The Weyl and Minkowski Problems, Revisited

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The Weyl and Minkowski problems are two inspiring sources for the theory of Monge-Ampère equation and fully nonlinear equations in general. The seminal works of Nirenberg [**23**], Pogorelov [**25, 28**] and Cheng-Yau [**6**] played important role in the development of geometric fully nonlinear PDEs. Though these two problems were solved longtime ago, there are many important geometric problems of current interest can be traced back to them. We discuss some recent work which are closely related to these two classical problems:

a. The intermediate Christoffel-Minkowski problem;
b. Isometric embedding of surfaces to 3-dimensional Riemannian manifolds.

The emphasis here is on issues of *regularity and convexity estimates* for solutions of non-linear PDEs.

1. The Minkowski problem

The classical Minkowski problem was considered by Minkowski in [**22**]. Suppose M is a closed strongly convex hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, the Gauss map $\nu : M \to \mathbb{S}^n$ is a diffeomorphism, where at any point $p \in M$, $\nu(p)$ is the unit outer normal at p. Let us denote $\kappa = (\kappa_1, \cdots, \kappa_n)$ to be the principal curvatures and $K = \kappa_1 \cdots \kappa_n$ the Gauss curvature of M respectively.

The Minkowski problem: *given a positive function φ on $\mathbb{S}^n$, find a closed strongly convex hypersurface whose Gauss curvature is $K = \frac{1}{\varphi}$ as a function on its outer normals.*

By the Divergence Theorem, φ has to satisfy equation

$$\int_{\mathbb{S}^n} x_i \varphi = \int_{\mathbb{S}^n} \frac{x_i}{K(x)} = \int_M \nu \cdot \vec{E}_i = 0, i = 1, \dots, n+1, \tag{1.1}$$

where x_i are the coordinate functions and $\vec{E}_i$ is the standard ith coordinate vector of $\mathbb{S}^n$.

The problem has been completely solved by Nirenberg [**23**] and Pogorelov [**25**] when $n = 2$, and by Cheng-Yau [**6**] and Pogorelov [**28**] for general dimensions.

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THEOREM 1. *Suppose $\varphi \in C^2(\mathbb{S}^n), \varphi(x) > 0, \quad \forall\, x \in \mathbb{S}^n$, and satisfies equation (1.1), then there is a $C^{3,\alpha}(\forall\ 0 < \alpha < 1)$ strongly convex hypersurface M in $\mathbb{R}^{n+1}$, such that $K(\nu_M^{-1}(x)) = \frac{1}{\varphi(x)}\ \forall\, x \in \mathbb{S}^n$. M is unique up to translations.*

Key to the proof of Theorem 1 is the a priori estimate of solutions to the problem. The problem can be deduced to a Monge-Ampère equation on $\mathbb{S}^n$.

Let's start with basic relationship between the convex body in $\mathbb{R}^{n+1}$ and its support function. A C^2 closed hypersurface M in $\mathbb{R}^{n+1}$ is called strongly convex if its Gauss curvature is positive everywhere. The Hadamard Theorem indicates that M is a boundary of a bounded convex domain. In turn, M can be parametrized by its inverse Gauss map over $\mathbb{S}^n$ with

$$y(x) = \nu_M^{-1}(x).$$

The support function of M is defined as

$$u(x) = \sup_{z \in M} x \cdot z = x \cdot y(x),\ \forall x \in \mathbb{S}^n.$$

Extending u as a homogeneous function of degree one in $\mathbb{R}^{n+1} \setminus \{0\}$, u is then a convex function in $\mathbb{R}^{n+1}$. Since $\frac{\partial y}{\partial x_j}$ is tangent to M for all j, and $x = \nu_M(y)$ is normal to M, we have $x \cdot \frac{\partial y}{\partial x_j} = 0$ for all j. It follows that

$$\nu_M^{-1}(x) = y(x) = \nabla_{\mathbb{R}^{n+1}} u(x). \tag{1.2}$$

Therefore, M can be recovered completely from u by above equation.

Let $e_{n+1} = x$ be the position vector on $\mathbb{S}^n$, let $e_1, \cdots, e_n$ is an orthonormal frame on S^n so that $e_1, \cdots, e_{n+1}$ is a positive oriented orthonormal frame in $\mathbb{R}^{n+1}$. Let ω^i and ω^i_j be the corresponding dual 1-forms and the connection forms respectively. We have

$$de_j = -\sum_{i=1}^n \omega^i_j e_i, \quad \forall j = 1, 2, \cdots, n, \quad and \quad de_{n+1} = \sum_{i=1}^n \omega^i e_i.$$

If u is a support function of M, by (1.2) the position vector of M as a function on $\mathbb{S}^n$ is

$$y(x) = \sum_{i=1}^n u_i e_i + u e_{n+1}.$$

One calculates that

$$dy = \sum_{i,j} (u_{ij} + u\delta_{ij}) e_i \otimes \omega_j \tag{1.3}$$

The identity (1.3) indicates that the differential dy maps $T_x(\mathbb{S}^n)$ to itself and it is self-adjoint. We have

$$dy = (d\nu_M)^{-1}, \tag{1.4}$$

so that the reverse Weingarten map at x coincides with the inverse of the Weingarten map at y. Since the eigenvaules of the Weingarten map are the principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_n)$ of M at y, the eigenvalues of reverse Weingarten map at $x = \nu_M(y)$ are exactly the principal radii at y.

Conversely, if $u(x)$ is a C^2 function on $\mathbb{S}^n$ with $(u_{ij}+u\delta_{ij})>0$, it is a support function of M defined as

$$M=\{\nabla_{\mathbb{R}^{n+1}}u(x)|x\in\mathbb{R}^{n+1}\setminus\{0\}\}=\{\sum_{i=1}^{n}u_i(x)e_i(x)+u(x)e_{n+1}(x)|x\in\mathbb{S}^n\}. \tag{1.5}$$

Equation (1.2) implies that u is C^2 if M is C^2 and its Gauss curvature is positive.

In summary,

PROPOSITION 1. *A strongly convex hypersurface M in $\mathbb{R}^{n+1}$ is C^2 if and only if its support function u is in $C^2(\mathbb{S}^n)$ with $(u_{ij}+u\delta_{ij})>0$. The eigenvalues of $(u_{ij}+u\delta_{ij})$ are the principal radii of M (parametrized by the inverse Gauss map over $\mathbb{S}^n$).*

In particular, the Gauss curvature K of M satisfies equation

$$\det(u_{ij}+u\delta_{ij})=\frac{1}{K}, \quad \text{on} \quad \mathbb{S}^n. \tag{1.6}$$

Furthermore, any function $u\in C^2(\mathbb{S}^n)$ with $(u_{ij}+u\delta_{ij})>0$ is a support function of a C^2 strongly convex hypersurface M in $\mathbb{R}^{n+1}$.

The proof Theorem 1 is method of continuity. Here we illustrate on how to obtain C^2 estimates for the solutions, setting a stage to dealing with the Christoffel-Minkowski problem in the next section.

For a solution u of equation (1.6), $u+\sum_{i=1}^{n+1}a_ix_i$ is also a solution. By proper choice of $\{a_i\}_{i=1}^n$, we may assume that u satisfies the following orthogonality condition:

$$\int_{\mathbb{S}^n}x_iu\,dx=0, \qquad \forall i=1,2,...,n+1. \tag{1.7}$$

If u is a support function of a closed hypersurface M which bounds a convex body Ω, condition (1.7) implies that the Steiner point of Ω coincides with the origin. First is the upper bound of the extrinsic diameter of M [**6**].

LEMMA 2. *Let $M\in C^2$, M be a closed convex hypersurface in $\mathbb{R}^{n+1}$, and let $\varphi=\frac{1}{K}$. If L is the extrinsic diameter of M, then*

$$L \;\le\; c_n\left(\int_{\mathbb{S}^n}\varphi\right)^{\frac{n+1}{n}}\left(\inf_{y\in\mathbb{S}^n}\int_{\mathbb{S}^n}\max(0,\langle y,x\rangle)\varphi(x)\right)^{-1},$$

where c_n is a positive constant depending only on n. In particular, if u is a support function of M satisfying (1.6) and (1.7), then

$$0\le\min u\le\max u\le c_{n,k}\left(\int_{\mathbb{S}^n}\varphi\right)^{\frac{n+1}{n}}\left(\inf_{y\in\mathbb{S}^n}\int_{\mathbb{S}^n}\max(0,\langle y,x\rangle)\varphi(x)\right)^{-1}.$$

PROOF. Let $p,q\in M$ such that the line segment joining p and q has length L. We may assume 0 is in the middle of the line segment. Let $\vec{y}$ be a unit vector in the direction of this line. Let v be the support function and $W = \{v_{ij} + v\delta_{ij}\}$. We have $\sigma_n(W) = \varphi$. Now, for $x \in \mathbb{S}^n$, we get

$$u(x) = \sup_{Z \in M} \langle Z, x\rangle \geq \frac{1}{2} L \max(0, \langle y, x\rangle).$$

If we multiply by φ and integrate over $\mathbb{S}^n$, we get

$$L \leq 2\left(\int_{\mathbb{S}^n} u\varphi\right)\left(\int_{\mathbb{S}^n} \max(0, \langle y, x\rangle)\varphi\right)^{-1}.$$

As $\int_{\mathbb{S}^n} u\sigma_n(W) = Vol(\Omega)$ and $\int_{\mathbb{S}^n} \sigma_n(W)$ is the surface area of $M = \partial\Omega$, by the isoperimetric inequality,

$$(\int_{\mathbb{S}^n} u\sigma_n(W))^{\frac{1}{n+1}} \leq C_n(\int_{\mathbb{S}^n} \sigma_n(W))^{\frac{1}{n}}.$$

In turn, we get

$$L \leq c_n \left(\int_{\mathbb{S}^n} \varphi\right)^{\frac{n+1}{n}} \left(\inf_{y \in \mathbb{S}^n} \int_{\mathbb{S}^n} \max(0, (y, x))\varphi\right)^{-1}.$$

If u satisfies (1.7), the Steiner point of M is the origin. The last inequality is a consequence of the above inequality. □

We precede to obtain C^2 estimate, using the **fact** that $\det^{\frac{1}{n}}(W)$ *is concave.*

PROPOSITION 2. *There is a constant $C > 0$ depending only on n, $\|K\|_{C^2(\mathbb{S}^n)}$ and $\min_{\mathbb{S}^n} K$, such that if u satisfies (1.7) and u is a solution of (1.6), then $\|u\|_{C^2(\mathbb{S}^n)} \leq C$. There is an explicit bound for the function $H := trace(u_{ij} + \delta_{ij}u) = \triangle u + nu$,*

$$\min_{x \in \mathbb{S}^n}(n\tilde{\varphi}(x)) \leq \max_{x \in \mathbb{S}^n} H(x) \leq \max_{x \in \mathbb{S}^n}(n\tilde{\varphi}(x) - \triangle\tilde{\varphi}(x)), \tag{1.8}$$

where $\tilde{\varphi} := \varphi^{\frac{1}{n}}$.

PROOF. Since $(u_{ij}+\delta_{ij}u)$ is positive definite, it is controlled by its trace by H. The first inequality follows from the Newton-MacLaurin inequality. Assume the maximum value of H is attained at a point $x_0 \in \mathbb{S}^n$. We choose an orthonormal local frame $e_1, e_2, ..., e_n$ near x_0 such that $u_{ij}(x_0)$ is diagonal. If $W = (u_{ij} + \delta_{ij}u)$, we define $G(W) := \sigma_n^{\frac{1}{n}}(W)$. Then equation (1.6) becomes

$$G(W) = \tilde{\varphi}. \tag{1.9}$$

By the commutator identity $H_{ii} = \triangle W_{ii} - nW_{ii} + H$ and the assumption that the matrix $W > 0$, so $(G^{ij}) = (\frac{\partial\sigma_n}{\partial W_{ij}})$ is positive definite. Since $(H_{ij}) \leq 0$, and (G^{ij}) is diagonal, by the above commutator identity, it follows that at x_0,

$$0 \geq G^{ij}H_{ij} = G^{ii}(\triangle W_{ii}) - nG^{ii}W_{ii} + H\sum_i^n G^{ii}. \tag{1.10}$$

As G is homogeneous of degree one, we have

$$G^{ii}W_{ii} = \tilde{\varphi}. \tag{1.11}$$

Next we apply the Laplace operator to equation (2.4) to obtain

$$G^{ij}W_{ijk} = \nabla_k\tilde{\varphi}, \quad G^{ij,rs}W_{ijk}W_{rsk} + G^{ij}\triangle W_{ij} = \triangle\tilde{\varphi}.$$

By fact that G is concave, at x_o

$$G^{ii}\triangle(W_{ii}) \geq \triangle\tilde{\varphi}. \tag{1.12}$$

Combining (1.11), (1.12) and (1.10),

$$0 \geq \triangle\tilde{\varphi} - n\tilde{\varphi} + H\sum_{i=1}^{n} G^{ii}. \tag{1.13}$$

As W is diagonal at the point, we may write $W = (W_{11}, ..., W_{nn})$ as a vector in $\mathbb{R}^n$. A simple calculation yields

$$\sum_{i=1}^{n} G^{ii} = \frac{\sigma_{n-1}(W)}{n\sigma_n^{1-\frac{1}{n}}(W)} \geq 1,$$

the last inequality follows from the Newton-MacLaurin inequality.

By (1.13), we have $H \leq n\tilde{\varphi} - \triangle\tilde{\varphi}$. □

This ends the a priori estimates for solutions of the Minkowski problem. It will serve as an introduction to the intermediate Christoffel-Minkowski problem in the next section.

2. The Christofell-Minkowski problem, regularity and convexity

The Minkowski problem was originated by Minkowski [**22**] related to the notions of area measures and curvature measures in convex geometry. The problem of prescribing area measures is called the Christoffel-Minkowski problem [**29**], we refer [**9**] (see also [**8**]) for the treatment of the problem of prescribing curvature measures.

For a convex body $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary, the n-th area measure of the convex body is $\frac{1}{K}dV_{S^n}$ where K is the Gauss curvature of $\partial\Omega$. For each $1 \leq k \leq n$, the k-th area measure of the body is $\sigma_k(W)dV_{S^n}$ (e.g., [**29**]), where σ_k the k-th elementary symmetric function. The problem of prescribing k-th area measure can be deduced to solve the following Hessian type equation

$$\sigma_k(u_{ij}(x) + u\delta_{ij}(x)) = \varphi(x), \forall x \in \mathbb{S}^n, \tag{2.1}$$

and

$$W(x) = (u_{ij}(x) + \delta_{ij}u(x)) > 0, \forall x \in \mathbb{S}^n. \tag{2.2}$$

We recall definition of admissible solutions [**5**].

Definition 3. *For $1 \leq k \leq n$, let Γ_k is a convex cone in $\mathbb{R}^n$ determined by*

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \quad \sigma_1(\lambda) > 0, ..., \sigma_k(\lambda) > 0\}.$$

$u \in C^2(\mathbb{S}^n)$ is called k-convex, if $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\} \in \Gamma_k$ for each $x \in S^n$. u is convex on $\mathbb{S}^n$ if W is n-convex. Furthermore, u is called an admissible solution of (2.1), if u is k-convex and satisfies (2.1).

The first step is to obtain regularity estimates for admissible solutions of equation (2.1).

2.1. A priori estimates. In the case of $k = 1$, equation (2.1) is a linear elliptic equation on the sphere. That C^2 a priori estimates hold for a solution u satisfying (1.7) in this case follows from standard linear elliptic theory. Therefore, we will restrict ourselves to the case $k \geq 2$. The proof of Proposition 2 can be adapted to get similar C^2 estimate as in (1.8) for solutions of equation (2.1). This type of estimate can be used to obtain C^0 estimate too, by a compactness argument.

Note that if $W \in \Gamma_k$, the $(\frac{\partial \sigma_k}{\partial W_{ij}})$ is positive definite and $\sigma_k^{\frac{1}{k}}(W)$ *is concave.*

PROPOSITION 3. *There is a constant $C > 0$ depending only on n, k, $\|\varphi\|_{C^2(\mathbb{S}^n)}$ and $\|\frac{1}{\varphi}\|_{C^0(\mathbb{S}^n)}$, such that if u satisfies (1.7) and u is an admissible solution of (2.1), then $\|u\|_{C^2(\mathbb{S}^n)} \leq C$. There is an explicit bound for the function $H := trace(u_{ij} + \delta_{ij}u) = \triangle u + nu$,*

$$\min_{x\in\mathbb{S}^n}(n\tilde{\varphi}(x)) \leq \max_{x\in\mathbb{S}^n} H(x) \leq \max_{x\in\mathbb{S}^n}(n\tilde{\varphi}(x) - \triangle\tilde{\varphi}(x)), \tag{2.3}$$

where $\tilde{\varphi} := (\frac{\varphi}{C_n^k})^{\frac{1}{k}}$, $C_n^k = \frac{n!}{k!(n-k)!}$.

PROOF. Since the entries $|u_{ij}+\delta_{ij}u|$ are controlled by eigenvalues $\{\lambda_i\}_{i=1}^n$ of $(u_{ij}+\delta_{ij}u)$. The eigenvalues are controlled by H since $(u_{ij} + \delta_{ij}u) \in \Gamma_k, k \geq 2$. Indeed,

$$\sum_{i=1}^n \lambda_i^2 = H^2 - 2\sigma_2(u_{ij} + \delta_{ij}u)) \leq H^2,$$

as $\sigma_2(u_{ij} + \delta_{ij}u)) > 0$ when $(u_{ij} + \delta_{ij}u) \in \Gamma_k, k \geq 2$.

The first inequality in (2.3) follows from the Newton-MacLaurin inequality. Assume the maximum value of H is attained at a point $x_0 \in \mathbb{S}^n$. We choose an orthonormal local frame $e_1, e_2, ..., e_n$ near x_0 such that $u_{ij}(x_0)$ is diagonal. If $W = (u_{ij} + \delta_{ij}u)$, we define $G(W) := (\frac{\sigma_k}{C_n^k})^{\frac{1}{k}}(W)$. Then equation (2.1) becomes

$$G(W) = \tilde{\varphi}. \tag{2.4}$$

For the standard metric on $\mathbb{S}^n$, one may easily check the commutator identity $H_{ii} = \triangle W_{ii} - nW_{ii} + H$. By the assumption that the matrix $W \in \Gamma_k$, so (G^{ij}) is positive definite. Since $(H_{ij}) \leq 0$, and (G^{ij}) is diagonal, it follows that at x_0,

$$0 \geq G^{ij}H_{ij} = G^{ii}(\triangle W_{ii}) - nG^{ii}W_{ii} + H\sum_i^n G^{ii}. \tag{2.5}$$

Next we apply the Laplace operator to equation (2.4) to obtain

$$G^{ij}W_{ijk} = \nabla_k\tilde{\varphi}, \quad G^{ij,rs}W_{ijk}W_{rsk} + G^{ij}\triangle W_{ij} = \triangle\tilde{\varphi}.$$

By the concavity of G, at x_o we have

$$G^{ii}\triangle(W_{ii}) \geq \triangle\tilde{\varphi}. \tag{2.6}$$

As $G^{ii}W_{ii} = \tilde{\varphi}$, by (2.6) and (2.5),

$$0 \geq \triangle\tilde{\varphi} - n\tilde{\varphi} + H\sum_{i=1}^{n} G^{ii}. \tag{2.7}$$

As W is diagonal at the point, we may write $W = (W_{11}, ..., W_{nn})$ as a vector in $\mathbb{R}^n$. A simple calculation yields

$$G^{ii} = \frac{\sigma_k(W)^{\frac{1}{k}-1}}{(C_n^k)^{\frac{1}{k}}}\frac{\partial\sigma_k(W)}{\partial W_{ii}} = \frac{\sigma_k(W)^{\frac{1}{k}-1}}{(C_n^k)^{\frac{1}{k}}}\sigma_{k-1}(W|i),$$

where $(W|i)$ is the vector given by W with W_{ii} deleted. It follows from the Newton-MacLaurin inequality that

$$\sum_{i=1}^{n} G^{ii} = (n-k+1)\frac{\sigma_k(W)^{\frac{1}{k}-1}}{(C_n^k)^{\frac{1}{k}}}\sigma_{k-1}(W) \geq 1.$$

By (2.7), we have $H \leq n\tilde{\varphi} - \triangle\tilde{\varphi}$.

Finally, we claim u is bounded if it satisfying condition (1.7). Suppose this is not true, there is a sequence u_l satisfying the equation with $\|u_l\|_{L^\infty} \to \infty$. We rescale, consider $\tilde{u}_l = \frac{u_l}{\|u_l\|_{L^\infty}}$, it satisfies (1.7) and (2.3) with $\|\tilde{\varphi}_l\|_{C^2} \to 0$. By compactness, there is a subsequence convergent to $\tilde{u}$ in $C^{1,\alpha}$ satisfying (1.7) and $\Delta\tilde{u} + n\tilde{u} = 0$, with $\|\tilde{u}\|_{L^\infty} = 1$. Contradiction. □

Once C^2 estimate is in hand, higher regularity estimates for admissible solutions of equation (2.1) by the Evens-Krylov Theorem. The existence of admissible solutions to equation (2.1) provided that φ satisfies (1.1) can be established [**14**].

The main question is when a solution of (2.1) is geometric. That is, when an admissible solution u satisfies the convexity condition (2.2). When $k < n$, an admissible solution of (2.1) may not satisfy (2.2) in general. The following example is essentially due to Alexandrov. Let

$$u(x) = 1 - \frac{x_{n+1}^2}{2}, \tag{2.8}$$

where x_{n+1} is the $(n+1)$-th coordinate function. It is straightforward to check that this function satisfies

$$W(x) = (u_{ij}(x) + \delta_{ij}u(x)) \geq 0, \forall x \in \mathbb{S}^n,$$

The spherical Hessian W is positive everywhere except on the great circle $x_{n+1} = 0$, the rank is $n-1$ there. For $1 \leq k < n$, there is $\delta_k > 0$, such that $u_\delta = u - \delta_k$ is an admissible solution to equation (2.1) for some positive analytic function φ, but one of eigenvalues of W is negative on the great circle.

2.2. Convexity. The following theorem in [**13**] provides a sufficient condition for convexity of solutions.

THEOREM 4. *Suppose $u \in C^4(\mathbb{S}^n)$ is a solution equation (2.1) with $W_u \geq 0$. Suppose φ satisfies*

$$(\varphi^{-\frac{1}{k}})_{ij}(x) + \delta_{ij}\varphi^{-\frac{1}{k}} \geq 0, \forall x \in \mathbb{S}^n, \tag{2.9}$$

then $W_u > 0$.

Set

$$\sigma_m^{\alpha\beta} = \frac{\partial \sigma_k(W)}{\partial W_{\alpha\beta}}, \qquad \sigma_m^{ij,rs} = \frac{\partial \sigma_k(W)}{\partial W_{ij}\partial W_{rs}}, \forall m = 1, \cdots, n. \tag{2.10}$$

Theorem 4 can be deduced from the Minkowski identity and the following proposition [**13**]. It is call the constant rank theorem, going back to the early works of Caffarelli-Friedman [**3**] and Yau [**30**], see also [**13, 4, 1**].

PROPOSITION 5. *Suppose $u \in C^4(\mathbb{S}^n)$ is a solution of (2.1) and $W(x) \geq 0, \forall x \in \mathbb{S}^n$. Let l be the minimal rank of $W(x)$ on $\mathbb{S}^n$ which is attained at x_0 and set $\phi(x) = \sigma_{l+1}(W(x))$. If φ satisfies condition (2.9), then there is a neighborhood O of x_0 and there are constants C_1, C_2 depending only on $||u||_{C^3}$, $||\varphi||_{C^{1,1}}$, n, k and $\sigma_l(W(x_0))$, such that differential inequality holds*

$$\sum_{\alpha,\beta}^{n} \sigma_k^{\alpha\beta}(x)\phi_{\alpha\beta}(x) \leq C_1|\nabla\phi(x)| + C_2\phi(x), \forall x \in O. \tag{2.11}$$

Recall that $\varphi(x) = \sigma_k(W(x))$, and $\phi(x) = \sigma_{l+1}(W(x))$. Since W is positive semi-definite and u is k-convex, $(\sigma_k^{\alpha\beta})$ is positive definite and (σ_{l+1}^{ij}) is positive semi-definite. One first observes that there are at least l positive eigenvalues of W with a controlled lower bound in a neighborhood O of x_0, and other $(n-l)$ eigenvalues are sufficient small. Let B be that part of the index set so arranged such that the W_{ii} might be small (controlled by ϕ) for $i \in B$ (see the proof below for the precise definition). In view of this observation, W_{ii} is negligible for each $i \in B$. The basic idea in the proof of Proposition 5 is to to explore the relationship between $\sum_{\alpha,\beta}^n \sigma_k^{\alpha\beta}\phi_{\alpha\beta}$ and $\varphi^{\frac{k+1}{k}}\sigma_l(W)\sum_i\{(\varphi^{-\frac{1}{k}})_{ii} + \delta_{ii}\varphi^{-\frac{1}{k}}\}$. One of the key terms to be handled will be $\sum_{i,\alpha}\sigma_{l+1}^{ii}\sigma_k^{\alpha\alpha}W_{ii\alpha\alpha}$. With the help of some basic properties of elementary symmetric functions, it turns out that some algebraic cancellations will occur after commuting covariant derivatives and re-arranging the terms to fit the right algebraic formats! Almost all of the computations in the proof are algebraic and the inequality in Lemma 9 in Appendix will be used in a crucial way in the last step of the proof.

Proof the Proposition. For two functions defined in an open set $O \subset \mathbb{S}^n$, $y \in O$, we say that $h(y) \lesssim k(y)$ provided there exist positive constants c_1 and c_2 such that

$$(h - k)(y) \leq (c_1|\nabla\phi| + c_2\phi)(y). \tag{2.12}$$

We also write $h(y) \sim k(y)$ if $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$. Next, we write $h \lesssim k$ if the above inequality holds in O, with the constants c_1, and c_2 depending only on $||u||_{C^3}$,

$||\varphi||_{C^2}$, n and C_0 (independent of y and O). Finally, $h \sim k$ if $h \lesssim k$ and $k \lesssim h$. We shall show that

$$\sum_{\alpha,\beta=1}^{n} \sigma_k^{\alpha\beta}\phi_{\alpha\beta} \lesssim 0. \tag{2.13}$$

For any $z \in O$, let $\lambda_1 \geq \lambda_2 ... \geq \lambda_n$ be the eigenvalues of W at z. Since $\sigma_l(W) \geq C_0 > 0$ and $u \in C^3$, for any $z \in \mathbb{S}^n$, there is a positive constant $C > 0$ depending only on $||u||_{C^3}$, $||\varphi||_{C^2}$, n and C_0, such that $\lambda_1 \geq \lambda_2 ... \geq \lambda_l \geq C$. Let $G = \{1, 2, ..., l\}$ and $B = \{l+1, ..., n\}$ be the "good" and "bad" sets of indices respectively, and define $\sigma_k(W|i) = \sigma_k((W|i))$ where $(W|i)$ means that the matrix W excluding the i-column and i-row, and $(W|ij)$ means that the matrix W excluding the i, j columns and i, j rows. Let $\Lambda_G = (\lambda_1, ..., \lambda_l)$ be the "good" eigenvalues of W at z; for convenience in notation, we also write $G = \Lambda_G$ if there is no confusion. In the following, all calculations are at the point z using the relation "$\lesssim$", with the understanding that the constants in (2.12) are under control.

For each fixed $z \in O$ fixed, we choose a local orthonormal frame $e_1, ..., e_n$ so that W is diagonal at z, and $W_{ii} = \lambda_i, \forall i = 1, ..., n$. Now we compute ϕ and its first and second derivatives in the direction e_α.

We note that σ_{l+1}^{ij} is diagonal at the point since W is diagonal. As $\phi = \sigma_{l+1}(W)$ and $\phi_\alpha = \sum_{i,j} \sigma_{l+1}^{ij} W_{ij\alpha}$, we find that (as W is diagonal at z),

$$0 \sim \phi(z) \sim (\sum_{i\in B} W_{ii})\sigma_l(G) \sim \sum_{i\in B} W_{ii}, \quad (\text{so} \quad W_{ii} \sim 0, \quad i \in B), \tag{2.14}$$

This relation yields that, for $1 \leq m \leq l$,

$$\sigma_m(W) \sim \sigma_m(G), \quad \sigma_m(W|j) \sim \begin{cases} \sigma_m(G|j), & \text{if } j \in G; \\ \sigma_m(G), & \text{if } j \in B. \end{cases} \tag{2.15}$$

$$\sigma_m(W|ij) \sim \begin{cases} \sigma_m(G|ij), & \text{if } i,j \in G; \\ \sigma_m(G|j), & \text{if } i \in B, j \in G; \\ \sigma_m(G), & \text{if } i,j \in B, i \neq j. \end{cases}$$

Also,

$$0 \sim \phi_\alpha \sim \sigma_l(G) \sum_{i\in B} W_{ii\alpha} \sim \sum_{i\in B} W_{ii\alpha} \tag{2.16}$$

and

$$\sigma_{l+1}^{ij} \sim \begin{cases} \sigma_l(G), & \text{if } i = j \in B, \\ 0, & \text{otherwise.} \end{cases} \tag{2.17}$$

$$\sigma_{l+1}^{ij,rs} = \begin{cases} \sigma_{l-1}(W|ir), & \text{if } i = j, r = s, i \neq r; \\ -\sigma_{l-1}(W|ij), & \text{if } i \neq j, r = j, s = i; \\ 0, & \text{otherwise.} \end{cases} \tag{2.18}$$

Since $\phi_{\alpha\alpha} = \sum_{i,j}(\sigma_{l+1}^{ij,rs} W_{rs\alpha} W_{ij\alpha} + \sigma_{l+1}^{ij} W_{ij\alpha\alpha})$, it follows from (2.18) that for any $\alpha \in \{1, 2, ..., n\}$

$$\begin{aligned}
\phi_{\alpha\alpha} &= \sum_{i\neq j} \sigma_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} - \sum_{i\neq j} \sigma_{l-1}(W|ij) W_{ij\alpha}^2 + \sum_i \sigma_{l+1}^{ii} W_{ii\alpha\alpha} \\
&= \Big(\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}}\Big) \sigma_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} \\
&- \Big(\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}}\Big) \sigma_{l-1}(W|ij) W_{ij\alpha}^2 + \sum_i \sigma_{l+1}^{ii} W_{ii\alpha\alpha}.
\end{aligned} \tag{2.19}$$

From (2.16) and (2.15),

$$\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} \sim \Big(\sum_{j\in G} \sigma_{l-1}(G|j) W_{jj\alpha}\Big) \sum_{i\in B} W_{ii\alpha} \sim 0. \tag{2.20}$$

Since $0 \leq W_{mm} \in C^2$ for any unit vector field, by Lemma 12,

$$|\nabla W_{mm}(x)| \leq C\sqrt{W_{mm}(x)}.$$

This implies that

$$|\nabla W_{ij}(x)| \leq C(\sqrt{W_{ii}(x)} + \sqrt{W_{jj}(x)}).$$

By (2.16), $\forall i \in B$ fixed and $\forall \alpha$, therefore,

$$\sum_{i,j\in B} \sigma_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} \sim 0. \tag{2.21}$$

and

$$\sum_{j\in G, i\in B} \sigma_{l-1}(W|ij) W_{ij\alpha}^2 \sim \sum_{i\in B, j\in G} \sigma_{l-1}(G|j) W_{ij\alpha}^2. \tag{2.22}$$

Inserting (2.20)-(2.22) into (2.19), by (2.15) we obtain

$$\phi_{\alpha\alpha} \sim \sum_i \sigma_{l+1}^{ii} W_{ii\alpha\alpha} - 2\sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j) W_{ij\alpha}^2. \tag{2.23}$$

Thus,

$$\begin{aligned}
&\sum_{\alpha,\beta} \sigma_k^{\alpha\beta} \phi_{\alpha\beta} = \sum_{\alpha=1}^n \sigma_k^{\alpha\alpha} \phi_{\alpha\alpha} \sim \sum_{\alpha=1}^n \sum_i \sigma_{l+1}^{ii} \sigma_k^{\alpha\alpha} W_{ii\alpha\alpha} \\
&-2 \sum_{\alpha\in G}^n \sum_{\substack{i\in B\\ j\in G}} \sigma_{l-1}(G|j) \sigma_k^{\alpha\alpha} W_{ij\alpha}^2.
\end{aligned} \tag{2.24}$$

By (2.14), (2.17) and homogeneity of σ_k and σ_{l+1} (since $|B| = n - l$)

$$\begin{aligned}\sum_{\alpha=1}^n\sum_{i=1}^n \sigma_{l+1}^{ii}\sigma_k^{\alpha\alpha}(W_{ii} - W_{\alpha\alpha}) &= (l+1)\phi\sum_{\alpha=1}^n \sigma_k^{\alpha\alpha} - k\varphi\sum_{i=1}^n \sigma_{l+1}^{ii} \\ &\sim -k\varphi\sum_{i\in B}\sigma_{l+1}^{ii} \sim -(n-l)k\varphi\sigma_l(G).\end{aligned}$$

Commuting the covariant derivatives, it follows that

$$\begin{aligned}\sum_{\alpha=1}^n\sum_{i=1}^n \sigma_{l+1}^{ii}\sigma_k^{\alpha\alpha}W_{ii\alpha\alpha} &= \sum_{\alpha=1}^n\sum_{i=1}^n \sigma_{l+1}^{ii}\sigma_k^{\alpha\alpha}(W_{\alpha\alpha ii} + W_{ii} - W_{\alpha\alpha}) \\ &\sim \sum_{\alpha=1}^n\sum_{i=1}^n \sigma_{l+1}^{ii}\sigma_k^{\alpha\alpha}W_{\alpha\alpha ii} - (n-l)k\varphi\sigma_l(G).\end{aligned}\tag{2.25}$$

Differentiating equation (2.1), we get

$$\varphi_{ii} = \sum_{\alpha,\beta,r,s}\sigma_k^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi} + \sum_{\alpha,\beta}\sigma_k^{\alpha\beta}W_{\alpha\beta ii}.$$

(2.15) and (2.17) yield,

$$\begin{aligned}&\sum_\alpha\sum_i \sigma_{l+1}^{ii}\sigma_k^{\alpha\alpha}W_{\alpha\alpha ii} = \sum_i \sigma_{l+1}^{ii}\{\varphi_{ii} - \sum_{\alpha,\beta,r,s}\sigma_k^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi}\} \\ &\sim \sum_{i\in B}\{-(\sum_{\substack{\alpha\in G\\ \beta\in B}} + \sum_{\substack{\alpha\in B\\ \beta\in G}} + \sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} + \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}})\sigma_{k-2}(W|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i} \\ &\quad +\varphi_{ii} + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^n \sigma_{k-2}(W|\alpha\beta)W_{\alpha\beta i}^2\}\sigma_l(G).\end{aligned}\tag{2.26}$$

It follows from (2.15) and (2.16) that for $1 \le m \le n$,

$$\sum_{\substack{\alpha\in B\\ \beta\in G}}\sigma_m(W|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i} \sim [\sum_{\beta\in G}\sigma_m(G|\beta)W_{\beta\beta i}]\sum_{\alpha\in B}W_{\alpha\alpha i} \sim 0.\tag{2.27}$$

In turn,

$$\begin{aligned}&\sum_{\alpha=1}^n\sum_{i=1}^n \sigma_{l+1}^{ii}\sigma_k^{\alpha\alpha}W_{\alpha\alpha ii} \sim \sigma_l(G)\sum_{i\in B}\{\varphi_{ii} - \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}\sigma_{k-2}(G|\alpha\beta)W_{\beta\beta i}W_{\alpha\alpha i} \\ &+ \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^n \sigma_{k-2}(W|\alpha\beta)W_{\alpha\beta i}^2\}.\end{aligned}\tag{2.28}$$

We note that $|B| = n - l$, so $\sum_{i\in B} k\varphi = (n-l)k\varphi$. Now inserting (2.28) and (2.25) to (2.24), by (2.15) and (2.15) we have

$$\begin{aligned}\sum_{\alpha,\beta}\sigma_k^{\alpha\beta}\phi_{\alpha\beta} &\sim \sigma_l(G)\sum_{i\in B}(\varphi_{ii}-k\varphi) - \sigma_l(G)\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}\sigma_{k-2}(G|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}\\ &+\sigma_l(G)\sum_{i\in B}\sum_{\alpha\neq\beta}\sigma_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} - 2\sum_{\alpha=1}^{n}\sum_{i\in B,\beta\in G}\sigma_{l-1}(G|\beta)\sigma_{k-1}(W|\alpha)W^2_{i\beta\alpha}. \qquad (2.29)\end{aligned}$$

When $\alpha,\beta\in G, \alpha\neq\beta$, as W is diagonal,

$$\begin{aligned}&\sigma_{l-1}(G|\beta)\sigma_{k-1}(G|\alpha) = \sigma_{l-1}(G|\beta)[\sigma_{k-1}(G|\alpha\beta) + W_{\beta\beta}\sigma_{k-2}(G|\alpha\beta)]\\ &\geq \sigma_{l-1}(G|\beta)W_{\beta\beta}\sigma_{k-2}(G|\alpha\beta) = \sigma_l(G)\sigma_{k-2}(G|\alpha\beta). \qquad (2.30)\end{aligned}$$

From (2.30), we get

$$\begin{aligned}&\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}\sigma_l(G)\sigma_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} - 2\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}\sigma_{l-1}(G|\beta)\sigma_{k-1}(G|\alpha)W^2_{\alpha\beta i}\\ &\lesssim -\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}\sigma_{l-1}(G|\beta)\sigma_{k-1}(G|\alpha)W^2_{\alpha\beta i}\leq 0. \qquad (2.31)\end{aligned}$$

As $W_{i\beta\alpha} = W_{\alpha\beta i}$ on the standard $\mathbb{S}^n$ (recall that $W_{\alpha\beta} = u_{\alpha\beta} + \delta_{\alpha\beta}u$). We have

$$\begin{aligned}&\sigma_l(G)\sum_{i\in B}\sum_{\alpha\neq\beta}\sigma_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} - 2\sum_{\alpha=1}^{n}\sum_{i\in B,\beta\in G}\sigma_{l-1}(G|\beta)\sigma_{k-1}(W|\alpha)W^2_{i\beta\alpha}\\ &\lesssim -2\sum_{i\in B}\sum_{\alpha\in G}\sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W^2_{\alpha\alpha i}.\end{aligned}$$

We note that $\sigma_m(W|\alpha\beta)\sim\sigma_m(G), \forall\alpha,\beta\in B$, putting the previous inequality into (2.29),

$$\begin{aligned}&\sum_{\alpha,\beta}^{n}\sigma_k^{\alpha\beta}\phi_{\alpha\beta} \lesssim \sigma_l(G)[\sum_{i\in B}(\varphi_{ii}-k\varphi) - \sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}\sigma_{k-2}(G|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}]\\ &-2\sum_{i\in B}\sum_{\alpha\in G}\sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W^2_{\alpha\alpha i}\\ &= \sigma_l(G)\sum_{i\in B}[\varphi_{ii} - \frac{k+1}{k}\frac{\varphi_i^2}{\varphi} - k\varphi] + I_1 + I_2, \qquad (2.32)\end{aligned}$$

where

$$I_1 = \sum_{i\in B}(\frac{\sigma_l(G)\varphi_i^2}{k\varphi} - \sum_{\alpha\in G}\sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W^2_{\alpha\alpha i}),$$

and

$$I_2 = \sum_{i\in B}\{\sigma_l(G)[\frac{\varphi_i^2}{\varphi} - \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} \sigma_{k-2}(G|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}] - \sum_{\alpha\in G}\sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2\}.$$

For $i\in B$,

$$\varphi_i = (\sum_{\alpha\in B}+\sum_{\alpha\in G})\sigma_{k-1}(W|\alpha)W_{\alpha\alpha i} \sim \sum_{\alpha\in G}\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}. \tag{2.33}$$

It follows that for any $i\in B$,

$$\varphi_i{}^2 \sim \sum_{\alpha\in G}\sigma_{k-1}^2(G|\alpha)W_{\alpha\alpha i}^2 + \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}\sigma_{k-1}(G|\alpha)\sigma_{k-1}(G|\beta)W_{\alpha\alpha i}W_{\beta\beta i}.$$

By Corollary 10, $I_2\lesssim 0$

By homogeneity of $\sigma_k(W)$ and (2.33),

$$\begin{aligned}
I_1 &\sim \frac{1}{k\varphi}(\sum_{\alpha\in G}\sigma_l^{\frac12}(G)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i})^2 - \sum_{\alpha\in G}\sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2\\
&= \frac{1}{k\varphi}[\sum_{\alpha\in G}\sigma_{l-1}^{\frac12}(G|\alpha)W_{\alpha\alpha}^{\frac12}\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}]^2 - \sum_{\alpha\in G}\sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2\\
&\leq \frac{1}{k\varphi}\sum_{\alpha,\beta\in G}\sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2W_{\beta\beta}\sigma_{k-1}(G|\beta)\\
&\quad - \sum_{\alpha\in G}\sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2\\
&\sim \sum_{\alpha\in G}\sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2 - \sum_{\alpha\in G}\sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2\\
&= 0.
\end{aligned}$$

The proof of the Proposition is complete. ☐

By Proposition 3 and Proposition 5, and compactness argument, we have

PROPOSITION 4. *Suppose $u\in C^4(\mathbb{S}^n)$ is a solution equation (2.1) with $W_u\geq 0$. Suppose φ satisfies condition (2.9), then there is $C>0$ depending only on n, $\|\varphi\|_{C^2}$, $\inf_{\mathbb{S}^n}\varphi$ such that*

$$W_u(x)\geq CI, \quad \forall x\in\mathbb{S}^n.$$

3. The Weyl problem, curvature estimates for immersed hypersurfaces

The Weyl problem [**34**] concerns the isometric embedding of positive curved surface $(\mathbb{S}^2, g)$ to $\mathbb{R}^3$. The problem was solved by Nirenberg in his landmark paper [**23**]. Prior to Nirenberg's work, Lewy [**16**] solved the problem when the metric g is analytic. The

Weyl isometric embedding problem in hyperbolic space was considered by Pogorelov [**26**], he also considered isometric embeddings of $(\mathbb{S}^2, g)$ to general 3-dimensional Riemannian manifolds [**27**]. Aside from geometric interest, such type of isometric embedding to general Riemannian manifolds has connections with quasi local mass in general relativity [**2, 31, 19, 20, 32, 33**].

As usual, one employs the method of continuity to obtain the isometric embedding. The openness is related to the infinitesimal rigidity, which is established by Li-Wang [**17**] for general ambient space. Here we only concentrate curvature estimates obtained in [**10**] for immersed hypersurfaces in warped product ambient space. This type estimate is valid for general dimensions and in degenerate case. For $n = 2$, there is also a work by Lu [**21**] where some refined estimates are proved for embedded surface (M^2, g) in $(\bar{N}^3, \bar{g})$ using Heinz system when the extrinsic scalar curvature is strictly positive.

A warped product space is a manifold $(N^{n+1}, \bar{g})$ for $n \geq 2$ equipped with warped product structure, where metric is of form

$$\bar{g} = dr^2 + \phi^2(r) d\sigma^2_{\mathbb{S}^n}, \tag{3.1}$$

where $\phi(r)$ is defined for $r \geq r_0 \geq 0$ and $d\sigma^2_{\mathbb{S}^n}$ is the standard metric on $\mathbb{S}^n$. $\phi(r) = r, \phi(r) = \sinh r$ and $\phi(r) = \sin r$ correspond to space form $\mathbb{R}^{n+1}, \mathbb{H}^{n+1}$ and $\mathbb{S}^{n+1}$ respectively.

Let (M^n, g) be an isometrically immersed hypersurface in an ambient space $(N^{n+1}, \bar{g})$ for $n \geq 2$. Denote Ric and $\bar{Ric}$ the Ricci curvature tensors of (M, g) and $(N, \bar{g})$ respectively, and denote R and $\bar{R}$ to be the scalar curvatures of M and N respectively. Fix a unit normal ν locally, let $\kappa_i, i = 1, \cdots, n$ be the principal curvatures of M with respect to ν. We call $\sigma_2(\kappa_1, \cdots, \kappa_n)$ the **extrinsic scalar curvature** of the immersed hypersurface. It is clear that it is independent the choice of unit normal ν as σ_2 is an even function. The Gauss equation yields,

$$\sigma_2(\kappa_1, \cdots, \kappa_n) = \frac{1}{2}(R - \bar{R}) + \bar{Ric}(\nu, \nu). \tag{3.2}$$

From the isometric immersing, one has C^1 estimate directly. We prove C^2 estimate by establishing the following curvature estimate of immersed hypersurafces in $(N^{n+1}, \bar{g})$.

THEOREM 6. *Let $(N, \bar{g})$ be a warped product space where $\bar{g}$ defined as in (3.1). Denote $\phi'(\rho) = \frac{d\phi}{d\rho}$ and $\Phi(\rho) = \int_0^\rho \phi(r)dr$. Suppose $X : (M^n, g) \to (N, \bar{g})$ is a C^4 immersed compact hypersurface with nonnegative extrinsic scalar curvature and $\phi' > 0$ in M, then there exists constant C depending only on n, $\|g\|_{C^4(M)}$, $\|\bar{g}\|_{C^4(\tilde{M})}$ (where $\tilde{M}$ is any open set in N containing $X(M)$), $\sup_{x\in M} \Phi(X(x))$ and $\inf_{x\in M} \phi'(X(x))$ such that*

$$\max_{x\in M, i=1,\ldots,n} |\kappa_i(X(x))| \leq C. \tag{3.3}$$

When $(N^{n+1}, \bar{g})$ is the standard Euclidean space $\mathbb{R}^{n+1}$, estimate (3.3) was proved in [**10**] for $n = 2$ and in [**18**] for general n with an explicit constant for embedded hypersurfaces with nonnegative sectional curvature. Estimate (3.3) does not depend on the lower bound of $\sigma_2(\kappa)$. We treat equation (3.2) as a degenerate fully nonlinear equation.

Let's denote R_{ijkl} and $\bar{R}_{abcd}$ to be the Riemannian curvatures of M and N respectively. For a fixed local frame $(e_1, \cdots, e_n)$ on M, let ν be a normal vector field of M, and let

$h = (h_{ij})$ be the second fundamental form of M with respect to ν. We have the Gauss equation and Codazzi equation,

$$R_{ijkl} = \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}, \quad (Gauss) \tag{3.4}$$

$$\nabla_k h_{ij} = \nabla_j h_{ik} + \bar{R}_{\nu ijk}. \quad (Codazzi) \tag{3.5}$$

The convention that R_{ijij} denotes the sectional curvature is used here.

Take trace of the Gauss equation,

$$Ric(i,i) = \bar{Ric}(i,i) - \bar{R}_{i\nu i\nu} + \sum_j \left(h_{ii}h_{jj} - h_{ij}^2\right),$$

and the scalar curvature of M is,

$$R = \bar{R} - 2\bar{Ric}(\nu,\nu) + 2\sigma_2(h).$$

Set

$$f(x,\nu(x)) = \frac{R(x) - \bar{R}(X(x))}{2} + \bar{Ric}_{X(x)}(\nu(x),\nu(x)), \tag{3.6}$$

we can write

$$\sigma_2(h(x)) = f(x,\nu(x)), \forall x \in M. \tag{3.7}$$

LEMMA 7. *Let $H = Trh$, then*

$$|\Delta_g f(x)| \le C(\sum_{i,j} |h_{ij}(x)|^2 + |\nabla H| + 1), \tag{3.8}$$

for any $x \in M$, where C depends on $\|g\|_{C^4}$ and $\|\bar{g}\|_{C^4}$.

PROOF. $\forall x_0 \in M \subset N$, fix a local orthonormal coordinates $(x_1, \cdots, x_n)$ at $x_0 \in M$, a local orthonormal coordinates $(X_1, \cdots, X_{n+1})$ of $x_0 \in N$. We may view $\bar{Ric}$ locally as a function in $C^2(N \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$. For each X fixed, $\bar{Ric}(\xi,\eta)$ is a bilinear function of $\xi, \eta \in \mathbb{R}^{n+1}$.

Denote $X_i^\alpha = \frac{\partial X^\alpha}{\partial x_i}, X_{ii}^\alpha = \frac{\partial^2 X^\alpha}{\partial x_i^2}$, and denote $\bar{Ric}_\alpha = \frac{\partial \bar{Ric}}{\partial X_\alpha}$. Direct computation yileds

$$\begin{aligned} f_{ii} &= \frac{R_{ii} - \bar{R}_{\alpha\beta}X_i^\alpha X_i^\beta - \bar{R}_\alpha X_{ii}^\alpha}{2} + 2\bar{Ric}\left(\frac{\partial^2 \nu}{\partial x_i^2}, \nu\right) + 2\bar{Ric}\left(\frac{\partial \nu}{\partial x_i}, \frac{\partial \nu}{\partial x_i}\right) \\ &\quad + \bar{Ric}_\alpha(\nu,\nu)X_{ii}^\alpha + 4\bar{Ric}_\alpha\left(\frac{\partial \nu}{\partial x_i}, \nu\right)X_i^\alpha + \bar{Ric}_{\alpha\beta}(\nu,\nu)X_i^\alpha X_i^\beta. \end{aligned}$$

Since

$$\frac{\partial \nu}{\partial x_i} = h_{ij}e_j, \quad \frac{\partial^2 \nu}{\partial x_i^2} = h_{iji}e_j - h_{ij}^2\nu,$$

and

$$|\frac{\partial X^\alpha}{\partial x_i}| \le C, \quad |\frac{\partial^2 X^\alpha}{\partial x_i^2}| \le C(\sum_j |h_{ij}| + 1).$$

Thus, by Codazzi equation

$$
\begin{aligned}
f_{ii} &= 2\sum_{jk} h_{ij}h_{ik}\bar{Ric}(j,k) - 2\sum_{j} h_{ij}^2\bar{Ric}(\nu,\nu) + 2\sum_{j} h_{iij}\bar{Ric}(j,\nu) \\
&\quad -O(\sum_{j}|h_{ij}|+1). \qquad (3.9)
\end{aligned}
$$

Sum over i, (3.8) follows directly. □

We need one more lemma.

LEMMA 8. *Suppose the second fundamental form (h_{ij}) is diagonalized at x_0, assume $h_{11} \geq h_{22} \cdots \geq h_{nn}$ and $\sigma_1 \geq 0$, then either $H \leq 1$ or $|h_{ii}| \leq \frac{C}{h_{11}}$ for $i \neq 1$, where C is a constant depending only on $\|g\|_{C^2}, \|\bar{g}\|_{C^2}$.*

PROOF. Suppose that $H > 1$, then $h_{11} \geq \frac{H}{n} \geq \frac{1}{n}$. By Gauss equation (3.4), $|h_{11}h_{ii}| = |R_{1i1i} - \bar{R}_{1i1i}| \leq C$, we deduce that $|h_{ii}| \leq \frac{C}{h_{11}}$. □

3.1. Proof of Theorem 6. Suppose $(N, \bar{g})$ is a warped product space with an ambient metric $\bar{g}$ as

$$
\bar{g} = d\rho^2 + \phi^2(\rho)ds^2_{\mathbb{S}^n} \qquad (3.10)
$$

where $ds^2_{\mathbb{S}^n}$ is the standard induced metric in $\mathbb{S}^n$, ρ represents the distance from the origin. The vector field $V = \phi(\rho)\frac{\partial}{\partial\rho}$ is a conformal Killing field in N. Denote $\Phi(\rho) = \int_0^\rho \phi(r)dr$.

PROOF. Denote by $\kappa(x) = (\kappa_1(x), \cdots, \kappa_n(x))$ the principal curvatures of $x \in M$. Set,

$$
\varphi = log|H| + \alpha\frac{\Phi}{m},
$$

where $H = \sigma_1(h)$ is the mean curvature, $m = \inf_{x\in M}\phi'(X(x))$ and α is a positive constant to be determined later. Suppose φ attains maximum at x_0. Without loss of generality, we may assume $|H|(x_0) \geq 1$, otherwise there's nothing to prove. With a suitable choice of local orthonormal frame $(e_1, \cdots, e_n)$, we may also assume $H(x_0) \geq 1$ and $h_{ij}(x_0)$ is diagonal so that $\kappa_i = h_{ii}$.

At x_0.

$$
\varphi_i = \frac{\sum_l h_{lli}}{H} + \alpha\frac{\Phi_i}{m} = 0, \qquad (3.11)
$$

$$
\varphi_{ii} = \frac{\sum_l h_{llii}}{H} - \frac{(\sum_l h_{lli})^2}{H^2} + \alpha\frac{\Phi_{ii}}{m}. \qquad (3.12)
$$

Commuting the derivatives,

$$
h_{llii} = h_{iill} - h_{ii}^2h_{ll} + h_{ll}^2h_{ii} + h_{ll}\bar{R}_{ilil} + h_{ii}\bar{R}_{illi} + \nabla_l\bar{R}_{iil\nu} + \nabla_i\bar{R}_{ill\nu}. \qquad (3.13)
$$

Put (3.13) into (3.12), at x_0,

$$\begin{aligned} \sigma_2^{ii}\varphi_{ii} &= \sum_l \frac{\sigma_2^{ii}\left(h_{iill} - h_{ii}^2 h_{ll} + h_{ll}\bar{R}_{ilil} + h_{ii}\bar{R}_{illi} + \nabla_l \bar{R}_{iil\nu} + \nabla_i \bar{R}_{ill\nu}\right)}{H} \\ &\quad + \frac{2fh_{ll}^2}{H} - \frac{\sigma_2^{ii}(\sum_l h_{lli})^2}{H^2} + \alpha\frac{\sigma_2^{ii}\Phi_{ii}}{m}. \end{aligned} \tag{3.14}$$

It follows from equation (3.7),

$$\sigma_2^{ii}h_{iik} = f_k, \quad \sigma_2^{ii}h_{iikk} + \sigma_2^{pq,rs}h_{pqk}h_{rsk} = f_{kk}.$$

Identity (3.14) becomes,

$$\begin{aligned} \sigma_2^{ii}\varphi_{ii} &= \sum_l \frac{\sigma_2^{ii}\left(h_{ll}\bar{R}_{ilil} + h_{ii}\bar{R}_{illi} + \nabla_l \bar{R}_{iil\nu} + \nabla_i \bar{R}_{ill\nu}\right) - \sigma_2^{pq,rs}h_{pql}h_{rsl}}{H} \\ &\quad - \sigma_2^{ii}h_{ii}^2 - \frac{\sigma_2^{ii}(\sum_l h_{lli})^2}{H^2} + \alpha\frac{\sigma_2^{ii}\Phi_{ii}}{m} + \frac{2fh_{ll}^2}{H} + \frac{f_{ll}}{H}. \end{aligned}$$

As $|\nabla_l \bar{R}_{ijk\nu}|, \forall i,j,k \le n$ are controlled by H, and $0 \le \sigma_2^{ii} \le CH$, at x_0,

$$0 \ge \frac{\sum_l (f_{ll} - \sigma_2^{pq,rs}h_{pql}h_{rsl})}{H} - \sigma_2^{ii}h_{ii}^2 - \frac{\sigma_2^{ii}(\sum_l h_{lli})^2}{H^2} + \alpha\frac{\sigma_2^{ii}\Phi_{ii}}{m} - CH, \tag{3.15}$$

where C is a constant depending on n, $\|g\|_{C^4}$, $\|\bar{g}\|_{C^4}$. In the rest of the proof, we denote C as a constant under contorl, which might change from line to line.

Replace Φ_{ii} by $\phi'(\rho) - h_{ii}u$ in (3.15),

$$\begin{aligned} 0 &\ge \frac{1}{H}\left(\sum_l (f_{ll} - \sigma_2^{pq,rs}h_{pql}h_{rsl})\right) - \sigma_2^{ii}h_{ii}^2 - \frac{\sigma_2^{ii}(\sum_l h_{lli})^2}{H^2} \\ &\quad + (\alpha - C)H - C\frac{\alpha\phi}{m}. \end{aligned} \tag{3.16}$$

By Lemma 7 and (3.11) and the assumption $H \ge 1$,

$$0 \ge (\alpha - C)H - \frac{\sum_l \sigma_2^{pq,rs}h_{pql}h_{rsl}}{H} - \sigma_2^{ii}h_{ii}^2 - \frac{\sigma_2^{ii}(\sum_l h_{lli})^2}{H^2} - C\frac{\alpha\phi}{m}. \tag{3.17}$$

Note that

$$-\sigma_2^{pq,rs}h_{pql}h_{rsl} = -\sum_{p\neq q}(h_{ppl}h_{qql} - h_{pql}^2), \tag{3.18}$$

It follows from Lemma 11 that,

$$-\sum_{p\neq q} h_{ppl}h_{qql} \ge \min\{-2\frac{(\sigma_2)_l(\sigma_1)_l}{\sigma_1}, 0\}$$

Together with critical condition (3.11) and the definition of σ_2,

$$-\sum_{p\neq q} h_{ppl}h_{qql} \ge -C\frac{\alpha\phi}{m}H \tag{3.19}$$

Combine (3.18), (3.19) and (3.17)

$$0 \geq \frac{\sum_{p\neq q} h_{pql}^2}{H} - \sigma_2^{ii} h_{ii}^2 - \frac{\sigma_2^{ii}(\sum_l h_{lli})^2}{H^2} + (\alpha - C)H - C\frac{\alpha\phi}{m}.$$

As $\sum_{j\neq i} h_{jj}h_{ii}$ is bounded by Gauss equation (3.4). Thus,

$$\sigma_2^{ii} h_{ii}^2 = \sum_{i=1}^n h_{ii}(\sum_{j\neq i} h_{jj}h_{ii}) \leq C\sum_{i=1}^n |h_{ii}| \leq CnH.$$

Therefore,

$$0 \geq \frac{1}{H}\sum_{p\neq q} h_{pql}^2 - \frac{\sigma_2^{ii}(\sum_l h_{lli})^2}{H^2} + (\alpha - C)H - C\frac{\alpha\phi}{m}. \tag{3.20}$$

By lemma 8, $\sigma_2^{11} \leq \frac{C}{H}$. Critical equation (3.11) yields

$$\varphi_i = \frac{H_i}{H} + \alpha\frac{\Phi_i}{m} = 0.$$

We have

$$\sigma_2^{11}\left(\frac{H_1}{H}\right)^2 \leq C\frac{\alpha^2\phi^2}{Hm^2}.$$

By Codazzi equation (3.5),

$$\begin{aligned}
0 \quad \geq \quad & \frac{\sum_{p\neq q} h_{pql}^2}{H} - \sum_{i\neq 1}\frac{\sigma_2^{ii}(\sum_l h_{lli})^2}{H^2} \\
& +(\alpha - C)H - C\frac{\alpha\phi}{m} - C\frac{\alpha^2\phi^2}{Hm^2} \\
\geq \quad & \frac{\sum_{l\neq i} 2h_{lli}^2}{H} - \sum_{i\neq 1}\frac{\sigma_2^{ii}\left(\sum_l h_{lli}^2 + \sum_{p\neq q} h_{ppi}h_{qqi}\right)}{H^2} \\
& +(\alpha - C)H - C\frac{\alpha\phi}{m} - C\frac{\alpha^2\phi^2}{Hm^2}.
\end{aligned} \tag{3.21}$$

It follows from (3.19) that,

$$-\sum_{i\neq 1}\frac{\sigma_2^{ii}\left(\sum_{p\neq q} h_{ppi}h_{qqi}\right)}{H^2} \geq -C\sum_{i\neq 1}\sigma_2^{ii}\frac{\alpha\phi}{mH} \geq -C\frac{\alpha\phi}{m}. \tag{3.22}$$

Insert (3.22) into (3.21),

$$0 \geq \frac{1}{H}\sum_{l\neq i} 2h_{lli}^2 - \sum_{i\neq 1}\frac{\sigma_2^{ii}h_{lli}^2}{H^2} + (\alpha - C)H - C\frac{\alpha\phi}{m} - C\frac{\alpha^2\phi^2}{Hm^2}.$$

By Lemma 8, $\sigma_2^{ii} \leq H + \frac{C}{H}$ for $i \neq 1$, we have

$$\begin{aligned} 0 \quad \geq \quad & \frac{1}{H}\sum_{l\neq i}(1-\frac{C}{H^2})h_{lli}^2 - \sum_{i\neq 1}\frac{(1+\frac{C}{H^2})h_{iii}^2}{H} \\ & +(\alpha - C)H - C\frac{\alpha\phi}{m} - C\frac{\alpha^2\phi^2}{Hm^2}. \end{aligned}$$

We deal with $\frac{h_{iii}^2}{H}$. Again by Gauss equation (3.4),

$$h_{11i}h_{ii} + h_{11}h_{iii} = R_{1i1i,i} - \bar{R}_{1i1i,i}$$

Thus $\forall i \neq 1$,

$$\frac{h_{iii}^2}{H} \leq \frac{2h_{ii}^2h_{11i}^2 + C}{Hh_{11}^2} \leq C\frac{h_{11i}^2}{H^5} + \frac{C}{H^3} \tag{3.23}$$

In turn,

$$0 \geq \frac{1}{H}\sum_{l\neq i}(1-\frac{C}{H^2})h_{lli}^2 + (\alpha - C)H - C\frac{\alpha\phi}{m} - C\frac{\alpha^2\phi^2}{Hm^2}. \tag{3.24}$$

Choose α big enough, we have $H \leq \frac{C\phi}{m}$ at the maximum point of φ. Since $\Phi(x_0) - \min\Phi \geq C\phi(\tilde{\rho})$, we have $H \leq Ce^{\frac{C}{m}(\Phi(x_0)-\min\Phi)}$.

As

$$\sum_i \kappa_i^2 = H^2(\kappa) - 2\sigma_2(\kappa),$$

we obtain a bound on the principal curvatures. The proof of Theorem 6 is complete. □

The curvature estimate in Theorem 6 also holds for a general class of Riemannian ambient spaces [**12**].

4. Appendix

We collect some technical lemmas here.

First is an algebraic lemma [**13**] regarding the elementary symmetric functions.

LEMMA 9. *For $1 \leq k \leq l$, $\lambda = (\lambda_1, ..., \lambda_l)$ and with $\lambda_i \geq 0$, for $1 \leq i \leq l$, $\forall \alpha \neq \beta$ and for all real numbers $\gamma_1, ..., \gamma_l$,*

$$\begin{aligned} & \sum_\alpha \sigma_k(\lambda|\alpha)\sigma_{l-1}(\lambda|\alpha)\sigma_{k-1}(\lambda|\alpha)\gamma_\alpha^2 \\ \geq \quad & \sigma_l(\lambda)\sum_{\alpha\neq\beta}(\sigma_{k-1}^2(\lambda|\alpha\beta) - \sigma_k(\lambda|\alpha\beta)\sigma_{k-2}(\lambda|\alpha\beta))\gamma_\alpha\gamma_\beta. \end{aligned} \tag{4.1}$$

Proof: For fixed α,

$$
\begin{aligned}
&\sum_{\beta\neq\alpha}\{\lambda_\beta\sigma_{k-1}^2(\lambda|\alpha\beta)-\lambda_\beta\sigma_k(\lambda|\alpha\beta)\sigma_{k-2}(\lambda|\alpha\beta)\}\\
&=\sum_{\beta\neq\alpha}[\sigma_{k-1}(\lambda|\alpha\beta)\sigma_k(\lambda|\alpha)-\sigma_k(\lambda|\alpha\beta)(\sigma_{k-1}(\lambda|\alpha\beta)+\lambda_\beta\sigma_{k-2}(\lambda|\alpha\beta))]\\
&=\sum_{\beta\neq\alpha}[\sigma_{k-1}(\lambda|\alpha\beta)\sigma_k(\lambda|\alpha)-\sigma_k(\lambda|\alpha\beta)\sigma_{k-1}(\lambda|\alpha)]\\
&=\sigma_k(\lambda|\alpha)[(l-k)\sigma_{k-1}(\lambda|\alpha)-(l-k-1)\sigma_{k-1}(\lambda|\alpha)]\\
&=\sigma_k(\lambda|\alpha)\sigma_{k-1}(\lambda|\alpha). \qquad (4.2)
\end{aligned}
$$

By the Cauchy inequality and (4.2) to prove (4.1), as

$$\sigma_l(\lambda)=\lambda_\alpha\lambda_\beta\sigma_{l-2}(\lambda|\alpha\beta),\forall\alpha\neq\beta,$$

we have

$$
\begin{aligned}
&\sigma_l(\lambda)\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}[\sigma_{k-1}^2(\lambda|\alpha\beta)-\sigma_k(\lambda|\alpha\beta)\sigma_{k-2}(\lambda|\alpha\beta)]\gamma_\alpha\gamma_\beta\\
&=\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\{\sigma_{l-2}(\lambda|\alpha\beta)[\sigma_{k-1}^2(\lambda|\alpha\beta)-\sigma_k(\lambda|\alpha\beta)\sigma_{k-2}(\lambda|\alpha\beta)]\}(\lambda_\beta\gamma_\alpha)(\lambda_\alpha\gamma_\beta)\\
&\le\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\{\sigma_{l-2}(\lambda|\alpha\beta)[\sigma_{k-1}^2(\lambda|\alpha\beta)-\sigma_k(\lambda|\alpha\beta)\sigma_{k-2}(\lambda|\alpha\beta)]\}\frac{\lambda_\beta^2\gamma_\alpha^2+\lambda_\alpha^2\gamma_\beta^2}{2}\\
&=\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\sigma_{l-2}(\lambda|\alpha\beta)\lambda_\beta[\sigma_{k-1}^2(\lambda|\alpha\beta)-\sigma_k(\lambda|\alpha\beta)\sigma_{k-2}(\lambda|\alpha\beta)]\lambda_\beta\gamma_\alpha^2\\
&=\sum_\alpha\sigma_{l-1}(\lambda|\alpha)\sum_{\beta,\beta\neq\alpha}\lambda_\beta[\sigma_{k-1}^2(\lambda|\alpha\beta)-\sigma_k(\lambda|\alpha\beta)\sigma_{k-2}(\lambda|\alpha\beta)]\gamma_\alpha^2\\
&=\sum_\alpha\sigma_k(\lambda|\alpha)\sigma_{l-1}(\lambda|\alpha)\sigma_{k-1}(\lambda|\alpha)\gamma_\alpha^2.
\end{aligned}
$$

This completes the proof of (4.1). ☐

The following corollary [**11**] indicates certain convexity of the k-th elementary symmetric functions in Γ_n, in contrast to the concavity property of $\sigma_k^{\frac{1}{k}}$ which was used in the proof of C^2 estimate for admissible solutions.

Corollary 10. *For $\lambda=(\lambda_1,\cdots,\lambda_l)\in\mathbb{R}^l$ with $\lambda_j>0,\forall j=1,\cdots,l$, for $1\le k\le l$, set $\eta(\lambda)=\log\sigma_k(\lambda)$. Then*

$$\sum_{\alpha,\beta}\frac{\partial^2\eta}{\partial\lambda_\alpha\partial\lambda_\beta}(\lambda)\gamma_\alpha\gamma_\beta+\sum_\alpha\frac{\partial\eta}{\partial\lambda_\alpha}\lambda_\alpha^{-1}\gamma_\alpha^2\ge 0,\quad\forall\gamma=(\gamma_1,\cdots,\gamma_l)\in\mathbb{R}^l. \qquad (4.3)$$

PROOF. By Lemma 9,

$$\begin{aligned}
&\sigma_l(\lambda)\sigma_k^2(\lambda)\big(\sum_{\alpha,\beta}\frac{\partial^2\eta}{\partial\lambda_\alpha\partial\lambda_\beta}(\lambda)\gamma_\alpha\gamma_\beta+\sum_\alpha\frac{\partial\eta}{\partial\lambda_\alpha}\lambda_\alpha^{-1}\gamma_\alpha^2\big)\\
&=\sum_\alpha[\sigma_k(G)\sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)-\sigma_l(G)\sigma_{k-1}^2(G|\alpha)]\gamma_\alpha^2\\
&+\sigma_l(\lambda)\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}[\sigma_k(\lambda)\sigma_{k-2}(\lambda|\alpha\beta)-\sigma_{k-1}(\lambda|\alpha)\sigma_{k-1}(\lambda|\beta)]\gamma_\alpha\gamma_\beta\\
&=\sum_\alpha\sigma_k(\lambda|\alpha)\sigma_{l-1}(\lambda|\alpha)\sigma_{k-1}(\lambda|\alpha)\gamma_\alpha^2\\
&+\sigma_l(\lambda)\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}[\sigma_k(\lambda|\alpha\beta)\sigma_{k-2}(\lambda|\alpha\beta)-\sigma_{k-1}^2(\lambda|\alpha\beta)]\gamma_\alpha\gamma_\beta\\
&\geq 0.
\end{aligned}$$

□

The next is a refined concavity property of σ_2 ([**9, 12**]).

LEMMA 11. *Let $W(x)=(W_{ij}(x))$ be a 2-symmetric tensor on a Riemannian manifold M, suppose that $p\in M$, $W(p)$ is diagonal, $0\leq\sigma_2(W(x))\in C^1$ in a neighborhood of point p, and $\sigma_1(W(p))\neq 0$. For each $m=1,...,n$, denote*

$$W_m(p)=(\nabla_m W_{11}(p),\cdots,\nabla_m W_{nn}(p)).$$

then at p,

$$-\sigma_2(W_m,W_m)\geq\min\{-2\frac{\nabla_m\sigma_2(W)\nabla_m\sigma_1(W)}{\sigma_1(W)}+2\frac{(\nabla_m\sigma_1(W))^2\sigma_2(W)}{\sigma_1^2(W)},0\}. \tag{4.4}$$

and

$$-\sigma_2(W_m,W_m)\geq\min\{-2\frac{\nabla_m\sigma_2(W)\nabla_m\sigma_1(W)}{\sigma_1(W)}+\frac{(\nabla_m\sigma_2(W))^2\sigma_2(I,I)}{\sigma_2^2(W,I)},0\}.$$

PROOF. We first prove
Claim: Suppose that W,V satisfy $\sigma_1(W)\neq 0$, $\sigma_2(W)\geq 0$ and $\sigma_2(V,W)=0$, then $\sigma_2(V,V)\leq 0$.

We may assume $\sigma_1(W)>0$ by switching W to $-W$ if necessary. The *claim* follows from the hyperbolicity of σ_2 in Γ_2 (see [**8**]) if $\sigma_2(W)>0$. The degenerate case $\sigma_2(W)=0$ can be dealt as follow. Set $W_\epsilon=W+\epsilon I$ and $V_\epsilon=V-\frac{\epsilon\sigma_1(V)I}{\sigma_1(W)+\epsilon\sigma_2(I,I)}$. Since $\sigma_1(W)>0$, $\forall\epsilon>0$, $W_\epsilon\in\Gamma_2$ and $\sigma_2(W_\epsilon,V_\epsilon)=0$. By the hyperbolicity of σ_2 in Γ_2, $\sigma_2(V_\epsilon,V_\epsilon)\leq 0$. The *claim* follows by taking $\epsilon\to 0$.

Denote $W_m=(\nabla_m W_{ii})$ and $\nabla_m\sigma_2(W)=(\sigma_2(W))_m$. If $\sigma_2(W(p))=0$, then at p, we have $0=(\sigma_2(W))_m=\sigma_2(W_m,W)$. By the assumption and the *claim*, $\sigma_2(W_m,W_m)\leq 0$ at p.

If $\sigma_2(W(p)) > 0$, we have $W(p) \in \Gamma_2$. Set, $V = W_m - \frac{\sigma_2(W,W_m)}{\sigma_2(W,I)} I$. So, $\sigma_2(W,V) = 0$. By Garding [**8**] $\sigma_2(V,V) \le 0$, that is ,

$$0 \ge \sigma_2(V,V) = \sigma_2(W_m,W_m) - 2\frac{\sigma_2(W,W_m)\sigma_2(W_m,I)}{\sigma_2(W,I)} + \frac{\sigma_2^2(W,W_m)\sigma_2(I,I)}{\sigma_2^2(W,I)}.$$

As $\sigma_2(W,W_m) = \nabla_m\sigma_2(W)$ and $\sigma_2(W_m,I) = (n-1)\nabla_m\sigma_1(W)$,

$$-\sigma_2(W_m,W_m) \ge -2\frac{\nabla_m\sigma_2(W)\nabla_m\sigma_1(W)}{\sigma_1(W)} + \frac{(\nabla_m\sigma_2(W))^2\sigma_2(I,I)}{\sigma_2^2(W,I)}.$$

This fulfills the second inequality. Now let's prove the first inequality. At point p, If $\sigma_1(W_m) = 0$, then $\sigma_2(W_m,W_m) \le 0$. Suppose now $\sigma_1(W_m) \ne 0$, let $V = W_m - \frac{\sigma_1(W_m)}{\sigma_1(W)} W$, then $\sigma_1(V) = 0$, thus $\sigma_2(V,V) \le 0$, i.e.

$$0 \ge \sigma_2(V,V) = \sigma_2(W_m,W_m) - 2\frac{\sigma_1(W_m)\sigma_2(W_m,W)}{\sigma_1(W)} + \frac{\sigma_1^2(W_m)\sigma_2(W,W)}{\sigma_1^2(W)}.$$

In turn,

$$-\sigma_2(W_m,W_m) \ge -2\frac{\nabla_m\sigma_1(W)\nabla_m\sigma_2(W)}{\sigma_1(W)} + 2\frac{(\nabla_m\sigma_1(W))^2\sigma_2(W)}{\sigma_1^2(W)}.$$

The lemma is now proved. □

The next lemma is due to Nirenberg-Treves [**24**].

LEMMA 12. *Let $f \ge 0$ be a C^2 function on a Riemannian manifold M, then if $\partial M \ne \emptyset$,*

$$|\nabla f(x)|^2 \le \frac{2||f||_{C^2(M)}(1 + d(x,\partial M))}{d(x,\partial M)} f(x), \forall x \in M; \tag{4.5}$$

if $\partial M = \emptyset$,

$$|\nabla f(x)|^2 \le 2||f||_{C^2(M)} f(x), \forall x \in M. \tag{4.6}$$

PROOF. We may assume $f > 0$ by working at $f_\epsilon = f + \epsilon$ for $\epsilon > 0$ if necessary. For each point x_0, pick any $r > 0$ such that $r < dist(x_0,\partial M)$ if $\partial M \ne \emptyset$. Set $B_r(x_0) = \{x \in M | dist(x,x_0) < r\}$.

Let's first assume $dist^2(x,x_0)$ is smooth in $B_r(x_0)$, Define $\rho(x)$ as follows:

$$\rho(x) = r^2 - dist^2(x,x_0), \quad x \in B_r(x_0); \qquad \rho(x) = 0, \quad otherwise.$$

Consider function $\rho\frac{|\nabla f|^2}{f}$, it is compactly supported in M^n. Thus it must have a maximum point in $\overline{B_r(x_0)}$ and we may assume it is positive. The maximum is attained interior, as $\rho = 0$ on $\partial B_r(x_0)$.

At the maximum point p, let e_1 be the direction of gradient of f, i.e. $|\nabla f| = f_1$, we have

$$\frac{\rho_1}{\rho} + \frac{2f_{11}f_1}{f_1^2} - \frac{f_1}{f} = 0.$$

Thus

$$\rho\frac{f_1^2}{f}(p) = 2f_{11}\rho + \rho_1 f_1 \leq 2||f||_{C^2(M)}r(1+r). \tag{4.7}$$

That is,

$$\frac{|\nabla f|^2}{f}(x_0) \leq 2||f||_{C^2(M)}\frac{1+r}{r}.$$

If $\partial M \neq \emptyset$, let $r \to dist(x_o, \partial M)$, if $\partial M = \emptyset$, let $r \to \infty$, the lemma is verified since x_0 is arbitrary, *provided that* ρ *is* C^1.

Function ρ may not be C^1 in general, but it is a Lipschitz function since $|\nabla dist(x, x_0)| = 1$. As $B_r(x_0) \subset\subset M$, we may approximate ρ by smooth nonnegative functions ρ_δ in $C^{0,1}(\bar{B}_{r_\delta}(x_0))$ with $supp(\rho_\delta) \subset \bar{B}_{r_\delta}$ with $r_\delta \to r$ and $B_{r_\delta} \to B_r(x_0)$ as $\delta \to 0$. Replace ρ by ρ_δ, and repeat the same argument as before, then take $\delta \to 0$. □

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