 1,\ \forall i = 1, \cdots, m$. More generally, suppose $f_1, \cdots, f_m$ satisfy conditions (1.4), (1.5) and (1.6), suppose $G : \mathbb{R}^m \to \mathbb{R}_+$ is a convex function, and

$$\frac{\partial G}{\partial r_i} > 0,\ \forall r_i > 0,\ \forall i = 1, \cdots, m,$$

then $f = G(f_1, \cdots, f_m)$ satisfies conditions (1.3)-(1.6). If in addition $\exists C_0 > 0$ such that

$$\sum_i \frac{\partial G(r)}{\partial r_i} r_i \leq C_0 G(r),\ \forall r_i > 0,\ \forall i = 1, \cdots, m,$$

and $f_1, \cdots, f_m$ satisfy condition (1.9), then $f = G(f_1, \cdots, f_m)$ satisfies condition (1.9).

Remark 5.2. *(1) If G_l satisfies (1.4) and*

$$G_l^{\alpha\beta,\gamma\eta}\xi_{\alpha\beta}\xi_{\gamma\eta} \leq -W^{\beta\gamma}G_l^{\alpha\eta}\xi_{\alpha\beta}\xi_{\gamma\eta},\ \forall W \in \Gamma^+,\ \forall \xi_{\alpha\beta},\ \forall l = 1, \cdots, N, \tag{5.1}$$

then

$$F(W) = -\sum_{l=1}^{N} e^{-G_l(W)}$$

satisfies (1.10). $G = s\log\left(\frac{\sigma_n(W)}{\sigma_k(W)}\right)$ satisfies (5.1) for $\forall s \in \mathbb{R}^+$. To see that, first it's easy to check that $\log\sigma_n(W)$ satisfies (5.1) with " = " holding. By the proof of Lemma 2 in [12],

$$\left[\sigma_k(W)^{\alpha\beta,\gamma\eta} + W^{\beta\gamma}\sigma_k(W)^{\alpha\eta}\right]\xi_{\alpha\beta}\xi_{\gamma\eta} \geq \frac{(\sigma_k(W)^{\alpha\beta}\xi_{\alpha\beta})^2}{\sigma_k(W)},\ \forall W \in \Gamma^+,\ \forall \xi_{\alpha\beta}.$$

This implies that $\log\left(\frac{\sigma_n(W)}{\sigma_k(W)}\right)$ satisfies (5.1).

(2) Condition (1.11) and conditions (1.3)-(1.6) are satisfied by $\sigma_n^s(\kappa)$, $\forall s > 0$. If $p(\kappa)$ satisfies these conditions, so is $f(\kappa) = G(p(\kappa))$ with $G : \mathbb{R}_+ \to \mathbb{R}_+$ a smooth convex function, $G'(r) > 0$, $\forall r > 0$, $G(0) = 0$. Thus, Theorem 1.3 holds for this type of inhomogeneous Gauss curvature flow. Condition (1.11) is restrictive, there should be some better conditions.

We note that the initial strictly convex condition on M in Theorems in Section 1 can be relaxed.

Proposition 5.1. *Suppose $M = \partial\Omega_0$ is closed, smooth and convex, denote*

$$\Gamma = \{\kappa(x),\ \forall x \in M\}.$$

Assume f is a positive, symmetric function on Γ^+ and extends smoothly to $(\bar{\Gamma}^+ \cap \Gamma) \cup \Gamma^+$ and satisfies conditions (1.4)-(1.6) on $(\bar{\Gamma}^+ \cap \Gamma) \cup \Gamma^+$, Then there is finite $T^* > 0$ such that flow (1.2) exists for $0 < t < T^*$, and solution $X(t)$ remains strictly convex and $X(t)$ converges to a point as $t \to T^*$.

The same conclusion holds for flow (1.7) if f satisfies (1.9) in addition.

Proof. By the initial assumption, flow exists for a short time $T > 0$ with conditions (1.4)-(1.6). The strict convexity of $X(t)$ follows from Theorem 1.4 in [5]. Then applying Theorem 1.1 to the flow starting from $t = T$. □

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