A WEIGHTED GRADIENT ESTIMATE FOR SOLUTIONS OF L^p CHRISTOFFEL-MINKOWSKI PROBLEM

PENGFEI GUAN

Dedicated to Professor Neil Trudinger on the occasion of his 80th birthday

Abstract. We extend the weighted gradient estimate for solutions of nonlinear PDE associated to the prescribed k-th L^p-area measure problem established in [8] to the case $0 < p < 1$. The estimate yields non-collapsing estimate for symmetric convex bodied with prescribed L^p-area measures.

1. Introduction

The classical Christoffel-Minkowski problem is a problem of prescribing k-th area measure on $\mathbb{S}^n$. Given a Borel measure $\mu = f d\sigma_{\mathbb{S}^n}$ on $\mathbb{S}^n$, one seeks a convex body $K \subset \mathbb{R}^{n+1}$ such that its k-th area measure $S_k(K, x) = \mu$. It is a fundamental problem in convex geometry. The problem plays important rule in the development of nonlinear geometric partial differential equations.

The Christoffel-Minkowski problem corresponds to solving the following fully nonlinear elliptic equation

$$\sigma_k(W(x)) = f(x), \quad W(x) > 0, \ \forall x \in \mathbb{S}^n, \tag{1.1}$$

where u is the support function of K defined on $\mathbb{S}^n$ and

$$W(x) = (u_{ij}(x) + u\delta_{ij}(x)), \quad \forall x \in \mathbb{S}^n.$$

The Christoffel problem and the Minkowski problem correspond to the cases $k = 1$ and $k = n$ respectively [15, 16, 2, 17, 1, 4, 7]. The notion of area measures in the Brunn-Minkowski theory is based on Minkowski summation. Lutwak [12] developed corresponding L^p Brunn-Minkowski-Firey theory based on Firey's p-sum [5]. L^p-Minkowski problem has attracted much attention, we refer [12, 13, 3, 6, 14] and references therein.

The focus of this paper is on the intermediate L^p-Christoffel-Minkowski problem. The problem is deduced to solve the following PDE on $\mathbb{S}^n$,

$$\sigma_k(W(x)) = u^{p-1} f(x), \quad W(x) > 0, \ \forall x \in \mathbb{S}^n. \tag{1.2}$$

$p = 1$ is the classical Christoffel-Minkowski problem [17, 7]. The case $p \geq k + 1$ was considered by Hu-Ma-Shen [9] and the case $1 < p < k + 1$ was considered by Guan-Xia [8]. Very little is known for equation (1.2) in the case $0 < p < 1$.

In general, admissible solutions to $\sigma_k(W) = f$ is not convex (i.e., $W > 0$) if $k < n$. The existence of geometric solutions of (1.2) relies on two ingredients:

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(1) A priori upper and lower bounds of solutions,
(2) Convexity of solutions (i.e., $W > 0$).

When $p-1 < k < n$, in general there is no direct non-collapsing estimate for convex body satisfying equation (1.2) when $k < n$. For $p \geq k+1$, maximum principle implies the upper and lower bounds of solutions [9]. When $p < k+1$, the lower bound of solutions are not true in general as discussed in examples in [8]. In [8], the upper and lower bounds for *even* solutions of (1.2) were obtained for $1 < p < k+1$. The estimate relies on a weighted gradient estimate for $\frac{|\nabla u|^2}{(u-m_u)^\gamma}$ where $m_u = \min_{x\in\mathbb{S}^n} u$. The purpose of this paper is to extend such estimate for the case $0 < p < 1$.

Similar to the classical intermediate Christoffel-Minkowski problem, one needs to impose appropriate appropriate conditions on the prescribed function f in equation (1.1) to ensure the convexity of solutions to (1.2). The key is the Constant Rank Theorem established by Guan-Ma in [7]. When $p > 1$, a corresponding condition was deduced in [9] from the Constant Rank Theorem in [7]. When $0 < p < 1$, it is an open problem to find a clean condition on f to guarantee the convexity of solutions to (1.2).

2. Weighted gradient estimate

In this section, we modify the arguments in [8] to establish a weighted gradient estimate for solutions of the intermediate Christoffel-Minkowski problem (1.2) for $0 < p \leq 1$. Specifically, we extend Proposition 3.1 in [8] to the case $0 < p < 1$. Recall Garding's cone

$$\Gamma_k = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0,\ \forall j = 1, \cdots, k.\}$$

A symmetric matrix W is called in Γ_k if its eigenvalue vector $\lambda_W \in \Gamma_k$. A positive function $u \in C^2(\mathbb{S}^n)$ is called an admissible solution to (1.2) if $W(x) \in \Gamma_k,\ \forall x \in \mathbb{S}^n$.

In the rest of the paper, we denote

$$(\lambda \mid 1) = (0, \lambda_2, \cdots, \lambda_n),\ \forall \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n.$$

Proposition 2.1. *Let $0 < p \leq 1$ and let u be a positive admissible solution to* (1.2)*. Denote $m_u = \min u$ and $M_u = \max u$. Set*

$$\gamma = \frac{2p}{k+4}. \tag{2.1}$$

Then there exist some positive constants A depending only on n, k, p and $\|\log f\|_{C^1}$, such that

$$\frac{|\nabla u|^2}{|u-m_u|^\gamma} \leq A M_u^{2-\gamma}. \tag{2.2}$$

The weighted gradient estimate for $\frac{|\nabla u|^2}{u^\gamma}$ was used in [6], later in [11, 10, 8]. It's useful tool to obtain lower bound of solution u.

Proof. After proper rescale, we may assume $\min_{x\in\mathbb{S}^n} f(x) = 1$. Maximum principle yields that there is $C_{n,k,p} > 0$, such that

$$M_u \geq C_{n,k,p}.$$

Set

$$\Phi = \frac{|\nabla u|^2}{(u-m_u)^\gamma},$$

where $0 < \gamma < 1$ as in (2.1). As pointed out in [8] that Φ is well-defined and it makes sense to define $\Phi = 0$ at the minimum point of u.

Let x_0 be a maximum point of Φ. Then $u(x_0) > m_u$ if u is not a constant. We may pick an orthonormal frame on $\mathbb{S}^n$ such that $u_1(x_0) = |\nabla u|(x_0)$ and $u_i(x_0) = 0$ for $i = 2, \cdots, n$. At x_0,

$$\frac{2u_l u_{li}}{|\nabla u|^2} = \gamma \frac{u_i}{u - m_u} \text{ for each } i.$$

Thus $u_{1i} = 0$ for $i = 2, \cdots, n$ and

$$u_{11} = \frac{\gamma}{2} \frac{u_1^2}{u - m_u} = \frac{\gamma}{2} \Phi \frac{1}{(u - m_u)^{1-\gamma}}. \tag{2.3}$$

Re-rotating the remaining $n - 1$ coordinates, we may assume

$$(u_{ij}) \text{ is diagonal, so are } (W_{ij}(x_0)) \text{ and } (F^{ij})(x_0) = (\frac{\partial \sigma_k}{\partial W_{ij}})(x_0).$$

We may assume $\frac{\Phi}{M_u^{2-\gamma}}$ is sufficiently large at x_0. In the rest of proof, constant C may change line by line, but under control.

$$W_{11} \leq u_{11}(1 + C(\frac{M_u^{2-\gamma}}{\Phi})). \tag{2.4}$$

At x_0, it follows from (2.3) and (1.2),

$$\begin{aligned}
0 &\geq F^{ii}(\log \Phi)_{ii} \\
&= F^{ii} \frac{2u_{ii}^2 + 2u_l u_{lii}}{|\nabla u|^2} - \gamma \frac{F^{ii} u_{ii}}{u - m_u} + \gamma(1 - \gamma) \frac{F^{ii} u_i^2}{(u - m_u)^2} \\
&= \frac{2F^{ii} u_{ii}^2}{u_1^2} + \frac{2F^{ii} u_1 (W_{ii1} - u_i \delta_{1i})}{u_1^2} - \gamma \frac{F^{ii} u_{ii}}{u - m_u} + \gamma(1 - \gamma) \frac{F^{ii} u_i^2}{(u - m_u)^2} \\
&= \frac{2F^{ii} u_{ii}^2}{u_1^2} + 2(p - 1) u^{p-2} f + \frac{2u^{p-1} f_1}{u_1} - 2F^{11} - \gamma \frac{F^{ii} u_{ii}}{u - m_u} + \gamma(1 - \gamma) \frac{F^{ii} u_i^2}{(u - m_u)^2} \\
&\geq \frac{2F^{ii} u_{ii}^2}{u_1^2} + 2(p - 1) u^{p-2} f + \gamma(1 - \gamma) \frac{F^{11} u_1^2}{(u - m_u)^2} + \frac{2u^{p-1} f_1}{u_1} - 2F^{11} - \gamma \frac{F^{ii} W_{ii}}{u - m_u} \\
&\geq \frac{2F^{ii} u_{ii}^2}{u_1^2} + 2(1 - \gamma) \frac{F^{11} u_{11}}{u - m_u} + \frac{2u^{p-1} f_1}{u_1} - 2F^{11} - (k\gamma - 2(p - 1)) \frac{\sigma_k(W)}{u - m_u} \\
&\geq 2(1 - \gamma) \frac{F^{11} u_{11}}{u - m_u} + \frac{2u^{p-1} f_1}{u_1} + 2F^{11}(\frac{u_{11}^2}{u_1^2} - 1) - (k\gamma - 2(p - 1)) \frac{\sigma_k(W)}{u - m_u}. \qquad (2.5)
\end{aligned}$$

It follows the definition of Φ,

$$\frac{2u^{p-1} f_1}{u_1} \geq -C u^{p-1} f \Phi^{-\frac{1}{2}} (u - m_u)^{-\frac{\gamma}{2}} \geq -C \frac{\sigma_k(W)}{u - m_u} \frac{M_u^{1-\frac{\gamma}{2}}}{\Phi^{\frac{1}{2}}}. \tag{2.6}$$

Note that $\frac{M_u^{2-\gamma}}{\Phi}$ sufficiently small by the assumption.

By (2.3) and (2.4),

$$\frac{u_{11}^2}{u_1^2} - 1 = \frac{\gamma}{2}\frac{u_{11}}{u - m_u} - 1 = \frac{\gamma}{2}\frac{W_{11}}{u - m_u}(1 - C\frac{M_u^{2-\gamma}}{\Phi}). \tag{2.7}$$

$$W_{11} \geq \frac{\gamma}{4}\frac{\Phi}{(u - m_u)^{1-\gamma}} \geq \frac{\gamma}{4}\frac{\Phi}{M_u^{2-\gamma}}\frac{M_u^{2-\gamma}}{(u - m_u)^{1-\gamma}}. \tag{2.8}$$

Put (2.6) and (2.7) to (2.5),

$$0 \geq (2 - \gamma - C\frac{M_u^{2-\gamma}}{\Phi})F^{11}\frac{W_{11}}{u - m_u} - (k\gamma - 2(p-1) + C\frac{M_u^{1-\frac{\gamma}{2}}}{\Phi^{\frac{1}{2}}})\frac{\sigma_k(W)}{u - m_u} \tag{2.9}$$

We divide in to two cases.

Case I.

$$\sigma_k(W|1) \leq \gamma\sigma_{k-1}(W|1)W_{11}.$$

We have,

$$\sigma_k(W) = \sigma_{k-1}(W|1)W_{11} + \sigma_k(W|1) \leq (1+\gamma)\sigma_{k-1}(W|1)W_{11} = (1+\gamma)F^{11}W_{11}.$$

Put this into (2.9), we obtain

$$0 \geq 2 - \gamma - (1+\gamma)(k\gamma - 2(p-1) + C\frac{M_u^{1-\frac{\gamma}{2}}}{\Phi^{\frac{1}{2}}}).$$

By the choice of γ in (2.1),

$$C\frac{M_u^{1-\frac{\gamma}{2}}}{\Phi^{\frac{1}{2}}} \geq \frac{p}{k+4}.$$

(2.2) is verified in this case.

Case II.

$$\sigma_k(W|1) > \gamma\sigma_{k-1}(W|1)W_{11}.$$

If $k \geq 2$, by the Newton-MacLaurin inequality,

$$\sigma_{k-1}^{\frac{k}{k-1}}(W|1) \geq C_{n,k}\sigma_k(W|1).$$

In turn,

$$\sigma_{k-1}^{\frac{k}{k-1}}(W|1) \geq C_{n,k}\sigma_k(W|1) > C_{n,k}\gamma\sigma_{k-1}(W|1)W_{11}.$$

Hence, $\sigma_{k-1}^{\frac{1}{k-1}}(W|1) \geq C_{n,k}\gamma W_{11}$. We now have,

$$u^{p-1}f = \sigma_k(W) = \sigma_k(W|1) + \sigma_{k-1}(W|1)W_{11} \geq (1+\gamma)\sigma_{k-1}(W|1)W_{11} \geq (C_{n,k}\gamma)^{k-1}W_{11}^k.$$

Note that the above inequality is trivial for $k = 1$ in this case. We obtain

$$W_{11} \leq (C_{n,k}\gamma)^{\frac{k-1}{k}}u^{\frac{p-1}{k}}f^{\frac{1}{k}}. \tag{2.10}$$

Then (2.2) follows from (2.10), (2.3) and (2.4).

□

When u is a convex solution of (1.2), estimate (2.2) in Proposition 2.1 can be refined. We will use this type of refined estimates to establish existence of convex even solutions for equation (1.2) when $0 < 1 - p$ is close to 0.

Proposition 2.2. *Let $0 < p \leq 1$ and let u be a positive convex solution to* (1.2).

a. *If $k = 1$, then*

$$M_u^{\gamma-2}\frac{|\nabla u(x)|^2}{(u(x)-m_u)^\gamma} \leq (\frac{2n}{\gamma})^{\frac{\gamma}{p}} e^{\frac{\gamma\pi}{p}\|\nabla \log f\|_{C^0}}, \ \forall 0<\gamma<1. \ \forall x \in \mathbb{S}^n. \tag{2.11}$$

b. *If $2 \leq k < n$, then there exists $A_{n,k,p}$ depending only on n, k, p, such that*

$$M_u^{\gamma-2}\frac{|\nabla u|^2}{|u-m_u|^\gamma} \leq A_{n,k,p} e^{\frac{\gamma\pi}{k-1+p}\|\nabla \log f\|_{C^0}}, \tag{2.12}$$

where

$$\gamma = \frac{p}{k+1}. \tag{2.13}$$

Proof. For $0 < \gamma < 1$, let $\Phi = \frac{|\nabla u|^2}{(u-m_u)^\gamma}$ as in the proof of Proposition 2.1. We may assume

$$\min_{x\in\mathbb{S}^n} f(x) = 1.$$

By equation (1.2),

$$M_u^{k+1-p} \geq \frac{(n-k)!k!}{n!}. \tag{2.14}$$

Set

$$q = 2 - \frac{\gamma}{p}, \ \beta = \frac{1}{p}(1-\gamma), \tag{2.15}$$

and

$$A_\gamma = \frac{\max_{x\in\mathbb{S}^n}\Phi(x)}{M_u^{2-\gamma}} = \frac{\Phi(x_0)}{M_u^{2-\gamma}}. \tag{2.16}$$

We want to estimate A_γ.

Suppose x_0 is a maximum point of Φ. Let $\eta > 0$ is a positive number to be determined. If,

$$(\frac{u(x_0)-m_u}{M_u})^{1-\gamma} \geq (\frac{\gamma}{\eta})^\beta,$$

then

$$(u(x_0)-m_u)^\gamma \geq M_u^\gamma(\frac{\gamma}{\eta})^{2-q}.$$

Since u is convex, $|\nabla u(x)|^2 \leq M_u^2, \ \forall x \in \mathbb{S}^n$. We have

$$A_\gamma = \frac{\Phi(x_0)}{M_u^{2-\gamma}} \leq \frac{M_u^\gamma}{(u-m_u)^\gamma} \leq (\frac{\eta}{\gamma})^{2-q}. \tag{2.17}$$

We now assume that at x_0,

$$(\frac{u-m_u}{M_u})^{1-\gamma} \leq (\frac{\gamma}{\eta})^\beta. \tag{2.18}$$

As in the proof of Proposition 2.1, one may pick an orthonormal frame on $\mathbb{S}^n$ near x_0, such that $|\nabla u(x_0)| = u_1(x_0)$, $(W_{ij}(x_0))$ is diagonal,

$$u_{11} = \frac{\gamma}{2}\frac{u_1^2}{u-m_u} = \frac{\gamma}{2}A_\gamma \frac{M_u^{2-\gamma}}{(u-m_u)^{1-\gamma}}, \tag{2.19}$$

and

$$W_{11} > u_{11} = \frac{\gamma}{2}A_\gamma \frac{M_u^{2-\gamma}}{(u-m_u)^{1-\gamma}}. \tag{2.20}$$

We first consider the simple case $k=1$.

Case $k=1$. Since $p \le 1$, $u^{p-1} \le (u-m_u)^{p-1}$. By (2.20), at maximum point x_0 of Φ,

$$(u-m_u)^{p-1} f \ge u^{p-1} f = \sigma_1(W) \ge W_{11} \ge u_{11} = \frac{\gamma}{2}A_\gamma \frac{M_u^{2-\gamma}}{(u-m_u)^{1-\gamma}}.$$

It follows

$$A_\gamma \le \frac{2n}{\gamma}(\frac{u-m_u}{M_u})^{p-\gamma} M_u^{p-2} f \le \frac{2n}{\gamma}(\frac{\gamma}{\eta})^{\frac{(p-\gamma)(2-q)}{\gamma}} f \le \frac{2n}{\gamma}(\frac{\gamma}{\eta})^{\frac{(p-\gamma)(2-q)}{\gamma}} e^{\pi\|\nabla \log f\|_{C^0}}, \tag{2.21}$$

here we used $\min_{x\in\mathbb{S}^n} f(x) = 1$ and (2.14) for $k=1$. Use (2.15) to equalize quantities on the right hand sides of (2.17) and (2.21), we pick

$$\eta = 2ne^{\pi\|\nabla \log f\|_{C^0}}.$$

Thus,

$$A_\gamma \le \gamma^{-\frac{\gamma}{p}}(2ne^{\pi\|\nabla \log f\|_{C^0}})^{\frac{\gamma}{p}}, \ \forall 0<\gamma<1.$$

(2.11) is proved. We may let $\gamma \to 1$,

$$\frac{|\nabla u(x)|^2}{u(x)-m_u} \le (2ne^{\pi\|\nabla \log f\|_{C^0}})^{\frac{1}{p}} M_u, \ \forall x \in \mathbb{S}^n. \tag{2.22}$$

We note that in this case, bound on $\|\nabla f\|$ can be replaced by ratio of $\frac{M_f}{m_f}$ in above estimate.

Case $2 \le k < n$. At x_0,

$$W_{11} = u_{11}(1 + \frac{2}{\gamma}A_\gamma^{-1}\frac{u(u-m_u)^{1-\gamma}}{M_u^{2-\gamma}}). \tag{2.23}$$

By (2.5),

$$0 \ge 2(1-\gamma)\frac{F^{11}u_{11}}{u-m_u} + \frac{2u^{p-1}f_1}{u_1} + 2F^{11}(\frac{u_{11}^2}{u_1^2} - 1) - (k\gamma - 2(p-1))\frac{\sigma_k(W)}{u-m_u}. \tag{2.24}$$

Since $\frac{f_1}{f} \ge -\|\nabla \log f\|_{C^0}$, (2.6) can be refined as

$$\begin{aligned} \frac{2u^{p-1}f_1}{u_1} &\ge -2u^{p-1} f \|\nabla \log f\|_{C^0} \Phi^{-\frac{1}{2}}(u-m_u)^{-\frac{\gamma}{2}} \\ &= -2\|\nabla \log f\|_{C^0} A_\gamma^{-\frac{1}{2}}(\frac{u-m_u}{M_u})^{1-\frac{\gamma}{2}}\frac{\sigma_k(W)}{u-m_u}. \end{aligned} \tag{2.25}$$

By (2.19), (2.23) and (2.20),

$$\frac{u_{11}^2}{u_1^2} - 1 = \frac{\gamma}{2}\frac{u_{11}}{u-m_u} - 1 \geq \frac{\gamma}{2}\frac{W_{11}}{u-m_u}(1 - \frac{8}{\gamma^2}A_\gamma^{-1}\frac{u(u-m_u)^{1-\gamma}}{M_u^{2-\gamma}}). \tag{2.26}$$

Put (2.25) and (2.26) to (2.24), as $p \leq 1$,

$$\begin{aligned} 0 \geq\ & (2-\gamma)\frac{F^{11}W_{11}}{u-m_u} - \Big\{k\gamma - 2(p-1) \\ & +\big(\frac{4}{\gamma}A_\gamma^{-1}\frac{u(u-m_u)^{1-\gamma}}{M_u^{2-\gamma}} + 2\|\nabla \log f\|_{C^0}A_\gamma^{-\frac{1}{2}}(\frac{u-m_u}{M_u})^{1-\frac{\gamma}{2}}\big)\Big\}\frac{\sigma_k(W)}{u-m_u}. \end{aligned} \tag{2.27}$$

Choose

$$\eta = (2^{2k-1}(n-k)^{k-1}\frac{n}{k^k}e^{\pi\|\nabla \log f\|_{C^0}})^{\frac{p}{k-1+p}}, \tag{2.28}$$

and

$$\gamma = \frac{p}{k+1},\ \ \delta = \frac{1}{2}\gamma^{\frac{1-p}{p}}. \tag{2.29}$$

We divide in to two subcases.

Subcase I. Assume that

$$\sigma_k(W|1) > \delta\sigma_{k-1}(W|1)W_{11}.$$

If $k \geq 2$, by the Newton-MacLaurin inequality,

$$\sigma_{k-1}^{\frac{k}{k-1}}(W|1) \geq C_{n,k}\sigma_k(W|1),$$

where

$$C_{n,k} = \frac{k}{n-k}(\frac{(n-1)!}{(n-k)!(k-1)!})^{\frac{1}{k-1}}. \tag{2.30}$$

In turn,

$$\sigma_{k-1}^{\frac{k}{k-1}}(W|1) \geq C_{n,k}\sigma_k(W|1) > C_{n,k}\delta\sigma_{k-1}(W|1)W_{11}.$$

Hence,

$$\sigma_{k-1}^{\frac{1}{k-1}}(W|1) \geq C_{n,k}\delta W_{11}.$$

By equation (1.2),

$$u^{p-1}f = \sigma_k(W) \geq \sigma_{k-1}(W|1)W_{11} \geq (C_{n,k}\delta)^{k-1}W_{11}^k. \tag{2.31}$$

Note that (2.31) is trivial for $k=1$ in this subcase. Thus it is true $\forall k \geq 1$. As $p \leq 1$, $u^{\frac{p-1}{k}} \leq (u-m_u)^{\frac{p-1}{k}}$, we deduce from (2.20) and (2.31) that,

$$A_\gamma \leq \frac{2}{\gamma}(C_{n,k}\delta)^{\frac{1-k}{k}}M_u^{-1+\frac{p-1}{k}}(\frac{u-m_u}{M_u})^{1-\gamma+\frac{p-1}{k}}f^{\frac{1}{k}}.$$

By (2.18), (2.14), (2.28), (2.29) and (2.30), and the fact that $\min f = 1$,

$$
\begin{aligned}
A_\gamma &\le \frac{2}{\gamma}(C_{n,k}\delta)^{\frac{1-k}{k}} M_u^{-1+\frac{p-1}{k}} (\frac{\gamma}{\eta})^{\frac{2-q}{\gamma}(1-\gamma+\frac{p-1}{k})} e^{\frac{\pi}{k}\|\nabla \log f\|_{C^0}} \\
&\le 2(\frac{C_{n,k}}{2})^{\frac{1-k}{k}} (\frac{n!}{(n-k)!k!})^{\frac{1}{k}} (\frac{1}{\eta})^{\frac{2-q}{\gamma}(1+\frac{p-1}{k})} e^{\frac{\pi}{k}\|\nabla \log f\|_{C^0}} (\frac{\gamma}{\eta})^{q-2} \\
&= (\frac{\gamma}{\eta})^{q-2}.
\end{aligned} \tag{2.32}
$$

Subcase II. Assume that

$$\sigma_k(W|1) \le \delta \sigma_{k-1}(W|1) W_{11}.$$

We have,

$$\sigma_k(W) = \sigma_{k-1}(W|1)W_{11} + \sigma_k(W|1) \le (1+\delta)\sigma_{k-1}(W|1)W_{11} = (1+\delta)F^{11}W_{11}.$$

Put this into (2.27), we obtain

$$0 \ge 2-\gamma-(1+\delta)\Big\{k\gamma - 2(p-1) + (\frac{4}{\gamma}A_\gamma^{-1}\frac{u(u-m_u)^{1-\gamma}}{M_u^{2-\gamma}} + 2\|\nabla \log f\|_{C^0} A_\gamma^{-\frac{1}{2}} (\frac{u-m_u}{M_u})^{1-\frac{\gamma}{2}})\Big\}.$$

From (2.13) and (2.29),

$$2-\gamma-(1+\delta)(k\gamma-2(p-1)) \ge \gamma(1+\delta).$$

Hence

$$0 \ge \gamma - \big(\frac{4}{\gamma}A_\gamma^{-1}\frac{u(u-m_u)^{1-\gamma}}{M_u^{2-\gamma}} + 2\|\nabla \log f\|_{C^0} A_\gamma^{-\frac{1}{2}} (\frac{u-m_u}{M_u})^{1-\frac{\gamma}{2}}\big).$$

Again by (2.13) and (2.29),

$$\frac{4}{\gamma}A_\gamma^{-1}\frac{u(u-m_u)^{1-\gamma}}{M_u^{2-\gamma}} + 2\|\nabla \log f\|_{C^0} A_\gamma^{-\frac{1}{2}} (\frac{u-m_u}{M_u})^{1-\frac{\gamma}{2}} \ge \gamma.$$

It follows from (2.18) that,

$$\frac{4}{\gamma}A_\gamma^{-1}(\frac{\gamma}{\eta})^{\frac{1-\gamma}{p}} + 2\|\nabla \log f\|_{C^0} A_\gamma^{-\frac{1}{2}} (\frac{\gamma}{\eta})^{\frac{1-\frac{\gamma}{2}}{p}} \ge \gamma.$$

We obtain

$$
\begin{aligned}
A_\gamma &\le 8\big(\eta^{-\frac{1}{p}}\gamma^{\frac{1}{p}-2} + \|\nabla \log f\|_{C^0}^2 \eta^{-\frac{2}{p}}\gamma^{\frac{2}{p}-2}\big)(\frac{\eta}{\gamma})^{\frac{\gamma}{p}} \\
&= 8\big(\eta^{-\frac{1}{p}}\gamma^{\frac{1}{p}-2} + \|\nabla \log f\|_{C^0}^2 \eta^{-\frac{2}{p}}\gamma^{\frac{2}{p}-2}\big)(\frac{\eta}{\gamma})^{2-q}.
\end{aligned} \tag{2.33}
$$

By (2.13) and (2.28), direct computation yields

$$\eta^{-\frac{1}{p}}\gamma^{\frac{1}{p}-2} + \|\nabla \log f\|_{C^0}^2 \eta^{-\frac{2}{p}}\gamma^{\frac{2}{p}-2} \le 4ek + 2\pi^{-2}e^{-2}k^4.$$

We obtain that

$$A_\gamma \le (4ek + 2\pi^{-2}e^{-2}k^4)(\frac{\eta}{\gamma})^{\frac{\gamma}{p}}, \tag{2.34}$$

where γ, η as in (2.13) and (2.28). □

Remark 2.1. *Constant $A_{n,k,p}$ in Proposition 2.2 can be computed explicitly. We observe that if u is even, (2.22) and (2.12) in Proposition 2.2 can be improved respectively as*

$$M_u^{\gamma-2}\frac{|\nabla u(x)|^2}{(u(x)-m_u)^\gamma} \le (\frac{2n}{\gamma})^{\frac{\gamma}{p}} e^{\frac{\gamma\pi}{2p}\|\nabla \log f\|_{C^0}}, \ \forall 0<\gamma<1, \quad \forall x \in \mathbb{S}^n. \tag{2.35}$$

and

$$M_u^{\gamma-2}\frac{|\nabla u|^2}{|u-m_u|^\gamma} \le A_{n,k,p} e^{\frac{\gamma\pi}{2(k-1+p)}\|\nabla \log f\|_{C^0}}. \tag{2.36}$$

This is due to the fact that one may choose maximum and minimum points of f such that the distance is at most $\frac{\pi}{2}$ in this case.

Remark 2.2. *It is of interest to obtain some form of weighted gradient estimate for equation (1.2) in the case $p = 0$.*

3. Non-collapsing estimate

In general, there is no positive lower bound for convex solutions of (1.2) when $p < k + 1$ [8]. We may obtain lower bound for *even convex* solutions of (1.2) in the case of $0 < p < 1$.

For convex body $\Omega \subset \mathbb{R}^{n+1}$, denote $\rho_-(\Omega)$ and $\rho_+(\Omega)$ to be the inner radius and outer radius of Ω respectively.

Lemma 3.1. *If u is a positive convex function on $\mathbb{S}^n$ satisfying condition*

$$\frac{|\nabla u(x)|^2}{(u(x)-m_u)^\gamma} \le AM_u^{2-\gamma}, \ \forall x \in \mathbb{S}^n, \tag{3.1}$$

for some $\gamma > 0$, $A > 0$. Let Ω_u be the convex body with support function u, and suppose there is an ellipsoid E centred at the origin such that

$$E \subset \Omega_u \subset \beta E. \tag{3.2}$$

Then the following non-collapsing estimate holds,

$$\frac{\rho_+(\Omega_u)}{\rho_-(\Omega_u)} \le \beta^{\frac{2}{\gamma}+1} A^{\frac{1}{\gamma}} 2^{\frac{4}{\gamma(2-\gamma)}}. \tag{3.3}$$

Proof. Write E

$$\frac{x_1^2}{a_1^2} + \cdots + \frac{x_{n+1}^2}{a_{n+1}^2} \le 1$$

with longest axis a_1, and the shortest axis a_{n+1}. We have

$$a_1 \le M_u \le \beta a_1, \quad a_{n+1} \le m_u \le \beta a_{n+1}.$$

Recall that

$$u_E(x) = \sqrt{a_1^2x_1^2 + a_2^2x_2^2 + \cdots + a_{n+1}^2x_{n+1}^2}, \qquad x \in \mathbb{S}^n$$

By (3.2), support functions of Ω and E are equivalent.

$$u_E(x) \le u(x) \le (n+1)u_E(x), \ \forall x \in \mathbb{S}^n.$$

Restrict the support function u_E, u to the slice $S := \{x \in \mathbb{S}^n | x = (x_1, 0, \ldots, 0, x_{n+1})\}$. Set

$$v(s) := u_E(s, 0, \ldots, 0, \sqrt{1-s^2}) = \sqrt{a_1^2 s^2 + a_{n+1}^2(1-s^2)} = \sqrt{a_{n+1}^2 + (a_1^2 - a_{n+1}^2)s^2}.$$

We have

$$ta_1^{\frac{\gamma}{2}} a_{n+1}^{\frac{2-\gamma}{2}} \le v(t(\frac{a_{n+1}}{a_1})^{\frac{2-\gamma}{2}}), \ \forall t \in [0,1].$$

On the other hand, set $q(s) = (u(s,0,\dots,0,\sqrt{1-s^2}) - m_u)^{\frac{2-\gamma}{2}}$. By the weighted gradient estimate (3.1),

$$|\frac{d}{ds}q(s)| \le A^{\frac{1}{2}} M_u^{1-\frac{\gamma}{2}} \le A^{\frac{1}{2}} \beta^{1-\frac{\gamma}{2}} a_1^{1-\frac{\gamma}{2}}.$$

This implies, $\forall 0 < t \le 1$,

$$q(t(\frac{a_{n+1}}{a_1})^{\frac{2-\gamma}{2}}) \le tA^{\frac{1}{2}}\beta^{1-\frac{\gamma}{2}}(\frac{a_{n+1}}{a_1})^{\frac{2-\gamma}{2}} a_1^{1-\frac{\gamma}{2}} + q(0) = t\beta^{1-\frac{\gamma}{2}} A^{\frac{1}{2}} a_{n+1}^{\frac{2-\gamma}{2}} + q(0).$$

As $q(0) \le \beta^{\frac{2-\gamma}{2}} a_{n+1}^{\frac{2-\gamma}{2}}$,

$$q(t(\frac{a_{n+1}}{a_1})^{\frac{2-\gamma}{2}}) \le (t\beta^{1-\frac{\gamma}{2}} A^{\frac{1}{2}} + \beta^{\frac{2-\gamma}{2}}) a_{n+1}^{\frac{2-\gamma}{2}}.$$

Thus,

$$u((\frac{a_{n+1}}{a_1})^{\frac{2-\gamma}{2}}, 0, \dots, 0, 1 - (\frac{a_{n+1}}{a_1})^{2-\gamma}) \le \beta^{1-\frac{\gamma}{2}}(tA^{\frac{1}{2}} + 1)^{\frac{2}{2-\gamma}} a_{n+1}.$$

Since $u(x) \ge u_E(x)$, we obtain

$$ta_1^{\frac{\gamma}{2}} a_{n+1}^{\frac{2-\gamma}{2}} \le \beta(tA^{\frac{1}{2}} + 1)^{\frac{2}{2-\gamma}} a_{n+1}.$$

This yields

$$\frac{a_1}{a_{n+1}} \le \big(\frac{\beta}{t}(tA^{\frac{1}{2}} + 1)^{\frac{2}{2-\gamma}}\big)^{\frac{2}{\gamma}}.$$

Choose $t = A^{-\frac{1}{2}}$,

$$\frac{a_1}{a_{n+1}} \le \beta^{\frac{2}{\gamma}} A^{\frac{1}{\gamma}} 2^{\frac{4}{\gamma(2-\gamma)}}. \tag{3.4}$$

□

Corollary 3.1. *If u is a positive, even, convex solution to (1.2) for $0 < p < k+1$. Then*

$$\frac{M_u}{m_u} \le \big(A_{n,k,p} e^{\frac{\gamma\pi}{2(k-1+p)}\|\nabla \log f\|_{C^0}}\big)^{\frac{1}{\gamma}} (n+1)^{\frac{1}{\gamma}+\frac{1}{2}} 2^{\frac{4}{\gamma(2-\gamma)}}, \tag{3.5}$$

where γ and $A_{n,k,p}$ as in Proposition 2.2. As a consequence,

$$\frac{|\nabla u(x)|^2}{u^2(x)} \le \big(A_{n,k,p} e^{\frac{\gamma\pi}{2(k-1+p)}\|\nabla \log f\|_{C^0}}\big)^{\frac{2-\gamma}{\gamma}+1} (n+1)^{\frac{4-\gamma^2}{2\gamma}} 2^{\frac{4}{\gamma}}. \tag{3.6}$$

In the case $k = 1$,

$$\frac{|\nabla u(x)|^2}{u^2(x)} \le 8(n+1)^{\frac{3}{2}} (2n)^{\frac{2}{p}} e^{\frac{\pi}{p}\|\nabla \log f\|_{C^0}}. \tag{3.7}$$

Moreover, there exist positive constant C_1, C_2 depending only on $n, k, p, \|\log f\|_{C^1}$, such that

$$C_1 \le u(x) \le C_2 > 0, \ \forall x \in \mathbb{S}^n; \quad \|u\|_{C^1(\mathbb{S}^n)} \le C.$$

Proof. Since Ω_u is even, we may pick $\beta = \sqrt{n+1}$ in (3.2). We let $A = A_{n,k,p} e^{\frac{\gamma\pi}{2(k-1+p)}\|\nabla \log f\|_{C^0}}$ as in (2.36). (3.5) follows Lemma 3.1. By (3.5),

$$\begin{aligned}\frac{|\nabla u(x)|^2}{u^2(x)} &= \frac{|\nabla u(x)|^2}{u^\gamma(x)} M_u^{-2+\gamma}(\frac{M_u}{u})^{2-\gamma}\\ &\leq \frac{|\nabla u(x)|^2}{(u-m_u)^\gamma} M_u^{-2+\gamma}(\frac{M_u}{m_u})^{2-\gamma}\\ &\leq \left(A_{n,k,p} e^{\frac{\gamma\pi}{2(k-1+p)}\|\nabla \log f\|_{C^0}}\right)^{\frac{2-\gamma}{\gamma}+1}(n+1)^{\frac{4-\gamma^2}{2\gamma}} 2^{\frac{4}{\gamma}}.\end{aligned}$$

Inequality (3.7) follows from (2.35). By equation (1.2), m_u is bounded from above and M_u is bounded from below. Therefore, u is bounded from below and above by (3.5). ☐

Lemma 3.1 yields a direct estimate of inner radius of the classical Christoffel-Minkowski problem: convex solutions to equation (1.1). When $k = n$, such estimate was proved in [2], it also follows from John's lemma. For $k < n$, we are not aware any such estimate in the literature.

Lemma 3.2. *Suppose u is convex solution to* (1.1). *Let Ω be the convex body determined by u as the support function, let $\rho_-(\Omega)$ be the inner radius of Ω. Then there exist positive constants C_1, C_2 depending only on n, k and $\|\log f\|_{C^1}$, such that*

$$C_2 \geq \rho_+(\Omega) \geq \rho_-(\Omega) \geq C_1.$$

Proof. As we may shift the origin to the center of the ellipsoid E in (3.2) with $\beta = n+1$. Lemma follows Lemma 3.1, since m_u is bounded from above and M_u is bounded from below by (1.1). ☐

With the upper and lower bounds of u for solutions of (1.2), the maximum principle (e.g., [8]) yields C^2 estimate. Higher regularity a priori estimates follows the standard elliptic theory.

Proposition 3.1. *Let u be a positive, even convex solution to* (1.2). *For any $l \in \mathbb{Z}^+$ and $0 < \alpha < 1$, there exists some positive constant C, depending on n, k, p, l, α and $\|\log f\|_{C^l}$, such that*

$$\|u\|_{C^{l+1,\alpha}(\mathbb{S}^n)} \leq C. \tag{3.8}$$

4. The issue of convexity

For L^p Christoffel-Minkowski problem, we want to find solution u of (1.2) which is convex, i.e., $W > 0$. The sufficient condition introduced in [7] for convexity of solution u to equation (1.1) is

$$((f^{\frac{-1}{k}})_{ij}(x) + f^{\frac{-1}{k}}(x)\delta_{ij}) \geq 0, \ \forall x \in \mathbb{S}^n. \tag{4.1}$$

Corresponding condition for (1.2) for $p > 1$ is

$$((\tilde{f}^{\frac{-1}{k}})_{ij}(x) + \tilde{f}^{\frac{-1}{k}}(x)\delta_{ij}) \geq 0, \ \forall x \in \mathbb{S}^n, \tag{4.2}$$

where $\tilde{f} = u^{p-1}f$. Write $\tilde{h} = \log \tilde{f} = (p-1)\log u + \log f$, (4.2) is equivalent to

$$\frac{1}{k}(\tilde{h}')^2 + k - \tilde{h}''(x) \geq 0, \ \forall x \in \mathbb{S}^n, \tag{4.3}$$

where derivatives are along any geodesic passing through x. Denote $\phi=\log f$, (4.3) is equivalent to

$$\frac{1}{k}(\phi')^2+k-\phi''+(p-1)\big\{-\frac{u''}{u}+(1+\frac{p-1}{k})(\frac{u'}{u})^2+\frac{2}{k}\frac{u'}{u}\phi'\big\}\geq 0. \tag{4.4}$$

In the case $p\geq 1$, it was observed in [9] that (4.2) would be valid if f satisfies

$$((f^{\frac{-1}{k+p-1}})_{ij}(x)+f^{\frac{-1}{k+p-1}}(x)\delta_{ij})>0,\ \forall x\in\mathbb{S}^n. \tag{4.5}$$

This relies on the fact that the coefficient $p-1+\frac{(p-1)^2}{k}$ in front of term $(\frac{u'}{u})^2$ in (4.4) is nonnegative when $p\geq 1$. In the case $0<p<1$, $p-1+\frac{(p-1)^2}{k}<0$. If

$$k-1+p-\phi''+(p-1)(\frac{u'}{u})^2\geq 0, \tag{4.6}$$

then (4.4) holds, as W is assumed semi-positive definite.

The main problem is to control $(p-1)(\frac{u'}{u})^2$ in (4.6) when $p<1$. When $0\leq 1-p$ is small, one may impose a condition that f is a positive C^2 even function on $\mathbb{S}^n$ satisfying

$$k-1+p-\phi''+(p-1)\big(A_{n,k,p}e^{\frac{\gamma\pi}{2(k-1+p)}\|\nabla\phi\|_{C^0}}\big)^{\frac{2-\gamma}{\gamma}+1}(n+1)^{\frac{4-\gamma^2}{2\gamma}}2^{\frac{4}{\gamma}}\geq 0. \tag{4.7}$$

By Corollary 3.1, Condition (4.7) implies Condition (4.6). The Constant Rank Theorem in [7] implies that there is a convex even solution $u\in C^{3,\alpha}(\mathbb{S}^n),\ \forall 0<\alpha<1$ of (1.2).

In the case $k=1$, one may use (3.7) to deduce a simpler condition for convex even solutions to L^p Christoffel problem:

$$p-\phi''+8(p-1)(n+1)^{\frac{3}{2}}(2n)^{\frac{2}{p}}e^{\frac{\pi}{p}\|\nabla\log f\|_{C^0}}\geq 0, \tag{4.8}$$

Conditions (4.7) and (4.8) are not satisfactory. It only makes some sense when $1-p$ is small. It is an open problem to find a clean pointwise condition on f for existence of convexity solutions to equation (1.2), $0<p<1$.

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Department of Mathematics and Statistics, McGill University, Montreal, H3A 0B9, Canada
Email address: pengfei.guan@mcgill.ca