MONGE-AMPE`ERE TYPE EQUATIONS AND RELATED GEOMETRIC PROBLEMS

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The material in this notes was taken from two lectures under the same title delivered at the Mini-school on Nonlinear Equations at the Center of Mathematical Sciences and Applications at Harvard University, December 3-4, 2016.

The Minkowski problem is one of the inspiring source for the development of the theory of Monge-Ampère equation and fully nonlinear equations in general. The emphasis of the lectures is on the regularity estimates, exemplified through two nonlinear partial differential equations arising from the classical problems of prescribing area measures (the Christoffel-Minkowski problem) and curvature measures in differential geometric setting (the Alexandrov problem for curvature measures). These problems lead to the fundamental issues of regularity and convexity for solutions of general nonlinear PDEs. The ideas and techniques developed around these special nonlinear equations have been found useful to deal with other geometric problems. The literature devoting to these problems is vast, we list some of them at the end of the notes. They are by no means to be complete.

1. The Minkowski Problem

We start with the classical Minkowski problem. Suppose $M$ is a closed strongly convex hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, the Gauss map $\bar{n}: M \to \mathbb{S}^n$ is a diffeomorphism, where at any point $p \in M$, $\nu(p)$ is the unit outer normal at $p$. In this way, the Gauss curvature can be viewed as a positive function $k(\nu^{-1}(x))$ on $\mathbb{S}^n$. Let us denote $\kappa = (\kappa_1, \cdots, \kappa_n)$ be the principal curvatures and $K = \kappa_1 \cdots \kappa_n$ the Gauss curvature of $M$ respectively. The Minkowski problem is a problem of prescribing Gauss curvature on the outer normals of convex hypersurfaces. To be more precise, the question is: given a positive function $\varphi$ on $\mathbb{S}^n$, is there a closed strongly convex hypersurface whose Gauss curvature is $\frac{1}{\varphi}$ as a function on its outer normals? By the Divergence Theorem, $\varphi$ has to satisfy equation

$$
(1.1) \quad \int_{\mathbb{S}^n} x_i \varphi = \int_{\mathbb{S}^n} \frac{x_i}{K(x)} = \int_{M} \nu \cdot \bar{E}_i = 0, i = 1, \ldots, n + 1,
$$

where $x_i$ are the coordinate functions and $\bar{E}_i$ is the standard $i$th coordinate vector of $\mathbb{S}^n$.

A $C^{2}$ closed hypersurface $M$ in $\mathbb{R}^{n+1}$ is called strongly convex if its Gauss curvature is positive everywhere. By the Hadamard Theorem, $M$ is a boundary of a convex domain. In turn, $M$ can be parametrized by its inverse Gauss map over $\mathbb{S}^n$ with $y(x) = \nu_M^{-1}(x)$. The following is a classical result.

Research of the first author was supported in part by an NSERC Discovery Grant.
Theorem 1.1. Suppose $\varphi \in C^2(S^n), \varphi(x) > 0, \forall x \in S^n,$ and satisfies equation (1.1), then there is a $C^{3,\alpha}(\forall 0 < \alpha < 1)$ strongly convex surface $M$ in $\mathbb{R}^{n+1},$ such that $K(\nu_M^{-1}(x)) = \frac{1}{\varphi(x)} \forall x \in S^n.$ $M$ is unique up to translations.

1.1. The support function. Let $M$ be a closed strongly convex hypersurface. The support function of $M$ is defined as

$$u(x) = \sup_{z \in M} x \cdot z = x \cdot y(x), \forall x \in S^n.$$ 

We extend $u$ as a homogeneous function of degree one in $\mathbb{R}^{n+1}\setminus\{0\}$. It is easy to check that $u$ is a convex function in $\mathbb{R}^{n+1}$. Since $\frac{\partial y}{\partial x}$ is tangent to $M$ for all $j$, and $x = \nu_M(y)$ is normal to $M$, we have $x \cdot \frac{\partial y}{\partial x_j} = 0$ for all $j$. It follows that

(1.2) $$y(x) = \nabla_{\mathbb{R}^{n+1}} u(x).$$

Therefore, $M$ can be recovered completely from $u$ by above equation. The relation $y(x) = \nu_M^{-1}(x)$ and (1.2) yield

(1.3) $$\nabla_{\mathbb{R}^{n+1}} u(x) = \nu_M^{-1}(x).$$

Equation (1.3) implies that $u$ is $C^2$ if $M$ is $C^2$ and its Gauss curvature is positive.

Let $e_{n+1} = x$ be the position vector on $S^n$, let $e_1, \cdots, e_n$ is an orthonormal frame on $\sigma_{n+1}$ so that $e_1, \cdots, e_{n+1}$ is a positive oriented orthonormal frame in $\mathbb{R}^{n+1}$. Let $\omega^i$ and $\omega^i_j$ be the corresponding dual 1-forms and the connection forms respectively. We have

$$de_j = -\sum_{i=1}^n \omega^j_i e_i, \quad \forall j = 1,2,\cdots,n, \quad \text{and} \quad de_{n+1} = \sum_{i=1}^n \omega^i e_i.$$

For each function $u \in C^2(\sigma_{n+1}^n)$, let $u_i$ be the covariant derivative of $u$ with respect to $e_i$. Define a vector valued function

$$Y = \sum_{i=1}^n u_i e_i + u e_{n+1}.$$

We note that $Y$ is independent of the choice of the orthonormal frames. We calculate that,

$$dY = \sum_{i=1}^n (du_i e_i + u_i de_i) + du e_{n+1} + u de_{n+1}$$

$$= \sum_{i=1}^n \left( \sum_{j=1}^n u_{ij} \omega^j_i - \sum_{j=1}^n u_j \omega^i_j \right) e_i + \sum_{i=1}^n (\sum_{\alpha=1}^{n+1} u_i \omega^\alpha_i e_\alpha)$$

$$+ \sum_{i=1}^n (u_i \omega^i) e_{n+1} + u \sum_{i=1}^n \omega^i e_i$$

$$= \sum_{j=1}^n \left( \sum_{i=1}^n (u_{ij} + \delta_{ij} u) e_i \right) \omega^j.$$
In particular, if \( u \) is a support function of \( M \), by (1.2) the position vector of \( M \) is
\[
y(x) = \sum_{i=1}^{n} u_i e_i + u e_{n+1}.
\]
In turn,
\[
(1.4) \quad dy = \sum_{i,j} (u_{ij} + u \delta_{ij}) e_i \otimes \omega_j
\]
The identity (1.4) indicates that the differential \( dy \) maps \( T_x(S^n) \) to itself and it is self-adjoint. \( dy \) is sometimes called the reverse Weingarten map. Since the Gauss curvature \( K \) is positive, the Gauss map \( n_M \) is invertible at \( y = \nu^{-1}_M(x) \). We have
\[
(1.5) \quad dy = (d\nu_M)^{-1},
\]
so that the reverse Weingarten map at \( x \) coincides with the inverse of the Weingarten map at \( y \). Since the eigenvalues of the Weingarten map are the principal curvatures \( \kappa = (\kappa_1, \cdots, \kappa_n) \) of \( M \) at \( y \), the eigenvalues of reverse Weingarten map at \( x = \nu_M(y) \) are exactly the principal radii at \( y \).

Conversely, if \( u(x) \) is a \( C^2 \) function on \( S^n \) with \( (u_{ij} + u \delta_{ij}) > 0 \), we claim that there is a strongly convex hypersurface \( M \) such that its support function is \( u \). Again, we extend \( u \) as a homogeneous function of degree one in \( \mathbb{R}^{n+1} \setminus \{0\} \). It is clear that \( M \) should be defined as in (1.2), that is,
\[
(1.6) \quad M = \{ \nabla_{\mathbb{R}^{n+1}} u(x) | x \in \mathbb{R}^{n+1} \setminus \{0\} \} = \{ \sum_{i=1}^{n} u_i(x) e_i(x) + u(x) e_{n+1}(x) | x \in S^n \}.
\]
Since \( (u_{ij} + u \delta_{ij}) > 0 \) is non-singular, we may read off from (1.4) that the tangent space of \( M \) in \( \mathbb{R}^{n+1} \) at \( y(x) = \sum_{i=1}^{n} u_i(x) e_i(x) + u(x) e_{n+1}(x) \) is \( \text{span}\{e_1, \cdots, e_n\} \). Moreover, from \( \det(u_{ij} + u \delta_{ij}) > 0 \) and
\[
dy \wedge \cdots \wedge dy \wedge e_{n+1} = \det(u_{ij} + u \delta_{ij}) d\omega_1 \wedge \cdots \wedge d\omega_n,
\]
we conclude that \( e_{n+1} = x \) is a normal vector at \( y(x) = \sum_{i=1}^{n} u_i(x) e_i(x) + u(x) e_{n+1}(x) \) of \( M \). This provides a global orientation of \( M \) and also gives a global inverse of the map from \( M \) (defined in (1.6)) to \( S^n \). That is, the map \( y(x) = \sum_{i=1}^{n} u_i(x) e_i(x) + u(x) e_{n+1}(x) \) is globally invertible and \( M \) is an embedded hypersurface in \( \mathbb{R}^{n+1} \). Equation (1.6) implies \( u(x) = x \cdot y(x) \). By (1.5), the principal curvatures at \( y \) is exactly the reciprocals of the eigenvalues of \( (u_{ij} + u \delta_{ij}) \). In particular, the Gauss curvature of \( M \) does not vanish. Because \( M \) is a compact hypersurface, the Gauss curvature is positive at some point, therefore must be positive at every point. By the Hadamard Theorem, \( M \) is strongly convex. And \( u(x) = x \cdot y(x) = x \cdot \nu^{-1}_M(x) \) is the support function of \( M \).

From the above discussion, the support function carries all the information of \( M \). In summary,

**Proposition 1.** A strongly convex hypersurface \( M \) in \( \mathbb{R}^{n+1} \) is \( C^2 \) if and only if its support function \( u \) is in \( C^2(S^n) \) with \( (u_{ij} + u \delta_{ij}) > 0 \). The eigenvalues of \( (u_{ij} + u \delta_{ij}) \) are the principal radii of \( M \) (parametrized by the inverse Gauss map over \( S^n \)).
In particular, the Gauss curvature $K$ of $M$ satisfies equation
\begin{equation}
\det(u_{ij} + u\delta_{ij}) = \frac{1}{K}, \quad \text{on } S^n.
\end{equation}

Furthermore, any function $u \in C^2(S^n)$ with $(u_{ij} + u\delta_{ij}) > 0$ is a support function of a $C^2$ strongly convex hypersurface $M$ in $\mathbb{R}^{n+1}$.

1.2. A priori estimates. The proof Theorem 1.1 is method of continuity, here we only concentrate on the closeness. That is, the a priori regularity estimates for equation (1.7). Equation (1.7) is elliptic at any $u$ with $W = (u_{ij} + u\delta_{ij}) > 0$. The focus here is to derive $C^2$ a priori estimates for equation (1.7).

For a solution $u$ of equation (1.7), $u + \sum_{i=1}^{n+1} a_i x_i$ is also a solution. By proper choice of $\{a_i\}_{i=1}^{n}$, we may assume that $u$ satisfies the following orthogonality condition:
\begin{equation}
\int_{S^n} x_i u \, dx = 0, \quad \forall i = 1, 2, ..., n+1.
\end{equation}

If $u$ is a support function of a closed hypersurface $M$ which bounds a convex body $\Omega$, condition (1.8) implies that the Steiner point of $\Omega$ coincides with the origin.

We first estimate the extrinsic diameter of $M$.

**Lemma 1.2.** Let $M \in C^2$, $M$ be a closed convex hypersurface in $\mathbb{R}^{n+1}$, and let $\varphi = \frac{1}{K}$. If $L$ is the extrinsic diameter of $M$, then
\begin{equation}
L \leq c_n \left( \int_{S^n} \varphi \right)^{\frac{n+1}{n}} \left( \inf_{y \in S^n} \int_{S^n} \max(0, \langle y, x \rangle) \varphi(x) \right)^{-1},
\end{equation}
where $c_n$ is a positive constant depending only on $n$. In particular, if $u$ is a support function of $M$ satisfying (1.7) and (1.8), then
\begin{equation}
0 \leq \min u \leq \max u \leq c_{n,k} \left( \int_{S^n} \varphi \right)^{\frac{n+1}{n}} \left( \inf_{y \in S^n} \int_{S^n} \max(0, \langle y, x \rangle) \varphi(x) \right)^{-1}.
\end{equation}

**Proof.** Let $p, q \in M$ such that the line segment joining $p$ and $q$ has length $L$. We may assume $0$ is in the middle of the line segment. Let $\vec{y}$ be a unit vector in the direction of this line. Let $v$ be the support function and $W = \{v_{ij} + v\delta_{ij}\}$. We have $\sigma_k(W) = \varphi$. Now, for $x \in S^n$, we get
\begin{equation}
u(x) = \sup_{Z \in M} \langle Z, x \rangle \geq \frac{1}{2} L \max(0, \langle y, x \rangle).
\end{equation}

If we multiply by $\varphi$ and integrate over $S^n$, we get
\begin{equation}
L \leq 2 \left( \int_{S^n} u \varphi \right) \left( \int_{S^n} \max(0, \langle y, x \rangle) \varphi \right)^{-1}.
\end{equation}
As $\int_{S^n} u \sigma_n(W) = \text{Vol}(\Omega)$ and $\int_{S^n} \sigma_n(W)$ is the surface area of $M = \partial \Omega$, by the isoperimetric inequality,
\begin{equation}(\int_{S^n} u \sigma_n(W))^\frac{1}{n+1} \leq C_n(\int_{S^n} \sigma_n(W))^{\frac{1}{n}}.
\end{equation}
In turn, we get

\[ L \leq c_n \left( \int_{S^n} \varphi \right)^{\frac{n+1}{n}} \left( \inf_{y \in S^n} \int_{S^n} \max(0, (y, x)) \varphi \right)^{-1}. \]

If \( u \) satisfies (1.8), the Steiner point of \( M \) is the origin. The last inequality is a consequence of the above inequality.

We precede to obtain \( C^2 \) estimate. The crucial fact is that \( \det \tilde{\varphi} = \varphi \). We have

\[ \frac{1}{n} \left( \sum_{i,j=1}^{n} G^{ij} H_{ij} \right) \geq \varphi. \]

Proposition 2. There is a constant \( C > 0 \) depending only on \( n \), \( \| K \|_{C^2(S^n)} \) and \( \min_{S^n} K \), such that if \( u \) satisfies (1.8) and \( u \) is a solution of (1.7), then \( \| u \|_{C^2(S^n)} \leq C \). There is an explicit bound for the function \( H := \frac{1}{n} \nabla \nabla \tilde{\varphi} \).

Proof. Since \( (u_{ij} + \delta_{ij} u) \) is positive definite, it is controlled by its trace by \( H \). The first inequality follows from the Newton-MacLaurin inequality. Assume the maximum value of \( H \) is attained at a point \( x_0 \in S^n \). We choose an orthonormal local frame \( e_1, e_2, ..., e_n \) near \( x_0 \) such that \( u_{ij}(x_0) \) is diagonal. If \( W = (u_{ij} + \delta_{ij} u) \), we define \( G(W) := \sigma_{ii}(W) \). Then equation (1.7) becomes

\[ G(W) = \tilde{\varphi}. \]

By the commutator identity \( H_{ii} = \Delta W_{ii} - n W_{ii} + H \) and the assumption that the matrix \( W > 0 \), so \( (G^{ij}) = \left( \frac{\partial G}{\partial W_{ij}} \right) \) is positive definite. Since \( (H_{ij}) \leq 0 \), and \( (G^{ij}) \) is diagonal, by the above commutator identity, it follows that at \( x_0 \),

\[ 0 \geq G^{ij} H_{ij} = G^{ii}(\Delta W_{ii}) - n G^{ii} W_{ii} + H \sum_{i} G^{ii}. \]

As \( G \) is homogeneous of degree one, we have

\[ \frac{1}{n} \left( \sum_{i,j=1}^{n} G^{ij} W_{ij} \right) = \tilde{\varphi}. \]

Next we apply the Laplace operator to equation (2.3) to obtain

\[ G^{ij} W_{ijk} = \nabla_k \varphi, \quad G^{ij,rs} W_{ijk} W_{rsk} + G^{ij} \Delta W_{ij} = \Delta \tilde{\varphi}. \]

By fact that \( G \) is concave, at \( x_0 \)

\[ G^{ii} \nabla (W_{ii}) \geq \Delta \tilde{\varphi}. \]

Combining (1.12), (1.13) and (1.11),

\[ 0 \geq \Delta \tilde{\varphi} - n \tilde{\varphi} + H \sum_{i=1}^{n} G^{ii} \]
As $W$ is diagonal at the point, we may write $W = (W_{11}, \ldots, W_{nn})$ as a vector in $\mathbb{R}^n$. A simple calculation yields
\[
\sum_{i=1}^{n} G^{ii} = \frac{\sigma_{n-1}(W)}{n \sigma_{n-\frac{1}{2}}(W)} \geq 1,
\]
the last inequality follows from the Newton-MacLaurin inequality.

By (1.14), we have $H \leq n\tilde{\varphi} - \Delta \tilde{\varphi}$.

By Proposition 2, Evans-Krylov Theorem [9, 19], and the standard theory of elliptic equations, we have the following higher order a priori estimates.

**Proposition 3.** For each integer $l \geq 1$ and $0 < \alpha < 1$, there exist a constant $C$ depending only on $n, l, \alpha, \min \varphi$, and $||\varphi||_{C^{l,1}(S^n)}$ such that
\[
||u||_{C^{l+1,\alpha}(S^n)} \leq C,
\]
for all solutions of (1.7) satisfying the condition (1.8).

2. The Christoffel-Minkowski Problem, Issue of Convexity

The Minkowski problem is one of the the prescribing area measure problems in convex geometry [25]. More precisely, $\frac{1}{n} dV_{\sigma_{i+1}}$ is the $n$-th area measure of convex body. For $1 \leq k \leq n$, the $k$-th area measure is $\sigma_k(W) dV_{\sigma_{i+1}}$. The prescribing $k$-th area measure corresponds to the following equation
\[
\sigma_k(u_{ij} + u \delta_{ij}) = \varphi \quad \text{on} \quad S^n.
\]

**Definition 2.1.** For $1 \leq k \leq n$, let $\Gamma_k$ is a convex cone in $\mathbb{R}^n$ determined by
\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0 \}.
\]
Suppose $u \in C^2(\sigma_{i+1}^n)$. We say $u$ is $k$-convex, if $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\}$ is in $\Gamma_k$ for each $x \in \sigma_{i+1}^n$. $u$ is convex on $\sigma_{i+1}^n$ if $W$ is convex. Furthermore, $u$ is called an admissible solution of (2.1), if $u$ is $k$-convex and satisfies (2.1).

We recall some basic facts of $\Gamma_k$. If $W \in \Gamma_k$, the $(\partial\sigma_k \partial W_{ij})$ is positive definite and $\sigma_k^\frac{1}{n}(W)$ is concave.

In the case of $k = 1$, equation (2.1) is a linear elliptic equation on the sphere. That $C^2$ a priori estimates hold for a solution $u$ satisfying (1.8) in this case follows from standard linear elliptic theory. Therefore, we will restrict ourselves to the case $k \geq 2$. The proof of Proposition 2 can be adapted to get similar $C^2$ estimate as in (1.9) for solutions of equation (2.1). This type of estimate can be used to obtain $C^0$ estimate too, by a compactness argument. Below is the statement.

**Proposition 4.** There is a constant $C > 0$ depending only on $n, k, ||\varphi||_{C^2(S^n)}$ and $\min_{S^n} \varphi$, such that if $u$ satisfies (1.8) and $u$ is an admissible solution of (2.1), then $||u||_{C^2(S^n)} \leq C$.

There is an explicit bound for the function $H := \text{trace}(u_{ij} + \delta_{ij}u) = \Delta u + nu$,
\[
\min_{x \in S^n}(n\tilde{\varphi}(x)) \leq \max_{x \in S^n} H(x) \leq \max_{x \in S^n}(n\tilde{\varphi}(x) - \Delta \tilde{\varphi}(x)),
\]
where \( \tilde{\varphi} := \frac{\varphi}{C_n^k} \), \( C_n^k = \frac{n!}{k!(n-k)!} \).

**Proof.** Since the entries \(|u_{ij} + \delta_{ij}u|\) are controlled by eigenvalues \( \{\lambda_i\}_{i=1}^n \) of \((u_{ij} + \delta_{ij}u)\). The eigenvalues are controlled by \( H \) since \((u_{ij} + \delta_{ij}u) \in \Gamma_k, k \geq 2 \). Indeed, we may write \( \lambda_1 \leq \cdots \leq \lambda_n \). Since \( k \geq 2 \), \( \sum_{i=1}^{n-1} \lambda_i > 0 \). This gives \( \lambda_n < H \). In turn, \( \lambda_i < H \) for all \( i = 1, \cdots, n \).

The first inequality follows from the Newton-MacLaurin inequality. Assume the maximum value of \( H \) is attained at a point \( x_0 \in S^n \). We choose an orthonormal local frame \( e_1, e_2, \ldots, e_n \) near \( x_0 \) such that \( u_{ij}(x_0) \) is diagonal. If \( W = (u_{ij} + \delta_{ij}u) \), we define \( G(W) := (\frac{n}{C_n^k})^{\frac{1}{k}}(W) \). Then equation (2.1) becomes

\[
G(W) = \tilde{\varphi}. 
\]

(2.3)

For the standard metric on \( S^n \), one may easily check the commutator identity \( H_{ii} = \Delta W_{ii} - nW_{ii} + H \). By the assumption that the matrix \( W \in \Gamma_k \), so \((G^{ij})\) is positive definite. Since \((H_{ij}) \leq 0 \), and \((G^{ij})\) is diagonal, by the above commutator identity, it follows that at \( x_0 \),

\[
0 \geq G^{ij}H_{ij} = G^{ii}(\Delta W_{ii}) - nG^{ii}W_{ii} + H \sum_i G^{ii}. 
\]

(2.4)

As \( G \) is homogeneous of degree one, we have

\[
G^{ii}W_{ii} = \tilde{\varphi}. 
\]

(2.5)

Next we apply the Laplace operator to equation (2.3) to obtain

\[
G^{ij}W_{ijk} = \nabla_k \tilde{\varphi}, \quad G^{ij,rs}W_{ijk}W_{rsk} + G^{ij}\Delta W_{ij} = \Delta \tilde{\varphi}. 
\]

By the concavity of \( G \), at \( x_0 \) we have

\[
G^{ii} \Delta(W_{ii}) \geq \Delta \tilde{\varphi}. 
\]

(2.6)

Combining (2.5), (2.6) and (2.4), we see that

\[
0 \geq \Delta \tilde{\varphi} - n\tilde{\varphi} + H \sum_{i=1}^n G^{ii}. 
\]

(2.7)

As \( W \) is diagonal at the point, we may write \( W = (W_{11}, \ldots, W_{nn}) \) as a vector in \( \mathbb{R}^n \). A simple calculation yields

\[
G^{ii} = \frac{\sigma_k(W)^{\frac{1}{k}} - \sigma_{k-1}(W)}{(C_n^k)^{\frac{1}{k}}} \frac{\partial \sigma_k(W)}{\partial W_{ii}} = \frac{\sigma_k(W)^{\frac{1}{k}} - \sigma_{k-1}(W)}{(C_n^k)^{\frac{1}{k}}} \sigma_{k-1}(W) 
\]

where \((W|i)\) is the vector given by \( W \) with \( W_{ii} \) deleted. It follows from the Newton-MacLaurin inequality that

\[
\sum_{i=1}^n G^{ii} = (n - k + 1)\frac{(n-k)!}{(C_n^k)^{\frac{1}{k}}} \sigma_{k-1}(W) \geq 1. 
\]

By (2.7), we have \( H \leq n\tilde{\varphi} - \Delta \tilde{\varphi} \).
Finally, we claim $u$ is bounded if it satisfying condition (1.8). Suppose this is not true, there is a sequence $u_l$ satisfying the equation with $\|u_l\|_{L^\infty} \to \infty$. We rescale, consider $\tilde{u}_l = \frac{u_l}{\|u_l\|_{L^\infty}}$, it satisfies (1.8) and (2.2) with $\|\tilde{u}_l\|_{C^2} \to 0$. By compactness, there is a subsequence convergent to $\tilde{u}$ satisfying (1.8) and $\Delta \tilde{u} + n\tilde{u} = 0$, with $\|\tilde{u}\|_{L^\infty} = 1$. Contradiction.

As in the case of the Minkowski problem, one may obtain higher regularity estimates for admissible solutions of equation (2.1). We may employ the degree theory to establish existence of admissible solutions to equation (2.1) provided that $\varphi$ satisfies (1.1), which is the necessary condition. This is an easy part. The main question is when a solution of (2.1) is geometric. By Proposition 1, solution $u$ is a support function of a convex body if and only if

$$W(x) = (u_{ij}(x) + \delta_{ij} u(x)) > 0, \forall x \in \mathbb{S}^n.$$  

When $k < n$, an admissible solution of (2.1) may not satisfy (2.8) in general. The following example is essentially due to Alexandrov. Let

$$u(x) = 1 - \frac{x_{n+1}^2}{2},$$

where $x_{n+1}$ is the $(n+1)$-th coordinate function. It is straightforward to check that this function satisfies

$$W(x) = (u_{ij}(x) + \delta_{ij} u(x)) \geq 0, \forall x \in \mathbb{S}^n,$$

The spherical Hessian $W$ is positive everywhere except on the great circle $x_{n+1} = 0$, the rank is $n-1$ there. For $1 \leq k < n$, there is $\delta_k > 0$, such that $u_\delta = u - \delta_k$ is an admissible solution to equation (2.1) for some positive analytic function $\varphi$, but one of eigenvalues of $W$ is negative on the great circle. The following theorem provides a sufficient condition.

**Theorem 2.2.** Suppose $0 < \varphi \in C^2(\mathbb{S}^n), \quad \forall x \in \mathbb{S}^n$, and satisfies equation (1.1). Suppose

$$(\varphi^\frac{1}{k})_{ij}(x) + \delta_{ij} \varphi^\frac{1}{k}(x) \geq 0, \forall x \in \sigma_{k+1}^n,$$

then there is a $C^{3,\alpha}(\forall 0 < \alpha < 1)$ strongly convex surface $M$ in $\mathbb{R}^{n+1}$, such that its support function satisfies equation (2.1). $M$ is unique up to translations.

We first prove two lemmas.

**Lemma 2.3.** For $1 \leq k \leq l$, $\lambda = (\lambda_1, ..., \lambda_l)$, $1 \leq i, j \leq l$, $i \neq j$, we denote by $\sigma_k(\lambda | i)$ the symmetric function with $\lambda_i = 0$ and $\sigma_k(\lambda | ij)$ the symmetric function with $\lambda_i = \lambda_j = 0$. Then the following hold,

$$\sigma_k(\lambda)\sigma_{l-1}(\lambda | \alpha)\sigma_{k-1}(\lambda | \alpha) - \sigma_l(\lambda)\sigma_{k-1}^2(\lambda | \alpha) = \sigma_l(\lambda | \alpha)\sigma_{l-1}(\lambda | \alpha)\sigma_{k-1}(\lambda | \alpha).$$

If $1 \leq k \leq l$, and $\alpha \neq \beta$,

$$\sigma_k(\lambda)\sigma_{k-2}(\lambda | \alpha \beta) - \sigma_{k-1}(\lambda | \alpha)\sigma_{k-1}(\lambda | \beta) = \sigma_k(\lambda | \alpha \beta)\sigma_{k-2}(\lambda | \alpha \beta) - \sigma_{k-1}^2(\lambda | \alpha \beta).$$
**Proof:** We first make a simple observation on $\sigma_l(\lambda)$ which will also be used repeatedly in the rest of the paper. As $l$ is equal to the size of $\lambda$, $\sigma_l(\lambda) = \lambda_1...\lambda_l$, and we have for $\alpha \neq \beta$ fixed,

$$\sigma_l(\lambda) = 0, \quad \sigma_l(\lambda) = \lambda_{\alpha}\sigma_{l-1}(\lambda|\alpha), \quad \sigma_l(\lambda) = \lambda_{\alpha\beta}\sigma_{l-2}(\lambda|\alpha\beta).$$  

From the definition of $\sigma_k(\lambda)$, we have the following identities:

$$\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i\sigma_{k-1}(\lambda|i),$$  

(2.12)

$$\sigma_k(\lambda) = \sigma_k(\lambda|ij) + \lambda_i\sigma_{k-1}(\lambda|ij) + \lambda_j\sigma_{k-1}(\lambda|ij) + \lambda_i\lambda_j\sigma_{k-2}(\lambda|ij).$$  

Now for any fixed $\alpha$,

$$\sigma_k(\lambda)\sigma_{l-1}(\lambda|\alpha)\sigma_{k-1}(\lambda|\alpha) - \sigma_l(\lambda)\sigma_{k-1}^2(\lambda|\alpha)$$

$$= [\lambda_{\alpha}\sigma_{k-1}(\lambda|\alpha) + \sigma_k(\lambda|\alpha)]\sigma_{l-1}(\lambda|\alpha)\sigma_{k-1}(\lambda|\alpha) - \sigma_l(\lambda)\sigma_{k-1}^2(\lambda|\alpha)$$

$$= \sigma_l(\lambda)\sigma_{k-1}(\lambda|\alpha)^2 + \sigma_k(\lambda|\alpha)\sigma_{l-1}(\lambda|\alpha)\sigma_{k-1}(\lambda|\alpha) - \sigma_l(\lambda)\sigma_{k-1}^2(\lambda|\alpha)$$

$$= \sigma_k(\lambda|\alpha)\sigma_{l-1}(\lambda|\alpha)\sigma_{k-1}(\lambda|\alpha).$$

The second identity in the lemma follows directly from the identities (2.12) and (2.13). □

**Lemma 2.4.** For $1 \leq k \leq l$, $\lambda = (\lambda_1,...,\lambda_l)$ and with $\lambda_i \geq 0$, for $1 \leq i \leq l$, $\forall \alpha \neq \beta$ and for all real numbers $\gamma_1,...,\gamma_l$,

$$\sum_{\alpha} \sigma_k(\lambda|\alpha)\sigma_{l-1}(\lambda|\alpha)\sigma_{k-1}(\lambda|\alpha)\gamma_\alpha^2$$

$$\geq \sigma_l(\lambda) \sum_{\alpha \neq \beta} (\sigma_{k-1}^2(\lambda|\alpha\beta) - \sigma_k(\lambda|\alpha\beta)\sigma_{k-2}(\lambda|\alpha\beta))\gamma_\alpha\gamma_\beta.$$  

(2.14)

**Proof:** For fixed $\alpha$,

$$\sum_{\beta \neq \alpha} \{\lambda_{\beta}\sigma_{k-1}^2(\lambda|\alpha\beta) - \lambda_{\beta}\sigma_k(\lambda|\alpha\beta)\sigma_{k-2}(\lambda|\alpha\beta)\}$$

$$= \sum_{\beta \neq \alpha} [\sigma_{k-1}(\lambda|\alpha\beta)\sigma_k(\lambda|\alpha) - \sigma_k(\lambda|\alpha\beta)(\sigma_{k-1}(\lambda|\alpha\beta) + \lambda_{\beta}\sigma_{k-2}(\lambda|\alpha\beta))]$$

$$= \sum_{\beta \neq \alpha} [\sigma_{k-1}(\lambda|\alpha\beta)\sigma_k(\lambda|\alpha) - \sigma_k(\lambda|\alpha\beta)\sigma_{k-1}(\lambda|\alpha)]$$

$$= \sigma_k(\lambda|\alpha)((l - k)\sigma_{k-1}(\lambda|\alpha) - (l - k - 1)\sigma_{k-1}(\lambda|\alpha))$$

(2.15)

$$= \sigma_k(\lambda|\alpha)\sigma_{k-1}(\lambda|\alpha).$$

Now we use the Cauchy inequality and (2.15) to prove (2.14). As

$$\sigma_l(\lambda) = \lambda_{\alpha}\lambda_{\beta}\sigma_{l-2}(\lambda|\alpha\beta), \forall \alpha \neq \beta,$$
we have

\[
\sigma_l(\lambda) \sum_{\alpha, \beta} \left[ \sigma_{k-1}^2(\lambda|\alpha\beta) - \sigma_k(\lambda|\alpha\beta) \sigma_{k-2}(\lambda|\alpha\beta) \right] \gamma_\alpha \gamma_\beta
\]

\[
= \sum_{\alpha, \beta} \left\{ \sigma_{l-2}(\lambda|\alpha\beta) \left[ \sigma_{k-1}^2(\lambda|\alpha\beta) - \sigma_k(\lambda|\alpha\beta) \sigma_{k-2}(\lambda|\alpha\beta) \right] \right\} (\lambda_\beta \gamma_\alpha) (\lambda_\alpha \gamma_\beta)
\]

\[
\leq \sum_{\alpha, \beta} \left\{ \sigma_{l-2}(\lambda|\alpha\beta) \left[ \sigma_{k-1}^2(\lambda|\alpha\beta) - \sigma_k(\lambda|\alpha\beta) \sigma_{k-2}(\lambda|\alpha\beta) \right] \right\} \frac{\lambda_\beta^2 \gamma_\alpha^2 + \lambda_\alpha^2 \gamma_\beta^2}{2}
\]

\[
= \sum_{\alpha, \beta} \sigma_{l-2}(\lambda|\alpha\beta) \lambda_\beta \left[ \sigma_{k-1}^2(\lambda|\alpha\beta) - \sigma_k(\lambda|\alpha\beta) \sigma_{k-2}(\lambda|\alpha\beta) \right] \lambda_\beta \gamma_\alpha^2
\]

\[
= \sum_{\alpha} \sigma_{l-1}(\lambda|\alpha) \sum_{\beta, \beta \neq \alpha} \lambda_\beta \left[ \sigma_{k-1}^2(\lambda|\alpha\beta) - \sigma_k(\lambda|\alpha\beta) \sigma_{k-2}(\lambda|\alpha\beta) \right] \gamma_\alpha^2
\]

\[
= \sum_{\alpha} \sigma_k(\lambda|\alpha) \sigma_{l-1}(\lambda|\alpha) \sigma_{k-1}(\lambda|\alpha) \gamma_\alpha^2.
\]

This completes the proof of (2.14). □

The following corollary indicates certain convexity of the \( k \)-th elementary symmetric functions, in contrast to the concavity property of \( \sigma_k^1 \) with used in the proof of \( C^2 \) estimate for admissible solutions.

**Corollary 2.5.** For \( \lambda = (\lambda_1, \cdots, \lambda_l) \in \mathbb{R}^l \) with \( \lambda_j > 0, \forall j = 1, \cdots, l \), for \( 1 \leq k \leq l \), set \( \eta(\lambda) = \log \sigma_k(\lambda) \). Then

\[
(2.16) \quad \sum_{\alpha, \beta} \frac{\partial^2 \eta}{\partial \lambda_\alpha \partial \lambda_\beta}(\lambda) \gamma_\alpha \gamma_\beta + \sum_\alpha \frac{\partial \eta}{\partial \lambda_\alpha}(\lambda) \lambda_\alpha^{-1} \gamma_\alpha^2 \geq 0, \quad \forall \gamma = (\gamma_1, \cdots, \gamma_l) \in \mathbb{R}^l.
\]

**Proof.** By Lemma 2.3 and Lemma 2.4,

\[
\sigma_l(\lambda) \sigma_k^2(\lambda) \sum_{\alpha, \beta} \frac{\partial^2 \eta}{\partial \lambda_\alpha \partial \lambda_\beta}(\lambda) \gamma_\alpha \gamma_\beta + \sum_\alpha \frac{\partial \eta}{\partial \lambda_\alpha}(\lambda) \lambda_\alpha^{-1} \gamma_\alpha^2
\]

\[
= \sum_\alpha \left[ \sigma_l(G) \sigma_{k-1}(G|\alpha) - \sigma_k(G) \sigma_{l-1}(G|\alpha) \sigma_{k-2}(G|\alpha) \right] \gamma_\alpha^2
\]

\[
+ \sigma_l(\lambda) \sum_{\alpha, \beta} \left[ \sigma_{k-1}(\lambda|\alpha) \sigma_{k-1}(\lambda|\beta) - \sigma_k(\lambda) \sigma_{k-2}(\lambda|\alpha\beta) \right] \gamma_\alpha \gamma_\beta
\]

\[
= - \sum_\alpha \sigma_k(\lambda|\alpha) \sigma_{l-1}(\lambda|\alpha) \sigma_{k-1}(\lambda|\alpha) \gamma_\alpha^2
\]

\[
+ \sigma_l(\lambda) \sum_{\alpha, \beta} \left[ \sigma_{k-1}^2(\lambda|\alpha\beta) - \sigma_k(\lambda|\alpha\beta) \sigma_{k-2}(\lambda|\alpha\beta) \right] \gamma_\alpha \gamma_\beta \leq 0.
\]
Theorem 2.2 can be deduced from following key Proposition, which is a form of strong maximum principle.

**Proposition 2.6.** Let \( O \subseteq \mathbb{S}^n \) be an open subset, suppose \( u \in C^4(O) \) is a solution of (2.1) in \( O \), and that the matrix \( W = (W_{ij}) \) is positive semi-definite. Suppose there is a positive constant \( C_0 > 0 \), such that for a fixed integer \((n - 1) \geq l \geq k \), \( \sigma_l(W(x)) \geq C_0 \) for all \( x \in O \). Let \( \phi(x) = \sigma_{l+1}(W(x)) \) and let \( \tau(x) \) be the largest eigenvalue of \( \{-(\varphi^{-\frac{1}{k}})_{ij}(x) - \delta_{ij}\varphi^{-\frac{1}{k}}(x)\} \). Then there are constants \( C_1, C_2 \) depending only on \( ||u||_{C^3} \), \( ||\varphi||_{C^{1,1}} \), \( n \), \( k \) and \( C_0 \), such that differential inequality

\[
(2.17) \sum_{\alpha,\beta} \sigma_{k}^{\alpha\beta}(x)\phi_{\alpha\beta}(x) \leq k(n - l)\varphi^{\frac{k+1}{k}}(x)\sigma_l(W(x))\tau(x) + C_1|\nabla \phi(x)| + C_2\phi(x)
\]

holds in \( O \), where

\[
(2.18) \quad \sigma_k^{\alpha\beta} = \frac{\partial \sigma_k(W)}{\partial W_{\alpha\beta}}, \quad \sigma_{k,ij}^{rs} = \frac{\partial \sigma_k(W)}{\partial W_{ij}}W_{rs}.
\]

We set

\[
(2.19) \quad \sigma_{l+1}^{ij} = \frac{\partial \sigma_{l+1}(W)}{\partial W_{ij}}, \quad \sigma_{l+1}^{ij,rs} = \frac{\partial^2 \sigma_{l+1}(W)}{\partial W_{ij}\partial W_{rs}}.
\]

Recall that \( \varphi(x) = \sigma_k(W(x)) \), and \( \phi(x) = \sigma_{l+1}(W(x)) \). Since \( W \) is positive semi-definite and \( u \) is \( k \)-convex, \( (\sigma_k^{\alpha\beta}) \) is positive definite and \( (\sigma_{l+1}^{ij}) \) is positive semi-definite. We observe that there are at least \( l \) positive eigenvalues of \( W \) with a controlled lower bound by the assumption \( \sigma_l(W) \geq C_0 \). Let \( B \) be that part of the index set so arranged such that the \( W_{ii} \) might be small (controlled by \( \phi \)) for \( i \in B \) (see the proof below for the precise definition). In view of this observation, \( W_{ii} \) is negligible for each \( i \in B \). The basic idea in the proof of Deformation Lemma is to explore the relationship between \( \sum_{\alpha,\beta} \sigma_k^{\alpha\beta}\phi_{\alpha\beta} \) and \( \varphi^{\frac{k+1}{k}}\sigma_l(W)\{\varphi^{-\frac{1}{k}}\}_{ii} + \delta_{ii}\varphi^{-\frac{1}{k}} \}. One of the key terms to be handled will be \( \sum_{i,\alpha} \sigma_{l+1}^{ii}\sigma_k^{\alpha\alpha}W_{i\alpha\alpha} \). With the help of some basic properties of elementary symmetric functions, it turns out that some algebraic cancellations will occur after commuting covariant derivatives and re-arranging the terms to fit the right algebraic formats! Almost all of the computations in the proof are algebraic and the inequality in Lemma 2.4 will be used in a crucial way in the last step of the proof.

**Proof the Proposition.** For two functions defined in an open set \( O \subseteq \mathbb{S}^n \), \( y \in O \), we say that \( h(y) \lesssim k(y) \) provided there exist positive constants \( c_1 \) and \( c_2 \) such that

\[
(2.20) \quad (h - k)(y) \leq (c_1|\nabla \phi| + c_2\phi)(y).
\]

We also write \( h(y) \sim k(y) \) if \( h(y) \lesssim k(y) \) and \( k(y) \lesssim h(y) \). Next, we write \( h \lesssim k \) if the above inequality holds in \( O \), with the constants \( c_1 \), and \( c_2 \) depending only on \( ||u||_{C^3} \),
This relation yields that, for \( 1 \leq l \leq k \) and \( k \leq h \). We shall show that

\[
\sum_{\alpha, \beta = 1}^{n} \sigma_k^{\alpha \beta} \phi_{\alpha \beta} \lesssim k(n - l) \varphi^{k+1} \sigma_l(W) \tau.
\]

For any \( z \in O \), let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) be the eigenvalues of \( W \) at \( z \). Since \( \sigma_l(W) \geq C_0 > 0 \) and \( u \in C^3 \), for any \( z \in \mathbb{S}^n \), there is a positive constant \( C > 0 \) depending only on \( \|u\|_{C^3} \), \( ||\varphi||_{C^2} \), \( n \) and \( C_0 \), such that \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq C \). Let \( G = \{1, 2, \ldots, l\} \) and \( B = \{l + 1, \ldots, n\} \) be the “good” and “bad” sets of indices respectively, and define \( \sigma_k(W[i]) = \sigma_k([W[i]]) \) where \( (W[i]) \) means that the matrix \( W \) excluding the \( i \)-column and \( i \)-row, and \( (W[ij]) \) means that the matrix \( W \) excluding the \( i, j \) columns and \( i, j \) rows. Let \( \Lambda_G = (\lambda_1, \ldots, \lambda_l) \) be the ”good” eigenvalues of \( W \) at \( z \); for convenience in notation, we also write \( G = \Lambda_G \) if there is no confusion. In the following, all calculations are at the point \( z \) using the relation \( \lesssim \), with the understanding that the constants in (2.20) are under control.

For each fixed \( z \in O \) fixed, we choose a local orthonormal frame \( e_1, \ldots, e_n \) so that \( W \) is diagonal at \( z \), and \( W_{ii} = \lambda_i, \forall i = 1, \ldots, n \). Now we compute \( \phi \) and its first and second derivatives in the direction \( e_a \).

We note that \( \sigma_{l+1}^{ij} \) in (2.19) is diagonal at the point since \( W \) is diagonal. As \( \phi = \sigma_{l+1}(W) \) and \( \phi_\alpha = \sum_{i,j} \sigma_{l+1}^{ij} W_{ija} \), we find that (as \( W \) is diagonal at \( z \)),

\[
0 \sim \phi(z) \sim \sum_{i \in B} W_{i} \sigma_l(G) \sim \sum_{i \in B} W_{ii}, \quad \text{(so \ } W_{ii} \sim 0, \ i \in B),
\]

This relation yields that, for \( 1 \leq m \leq l \),

\[
\sigma_m(W) \sim \sigma_m(G), \quad \sigma_m(W[j]) \sim \begin{cases} \sigma_m(G[j]), & \text{if } j \in G; \\ \sigma_m(G), & \text{if } j \in B. \end{cases}
\]

\[
\sigma_m(W[ij]) \sim \begin{cases} \sigma_m(G[ij]), & \text{if } i, j \in G; \\ \sigma_m(G[j]), & \text{if } i \in B, j \in G; \\ \sigma_m(G), & \text{if } i, j \in B, i \neq j. \end{cases}
\]

Also,

\[
0 \sim \phi_\alpha \sim \sigma_l(G) \sum_{i \in B} W_{i\alpha a} \sim \sum_{i \in B} W_{i\alpha a}
\]

and

\[
\sigma_{l+1}^{ij} \sim \begin{cases} \sigma_l(G), & \text{if } i = j \in B, \\ 0, & \text{otherwise}. \end{cases}
\]

\[
\sigma_{l+1}^{ij,rs} = \begin{cases} \sigma_{l-1}(W[ir]), & \text{if } i = j, r = s, i \neq r; \\ -\sigma_{l-1}(W[ij]), & \text{if } i \neq j, r = j, s = i; \\ 0, & \text{otherwise}. \end{cases}
\]
Since \( \phi_{\alpha\alpha} = \sum_{i,j} (\sigma_{l+1}^{ij} W_{rs} W_{ij} + \sigma_{l+1}^{ij} W_{ij\alpha\alpha}) \), it follows from (2.26) that for any \( \alpha \in \{1, 2, \ldots, n\} \)
\[
\phi_{\alpha\alpha} = \sum_{i \neq j} \sigma_{l-1}(W_{ij}) W_{iia} W_{jj} - \sum_{i \neq j} \sigma_{l-1}(W_{ij}) W_{iij}^2 + \sum_i \sigma_{l+1}^{ii} W_{ii\alpha\alpha}
\]
\[
= \left( \sum_{i \in G} + \sum_{i \in B} + \sum_{i \neq j} + \sum_{i \neq j} \right) \sigma_{l-1}(W_{ij}) W_{iia} W_{jj} - \left( \sum_{i \in G} + \sum_{i \in B} + \sum_{i \neq j} + \sum_{i \neq j} \right) \sigma_{l-1}(W_{ij}) W_{ij}^2 + \sum_i \sigma_{l+1}^{ii} W_{ii\alpha\alpha}.
\]

(2.27)

From (2.24) and (2.23),
\[
\sum_{i \in B} \sigma_{l-1}(W_{ij}) W_{iia} W_{jj} \sim \left( \sum_{j \in G} \sigma_{l-1}(G_{ij}) W_{jja} \right) \sum_{i \in B} W_{iia} \sim 0.
\]

Since \( 0 \leq W_{mm} \in C^2 \) for any unit vector field, we have \( \nabla W_{mm}(x) \leq C \sqrt{W_{mm}(x)} \). This implies that \( \nabla W_{ij}(x) \leq C(\sqrt{W_{ii}(x)} + \sqrt{W_{jj}(x)}) \). By (2.24), \( \forall i \in B \) fixed and \( \forall \alpha \), therefore,
\[
\sum_{i,j \in B} \sigma_{l-1}(W_{ij}) W_{iia} W_{jj} \sim 0.
\]
and
\[
\sum_{j \in G, i \in B} \sigma_{l-1}(W_{ij}) W_{ij}^2 \sim \sum_{i \in B, j \in G} \sigma_{l-1}(G_{ij}) W_{ij}^2.
\]

(2.30)

Inserting (2.28)-(2.30) into (2.27), by (2.23) we obtain
\[
\phi_{\alpha\alpha} \sim \sum_{i} \sigma_{l+1}^{ii} W_{iia\alpha} - 2 \sum_{i \in B} \sigma_{l-1}(G_{ij}) W_{ij}^2.
\]

(2.31)

Thus,
\[
 \sum_{\alpha, \beta} \sigma_{k}^{\alpha\beta} \phi_{\alpha\beta} = \sum_{\alpha=1}^{n} \sigma_{k}^{\alpha\alpha} \phi_{\alpha\alpha} \sim \sum_{\alpha=1}^{n} \sum_{i} \sigma_{l+1}^{ii} \sigma_{k}^{\alpha\alpha} W_{iia\alpha}
\]
\[
-2 \sum_{\alpha=1}^{n} \sum_{j \in G} \sigma_{l-1}(G_{ij}) \sigma_{k}^{\alpha\alpha} W_{ij}^2
\]

(2.32)

By (2.22), (2.25) and homogeneity of \( \sigma_k \) and \( \sigma_{l+1} \) (since \( |B| = n - l \))
\[
\sum_{\alpha=1}^{n} \sum_{i=1}^{n} \sigma_{l+1}^{ii} \sigma_{k}^{\alpha\alpha} (W_{ii} - W_{\alpha\alpha}) = (l + 1) \phi \sum_{\alpha=1}^{n} \sigma_{k}^{\alpha\alpha} - k \varphi \sum_{i=1}^{n} \sigma_{l+1}^{ii} 
\]
\[
\sim -k \varphi \sum_{i \in B} \sigma_{l+1}^{ii} \sim -(n - l) k \varphi \sigma_l(G).
\]
Commuting the covariant derivatives, it follows that
\[
\sum_{\alpha=1}^{n} \sum_{i=1}^{n} \sigma_{l+1}^{\alpha i} \alpha^a W_{i\alpha a} = \sum_{\alpha=1}^{n} \sum_{i=1}^{n} \sigma_{l+1}^{\alpha i} \alpha^a (W_{\alpha a i} + W_{i i} - W_{\alpha a i})
\]
(2.33)
\[
\sim \sum_{\alpha=1}^{n} \sum_{i=1}^{n} \sigma_{l+1}^{\alpha i} \alpha^a W_{\alpha a i} - (n - l)k\varphi \sigma_l(G).
\]

Differentiating equation (2.1), we get
\[
\varphi_{ii} = \sum_{\alpha,\beta} \sigma_{k}^{\alpha \beta, rs} W_{\alpha \beta} W_{rs i} + \sum_{\alpha,\beta} \sigma_{k}^{\alpha \beta} W_{\alpha \beta i i}.
\]
(2.23) and (2.25) yield,
\[
\sum_{\alpha} \sum_{i} \sigma_{l+1}^{\alpha i} \alpha^a W_{\alpha a i} \sim \sum_{\alpha} \sum_{i} \alpha \varphi_{ii} - \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \alpha - 2 \sum_{\alpha,\beta,\alpha,\beta} \alpha \varphi_{i i} - \sum_{\alpha} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \alpha - 2 \sum_{\alpha} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \alpha \varphi_{i i} + \alpha \varphi_{ii} + \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sigma_{k-2}(W_{|\alpha \beta} W_{\alpha \beta i i}) \sigma_l(G).
\]
(2.34)

It follows from (2.23) and (2.24) that for $1 \leq m \leq n$,
\[
\sum_{\alpha,\beta} \sigma_{m}(W_{|\alpha \beta} W_{\alpha \beta i i}) \sim \sum_{\alpha,\beta} \sigma_{m}(G_{|\beta} W_{\beta \beta i i}) \sum_{\alpha} W_{\alpha a i} \sim 0.
\]
(2.35)

In turn,
\[
\sum_{\alpha=1}^{n} \sum_{i=1}^{n} \sigma_{l+1}^{\alpha i} \alpha^a W_{\alpha a i} \sim \sigma_l(G) \sum_{i \in B} \varphi_{ii} - \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \alpha - 2 \sum_{\alpha} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \alpha \varphi_{i i} + \alpha \varphi_{ii} + \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sigma_{k-2}(W_{|\alpha \beta} W_{\alpha \beta i i}) \sigma_l(G) + \sum_{\alpha=1}^{n} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sigma_{k-2}(W_{|\alpha \beta} W_{\alpha \beta i i}) \sigma_l(G).
\]
(2.36)

We note that $|B| = n - l$, so $\sum_{i \in B} k\varphi = (n - l)k\varphi$. Now inserting (2.36) and (2.33) to (2.32), by (2.23) and (2.23) we have
\[
\sum_{\alpha,\beta} \sigma_{l}^{\alpha \beta} \varphi_{ii} - \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \alpha - 2 \sum_{\alpha} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \alpha \varphi_{i i} + \alpha \varphi_{ii} + \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sigma_{k-2}(W_{|\alpha \beta} W_{\alpha \beta i i}) \sigma_l(G) + \sum_{\alpha=1}^{n} \sum_{\alpha,\beta} \sum_{\alpha,\beta} \sigma_{k-2}(W_{|\alpha \beta} W_{\alpha \beta i i}) \sigma_l(G).
\]
(2.37)
When \( \alpha, \beta \in G, \alpha \neq \beta \), as \( W \) is diagonal,
\[
\sigma_{l-1}(G|\beta)\sigma_{k-1}(G|\alpha) = \sigma_{l-1}(G|\beta)[\sigma_{k-1}(G|\alpha) + W_{\beta\beta}\sigma_{k-2}(G|\alpha)]
\]
(2.38)
\[
g \geq \sigma_{l-1}(G|\beta)W_{\beta\beta}\sigma_{k-2}(G|\alpha) = \sigma_{l}(G)\sigma_{k-2}(G|\alpha).
\]
From (2.38), we get
\[
\sum_{i \in B} \sum_{\alpha,\beta \in G, \alpha \neq \beta} \sigma_{l}(G)\sigma_{k-2}(W|\alpha)W_{\alpha\beta i}^2 - 2\sum_{i \in B} \sum_{\alpha,\beta \in G, \alpha \neq \beta} \sigma_{l-1}(G|\beta)\sigma_{k-1}(G|\alpha)W_{\alpha\beta i}^2
\]
(2.39)
\[
\lesssim -\sum_{i \in B} \sum_{\alpha,\beta \in G, \alpha \neq \beta} \sigma_{l-1}(G|\beta)\sigma_{k-1}(G|\alpha)W_{\alpha\beta i}^2 \leq 0.
\]
As \( W_{i\beta\alpha} = W_{\alpha\beta i} \) on the standard \( S^n \) (recall that \( W_{\alpha\beta} = u_{\alpha\beta} + \delta_{\alpha\beta}u \)). We have
\[
\sigma_{l}(G)\sum_{i \in B} \sum_{\alpha \neq \beta} \sigma_{k-2}(W|\alpha)W_{\alpha\beta i}^2 - 2\sum_{\alpha = 1}^{n} \sum_{i \in B} \sum_{\beta \in G} \sigma_{l-1}(G|\beta)\sigma_{k-1}(W|\alpha)W_{\alpha\beta i}^2
\]
\[
\lesssim -2\sum_{i \in B} \sum_{\alpha \in G} \sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2.
\]
We note that \( \sigma_{m}(W|\alpha\beta) \sim \sigma_{m}(G), \forall \alpha, \beta \in B \), putting the previous inequality into (2.37),
\[
\sum_{\alpha,\beta} \sigma_{k}^{\alpha\beta} \phi_{\alpha\beta} \lesssim \sigma_{l}(G)\sum_{i \in B} (\varphi_{ii} - k\varphi) - \sum_{i \in B} \sum_{\alpha,\beta \in G, \alpha \neq \beta} \sigma_{k-2}(G|\alpha)W_{\alpha\alpha i}W_{\beta\beta i}]
\]
\[
-2\sum_{i \in B} \sum_{\alpha \in G} \sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2
\]
(2.40)
\[
= \sigma_{l}(G)\sum_{i \in B} (\varphi_{ii} - k\varphi - \frac{k + 1}{k}\varphi) + I_1 + I_2,
\]
where
\[
I_1 = \sum_{i \in B} \frac{\sigma_{l}(G)\varphi_{i}^2}{k\varphi} - \sum_{\alpha \in G} \sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2,
\]
and
\[
I_2 = \sum_{i \in B} \{\sigma_{l}(G)(\frac{\varphi_{i}^2}{\varphi} - \sum_{\alpha,\beta \in G, \alpha \neq \beta} \sigma_{k-2}(G|\alpha)W_{\alpha\alpha i}W_{\beta\beta i}] - \sum_{\alpha \in G} \sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2 \}.
\]
For \( i \in B \),
\[
\varphi_i = (\sum_{\alpha \in B} + \sum_{\alpha \in G}) \sigma_{k-1}(W|\alpha)W_{\alpha\alpha i} \sim \sum_{\alpha \in G} \sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}.
\]
It follows that for any $i \in B$,
\[
\varphi_i^2 \sim \sum_{\alpha \in G} \sigma_{k-1}^2(G|\alpha)W_{\alpha\alpha i}^2 + \sum_{\alpha, \beta \in G, \alpha \neq \beta} \sigma_{k-1}(G|\alpha)\sigma_{k-1}(G|\beta)W_{\alpha\alpha i}W_{\beta\beta i}.
\]

By Corollary 2.5, $I_2 \lesssim 0$.

By homogeneity of $\sigma_k(W)$ and (2.41),
\[
I_1 \sim \frac{1}{k^2} \left( \sum_{\alpha \in G} \sigma_{l-1}^2(G|\alpha)W_{\alpha\alpha i}^2 - \sum_{\alpha \in G} \sigma_{l-1}(G|\alpha)\sigma_{l-1}(G|\alpha)W_{\alpha\alpha i}^2 \right) - \sum_{\alpha \in G} \sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2
\]
\[
\leq \frac{1}{k^2} \sum_{\alpha, \beta \in G} \sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2 \sigma_{k-1}(G|\beta)
\]
\[
- \sum_{\alpha \in G} \sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2
\]
\[
\sim \sum_{\alpha \in G} \sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2 - \sum_{\alpha \in G} \sigma_{l-1}(G|\alpha)\sigma_{k-1}(G|\alpha)W_{\alpha\alpha i}^2
\]
\[
= 0.
\]

The proof of the Proposition is complete. \hfill \Box

3. Equation of the Prescribing Curvature Measure

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $M$. The Steiner formula yields that the $(n-k)$-th curvature measure of $\Omega$ is $\sigma_k(\kappa)\mu_M$, where $\mu_M$ is the surface area element of $M$. That is, for each Borel set $\beta \subset \mathbb{R}^n$ can be defined as
\[
\mathcal{C}_k(M, \beta) := \int_{\beta \cap M} \sigma_k(\kappa)d\mu_M.
\]

Following Alexandrov, one may propose the problem of prescribing $(n-k)$-th curvature measure on $\mathbb{S}^n$, via radial map by assuming the origin is inside $\Omega$ and $\Omega$ is starshaped (with respect to the origin). One may parametrize $M = \partial \Omega$ over $\mathbb{S}^n$ by positive radial function $\rho$. Due to the parametrization, the prescribe curvature measure problem for this class of domains can be reduced to a curvature type nonlinear partial differential equation of $\rho$ on $\mathbb{S}^n$. We want to establish the existence theorems of prescribing general $(n-k)$-th curvature measure problem with $k > 0$ on bounded $C^2$ star-shaped domains. When $k = n$, the prescribing curvature measure $\mathcal{C}_0$ is the Alexandrov problem corresponding to a Monge-Ampère type equation on $\mathbb{S}^n$, which won’t be treated here.

In order to make the problem in proper PDE setting, we need to impose some geometric condition on $\partial \Omega$.

**Definition 3.1.** A domain $\Omega$ is called $k$-convex if its principal curvature vector $\kappa(x) = (\kappa_1, \ldots, \kappa_n) \in \Gamma_k$ at every point $x \in \partial \Omega$.
For each star-shaped domain $\Omega$ with $M = \partial \Omega$, express $M$ as a radial graph of $\mathbb{S}^n$,
\[
R_M : \mathbb{S}^n \rightarrow M \quad z \mapsto \rho(z)z.
\]

We want to solve the following problem: given a positive function $f \in C^2(\mathbb{S}^n)$, find a closed hypersurface $M$ as a radial graph over $\mathbb{S}^n$, such that $C_{n-k}(M, \beta) = \int_{\beta} f d\mu$ for every Borel set $\beta$ in $\mathbb{S}^n$, where $d\mu$ is the standard volume element on $\mathbb{S}^n$.

For the $C^2$ graph $M$ on $\mathbb{S}^n$, denote the induced metric to be $g$ and the density function is $\sqrt{\det g}$. Then
\[
C_{n-k}(M, \beta) = \int_{M \cap \beta} \sigma_k d\mu_M = \int_{R^{-1}(\beta)} \sigma_k \sqrt{\det g} d\mathbb{S}^n.
\]

Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame on $\mathbb{S}^n$, and denote $e_{ij}$ the standard spherical metric with respect to this frame (which is the identity matrix). Denote $\nabla$ as the gradient operator with respect to standard metric on $\mathbb{S}^n$. To simplify notation, for any function $v$ on $\mathbb{S}^n$, we will write $\nabla e_i v = \tau_i$ as covariant derivative with respect to $e_i$ on $\mathbb{S}^n$, if there is no confusion arising. By the radial parametrization $X(x) = \rho(x)x$,
\[
X_i = \rho_i x + pe_i,
\]
\[
X_{ij} = \rho_{ij} x + \rho_i e_j + \rho_j e_i + \rho(e_i) e_j = \rho_{ij} x + \rho_i e_j + \rho_j e_i - pe_{ij} x.
\]
We deduce that,
\[
\nu = \frac{\rho^2 - \sqrt{\rho} \rho}{\sqrt{\rho^2 + \left| \nabla \rho \right|^2}}, \quad u = \frac{\rho^2}{\sqrt{\rho^2 + \left| \nabla \rho \right|^2}}
\]
\[
\begin{aligned}
g_{ij} &= \rho^2 \delta_{ij} + \rho_i \rho_j, & g^{ij} &= \frac{1}{\rho^2} (\delta^{ij} - \frac{\rho_i \rho_j}{\rho^2 + \left| \nabla \rho \right|^2}), & \sqrt{\det g} &= \rho^{n-1} \sqrt{\rho^2 + \left| \nabla \rho \right|^2}, \\
h_{ij} &= (\sqrt{\rho^2 + \left| \nabla \rho \right|^2})^{-1} (\rho \nabla_i \nabla_j \rho + 2 \rho_i \rho_j + \rho^2 e_{ij}) \\
h^i_j &= \frac{1}{\rho^2 \sqrt{\rho^2 + \left| \nabla \rho \right|^2}} (e^{ik} - \frac{\rho_i \rho_k}{\rho^2 + \left| \nabla \rho \right|^2}) (\rho \nabla_k \nabla_{ij} \rho + 2 \rho_k \rho_j + \rho^2 e_{kj}).
\end{aligned}
\]

It follows that, the prescribing $(n-k)$-th curvature measure problem is equivalent to the following curvature equation on $\mathbb{S}^n$:
\[
\sigma_k(\kappa_1, \cdots, \kappa_n) = \sigma_k(h^i_j) = \frac{f}{\rho^{n-1} \sqrt{\rho^2 + \left| \nabla \rho \right|^2}},
\]
where $f > 0$ is the given function on $\mathbb{S}^n$. A solution of (3.3) is called admissible if $\kappa(X) \in \Gamma_k$ at each point $X \in M$. Note that any positive $C^2$ function $\rho$ on $\mathbb{S}^n$ satisfying equation (3.3) is automatically an admissible solution. This is so because $\Gamma_k$ and $\mathbb{S}^n$ are connected, and the principal curvatures at a maximum point of $\rho$ are non-negative.

The following theorem provides unique solution to the problem.

**Theorem 3.2.** Let $n \geq 2$ and $1 \leq k \leq n - 1$. Suppose $f \in C^2(\mathbb{S}^n)$ and $f > 0$. Then there exists a unique $k$-convex star-shaped hypersurface $M \in C^{3,\alpha}$, $\forall \alpha \in (0,1)$ such that it satisfies (3.3). Moreover, there is a constant $C$ depending only on $k, n, \|f\|_{C^{1,1}}, \|1/f\|_{C^{\alpha}}$, and $\alpha$ such that,
\[
\|\rho\|_{C^{1,\alpha}} \leq C.
\]
It will be convenient to introduce a new variable $\gamma = \log \rho$. Set

$$\omega := \sqrt{1 + |\nabla \gamma|^2}.$$  

The unit outward normal and support function can be expressed as $\nu = \frac{1}{\omega}(1, -\gamma_1, \cdots, -\gamma_n)$ and $u = \frac{\omega}{\omega}$ respectively. Moreover,

$$g_{ij} = e^{2\gamma}(\delta_{ij} + \gamma_i \gamma_j),$$

$$g^{ij} = e^{-2\gamma}(e^{ij} - \frac{\gamma_i \gamma_j}{\omega^2}),$$

$$h_{ij} = \frac{e^{\gamma}}{\omega}(-\gamma_{ij} + \gamma_i \gamma_j + e_{ij})$$

$$h^i_j = \frac{e^{-\gamma}}{\omega}(e^{ik} - \frac{\gamma_k \gamma_i}{\omega^2})(-\gamma_{kj} + \gamma_k \gamma_j + e_{kj}).$$

The Weingarten tensor in (3.5) is in general not symmetric with respect local orthonormal frames $(e_1, \cdots, e_n)$ on $S^n$, even though it is symmetric with respect to local orthonormal frames on $M$. We observe that the symmetric matrix $(e^{ij} - \frac{\gamma_i \gamma_j}{\omega^2})$ has an obvious square root $S$. That is,

$$(3.6) \quad S = (\sigma_{ij}) = \left( e_{ij} - \frac{\gamma_i \gamma_j}{\omega (\omega + 1)} \right), \quad \left( e_{ij} - \frac{\gamma_i \gamma_j}{\omega^2} \right) = S^2.$$  

$S$ can be used to symmetrize the Weingarten tensor. The eigenvalues of $(h^i_j)$ is the same as eigenvalues of $\frac{e^{-\gamma}}{\omega} B$, with $B$ defined as

$$(3.7) \quad B = : (b_{ij}) = S(-\gamma_{lm} + \gamma_l \gamma_m + e_{lm}) S$$

$$= (-\gamma_{ij} + \delta_{ij} + \sum_l (\gamma_i \gamma_{lj} + \gamma_j \gamma_{li}) \gamma_l \omega (\omega + 1) - \frac{\gamma_i \gamma_j \gamma_l \gamma_m \gamma_l \gamma_m}{\omega^2 (1 + \omega)^2}).$$

Equation (3.3) can be rewritten as

$$(3.8) \quad \frac{e^{(n-k)\gamma}}{\omega^{k-1}} \sigma_k(B) = f.$$  

As $B$ is a function in $\nabla^2 \gamma, \nabla \gamma$ only, it is independent of $\gamma$. Set

$$(3.9) \quad \tilde{F}(\nabla^2 \gamma, \nabla \gamma) = -\sigma_k(B).$$

Denote $\sigma_{ij}^k(B) = \frac{\partial \sigma_k}{\partial b_{ij}}$, we compute

$$(3.10) \quad (\tilde{\sigma}_k)_{ij} = \left( \frac{\partial \tilde{F}}{\partial \gamma_{ij}} \right) = S(\sigma_k(B)) S.$$  

Since $S$ in (3.6) is positive definite, we have $(\frac{\partial \tilde{F}}{\partial \gamma_{ij}}) > 0$.

### 3.1. Uniqueness and $C^1$-estimates.

**Lemma 3.3.** Let $1 \leq k < n$. Let $L$ denote the linearized operator at a solution $\rho$ of (3.3), if $v$ satisfies $L(v) = 0$ on $S^n$, then $v \equiv 0$ on $S^n$. Moreover, suppose $\rho, \tilde{\rho}$ are two solutions of equation (3.3) and $\lambda(\rho_i) \in \Gamma_k$, for $i = 1, 2$. Then $\rho_1 \equiv \rho_2$.  

\textbf{Proof.} \eqref{eq:3.8} can be put in the form of
\begin{equation}
\frac{e^{(n-k)\gamma}}{\omega_k^{k-1}} \tilde{F}(\nabla^2 \gamma, \nabla \gamma) = -f.
\end{equation}
The linearized operator at $\gamma$ is
\begin{equation}
L(v) = \frac{e^{(n-k)\gamma}}{\omega_k^{k-1}} \tilde{\sigma}_{ij}^k v_{ij} + \sum_l b_l v_l - (n-k)f v,
\end{equation}
for some function $b_l, l = 1, \ldots, n$. The first statement in lemma follows immediately from the maximum principle.

Suppose $\gamma = \log \rho$ and $\tilde{\gamma} = \log \tilde{\rho}$ are two solutions of equation \eqref{eq:3.8}, denote $\tilde{\omega} = \sqrt{1 + |\nabla \tilde{\gamma}|^2}$ and $\tilde{B}$ to be the corresponding tensor $B$ in \eqref{eq:3.7} with $\gamma$ replaced by $\tilde{\gamma}$. For $t \in [0, 1]$, set
\begin{equation}
\gamma^t = t \gamma + (1-t)\tilde{\gamma}, \quad \omega_t = \sqrt{1 + |\nabla \gamma^t|^2}, \quad B^t = tB + (1-t)\tilde{B}.
\end{equation}
Set $v = \gamma - \tilde{\gamma}$, as $B^t \in \Gamma_k$,
\begin{align*}
0 &= \frac{e^{(n-k)\gamma}}{\omega_k^{k-1}} F(B) - \frac{e^{(n-k)\tilde{\gamma}}}{\omega_k^{k-1}} F(\tilde{B}) \\
&= \int_0^1 \frac{d}{dt} \left( \frac{e^{(n-k)\gamma^t}}{\omega_k^{k-1}} F(B^t) \right) dt \\
&= \int_0^1 (n-k) \frac{e^{(n-k)\gamma^t}}{\omega_k^{k-1}} F(B^t) dt + \int_0^1 \left( \frac{e^{(n-k)\gamma^t}}{\omega_k^{k-1}} \tilde{\sigma}_{ij}^k (B^t) \right) dt (b_{ij} - \tilde{b}_{ij}) + \text{mod(}\nabla v\text{)}.
\end{align*}
Write $S = (S_{ij}^k)$, and observe that $S$ only involves $\nabla \gamma, \nabla^2 \gamma$ (and so is $\tilde{S}$), by the Mean Value Theorem,
\begin{equation}
B - \tilde{B} = -S(\nabla^2 v)S + \text{mod(}\nabla v\text{)},
\end{equation}
and
\begin{equation}
0 = (\int_0^1 (n-k) \frac{e^{(n-k)\gamma^t}}{\omega_k^{k-1}} \tilde{F}(B^t) dt) v - \int_0^1 \left( \frac{e^{(n-k)\gamma^t}}{\omega_k^{k-1}} \tilde{\sigma}_{ij}^k (B^t) \right) dt S^\alpha S_{ij}^\beta v_{\alpha\beta} + \text{mod(}\nabla v\text{)}.
\end{equation}
Since $\int_0^1 \left( \frac{e^{(n-k)\gamma^t}}{\omega_k^{k-1}} \tilde{\sigma}_{ij}^k (B^t) \right) dt S^\alpha S_{ij}^\beta > 0$, $\int_0^1 (n-k) \frac{e^{(n-k)\gamma^t}}{\omega_k^{k-1}} \tilde{F}(B^t) dt > 0$, $v$ satisfies the following elliptic equation,
\begin{equation}
a_{ij}(x)v_{ij}(x) + b_k(x)v_k(x) + c(x)v(x) = 0, \quad \forall x \in \mathbb{S}^n,
\end{equation}
with $c(x) < 0$ for all $x \in \mathbb{S}^n$. The maximum principle yields $v \equiv 0$. That is $\rho = \tilde{\rho}$. \hfill $\square$

Recall $\bar{\sigma}_{ij}^k = \frac{\partial x_k}{\partial y_{ij}}$, define
\begin{equation}
(\bar{\sigma}_{ij}^k) = S(\bar{\sigma}_{ij}^k)S.
\end{equation}
Lemma 3.4. For any $C^1$ symmetric function $F(B)$, set $\phi = \frac{\nabla \gamma^2}{2}$, then there exist $c_m$ depending on $(\nabla^2 \gamma, \nabla \gamma, F)$, such that
\[
\tilde{\sigma}^{ij}_k \phi_{ij} = \sum_m c_m \phi_m - \sum_l \gamma_l(F(B))_l + \sigma^{ij}_k(\delta_{ij} |\nabla \gamma|^2 - \gamma_j \gamma_i + \delta_{ij} \gamma_i^2).
\]

Proof. By (3.7),
\[
\phi_{ij} = \sum_l (\gamma_l \gamma_{ij} + \gamma_i \gamma_{lj})
\]
\[
= \sum_l (\gamma_l (\gamma_{lij} + \delta_l \gamma_i - \gamma_j \delta_{il}) + \gamma_i \gamma_{lj})
\]
\[
= \sum_l (\gamma_l (\gamma_{ijl} + \delta_{ij} \gamma_l - \gamma_j \delta_{il}) + \gamma_i \gamma_{lj})
\]
\[
= \sum_l \gamma_l (-b_{ijl} + \left( \frac{\gamma_l \phi_{ij} + \gamma_j \phi_i}{\omega(\omega + 1)} - \frac{\gamma_i \gamma_j \sum_m \gamma_m \phi_m}{\omega^2(1 + \omega)^2} \right)_l)
\]
\[
+ \delta_{ij} |\nabla \gamma|^2 - \gamma_j \gamma_i + \gamma_i \gamma_{lj}^2
\]
\[
= \sum_l \gamma_l (-b_{ijl} + \left( \frac{\gamma_i \gamma_{lij} + \gamma_j \phi_{il}}{\omega(\omega + 1)} - \frac{\gamma_i \gamma_j \sum_m \gamma_m \phi_{ml}}{\omega^2(1 + \omega)^2} \right)_l)
\]
\[
+ \delta_{ij} |\nabla \gamma|^2 - \gamma_j \gamma_i + \gamma_i \gamma_{lj}^2 + c_m \phi_m,
\]
where we used the fact that tensor $A_{ij} = \gamma_{ij} + \gamma e_{ij}$ is Codazzi for any function $\gamma \in C^3(S^n)$.

The above identity can be rewritten as
\[
\phi_{ij} = \sum_l \left( \frac{\gamma_i \gamma_{lij} + \gamma_j \phi_{il}}{\omega(\omega + 1)} - \frac{\gamma_i \gamma_j \sum_m \gamma_m \phi_{ml}}{\omega^2(1 + \omega)^2} \right)_l c_m \phi_m
\]
\[
+ \delta_{ij} |\nabla \gamma|^2 - \gamma_j \gamma_i + \gamma_i \gamma_{lj}^2 - \sum_l \gamma_l b_{ijl},
\]
or equivalently
\[
S \nabla^2 \phi S - (c_m \phi_m) = |\nabla \gamma|^2 I - (\gamma_i \gamma_j) + (\nabla^2 \gamma^2) - (\sum_l \gamma_l b_{ijl}).
\]

Set $c_m = \sum_{ij} \sigma^{ij}_k c_m$, contracting above identity with $\sigma^{ij}_k$, it follows from (3.12),
\[
\tilde{\sigma}^{ij}_k \phi_{ij} - \sum_m c_m \phi_m = - \sum_l \sigma^{ij}_k (B) \gamma_l b_{ijl} + \sigma^{ij}_k (\delta_{ij} |\nabla \gamma|^2 - \gamma_j \gamma_i + \delta_{ij} \gamma_i^2)
\]
\[
= - \sum_l \gamma_l(F(B))_l + \sigma^{ij}_k (\delta_{ij} |\nabla \gamma|^2 - \gamma_j \gamma_i + \delta_{ij} \gamma_i^2).
\]

Proposition 3.5. If $M$ satisfies (3.3), then
\[
\left( \frac{\min_{S^n} f}{C_k} \right)^{\frac{1}{n-k}} \leq \min_{S^n} |X| \leq \max_{S^n} |X| \leq \left( \frac{\max_{S^n} f}{C_k} \right)^{\frac{1}{n-k}}
\].
Moreover, there exists a constant $C$ depending only on $n$, $k$, $\min_{S^n} f$, $|f|_{C^1}$ such that
\[
\max_{S^n} |\nabla \rho| \leq C.
\]

Proof. $(\gamma_{ij})$ is semi-negative definite at maximum point of $\rho$ and $\nabla \gamma = 0$. By (3.15),
\[
f = \frac{e^{(n-k)\gamma}}{\omega^{k-1}} \sigma_k(B) = e^{(n-k)\gamma} \sigma_k(B) \geq e^{(n-k)\gamma}.
\]
This yields an upper bound of $\gamma$. A lower bound of $\gamma$ follows similarly, as $(\gamma_{ij})$ is semi-positive definite at any minimum point of $\rho$.

To obtain an upper bound for $|\nabla \rho|$ is now equivalent to obtain an upper bound of $\phi = \frac{\nabla \gamma}{2}$. Suppose $p \in S^n$ is a maximum point of $\phi$. At $p$,
\[
(3.14) \quad \nabla|\nabla \gamma|^2 = 0, \quad \nabla \omega = 0, \quad B = (-\gamma_{ij} + \delta_{ij}).
\]
It follows from (3.13) with $F(B) = \sigma_k(B)$, at $p$,
\[
0 \geq \sum_{ij} \sigma_k^{ij} \phi_{ij} = -\sum_l \gamma_l (\sigma_k(B))_{il} + \sum_{ij} \sigma_k^{ij} (\delta_{ij} |\nabla \gamma|^2 - \gamma_i \gamma_j + \delta_{ij} \gamma_{ii})
\geq -\sum_l \gamma_l (e^{-(n-k)\gamma} \omega^{k-1} f)_l
\geq \frac{c(|\nabla \gamma|^2 - C|\nabla \gamma|) e^{(k-n)\gamma} \omega^{k-1}}{2}.
\]
where $c \geq \delta, C \leq \frac{1}{2}$ are two positive constants with $\delta$ depending only on $n, k, \inf f, |\nabla f|$. The gradient estimate follows from (3.15). \qed

3.2. $C^2$-estimates and the existence. We want to obtain curvature estimate for $M$. For this purpose, it is convenient to work directly on induced metric $g$ on $M \subset \mathbb{R}^{n+1}$. For $X \in M$, choose local orthonormal frame $\{e_1, \ldots, e_n\}$ on $M$, and $\nu = e_{n+1}$ is the unit outer normal of the hypersurface, such that $\{e_1, \ldots, e_{n+1}\}$ of $\mathbb{R}^{n+1}$ is a local orthonormal frame in $\mathbb{R}^{n+1}$. We use lower indices to denote covariant derivatives with respect to the induced metric.

The second fundamental form is the symmetric $(2, 0)$-tensor given by the matrix $\{h_{ij}\}$, and we denote the Weingarten tensor $\{h_{ij}\} = \{g^{jl}h_{li}\}$. Since $\{e_1, \ldots, e_n\}$ is an orthonormal frame on $M$, $g_{ij} = \delta_{ij}, h_{ij} = h^j_i$. The principal curvatures $(\kappa_1, \ldots, \kappa_n)$ are the eigenvalues of the second fundamental form with respect to the metric which satisfy
\[
\det(h_{ij} - \kappa g_{ij}) = 0.
\]
The curvature equation (3.3) on $S^n$ can also be equivalently expressed as a curvature equation on $M$,
\[
(3.16) \quad \sigma_k(\kappa_1, \ldots, \kappa_n)(X) = \frac{u(X)}{|X|^{n+1}} f\left(\frac{X}{|X|}\right), \quad \forall X \in M.
\]

Proposition 3.6. For $1 < k < n$, let $F \equiv \sigma_k = \Phi u$ and denote $H \equiv \sigma_1$, then at a maximum point of $H_u$,

\[
\sigma_k^{ij}(H_u)_{ij} = \frac{1}{u}[\Phi_{ss} u + 2\Phi_s u_s] - (H_u)\Phi_t(X, X_l) - (k-1)(H_u)\Phi + (k-1)\phi|A|^2 - \frac{1}{u}\sigma_k^{ijml}h_{ijls}h_{mls},
\]

where $A$ denotes the second fundamental form.

Proof. By definition, $u = \langle X, \nu \rangle$. Compute the first and second order covariant derivatives, we have

\[
\begin{align*}
    u_s &= h_{st}\langle X, X_l \rangle \\
    u_{ij} &= h_{ijl}(X, X_l) + h_{ij} - (h^2)_{ij}u
\end{align*}
\]

Also since $(h_{ij})$ is Codazzi, by Ricci identity and Gauss equation,

\[
\begin{align*}
    h_{ij;kl} &= h_{klij} + (h_{kij}h_{im} - h_{lm}h_{ik})h_{mj} + (h_{ij}h_{im} - h_{lm}h_{ij})h_{mk} \\
    \sigma_k^{ij} h_{ij;st} &= F_{st} - \sigma_k^{ijml}h_{ijls}h_{mls}.
\end{align*}
\]

At any maximum point $P \in M^n$ of $H_u$, $(\frac{H_u}{u})_i(P) = 0$. At $P$,

\[
\sigma_k^{ij}(H_u)_{ij} = \frac{1}{u}\sigma_k^{ij} H_{ij} - \frac{1}{u}(H_u)\sigma_k^{ij} u_{ij}.
\]

Apply formulas (3.18) and (3.19),

\[
\begin{align*}
    \frac{1}{u}\sigma_k^{ij} H_{ij} &= \frac{1}{u}\sigma_k^{ij} h_{ss;ij} \\
    &= \frac{1}{u}\sigma_k^{ij} [h_{ij;ss} + (h_{ij}h_{sm} - h_{jm}h_{sj})h_{ms} + (h_{js}h_{sm} - h_{jm}h_{ss})h_{mi}] \\
    &= \frac{1}{u}\sigma_k^{ij} h_{ij;ss} + k\Phi|A|^2 - \frac{1}{u}\sigma_k^{ij}(h^2)_{ij}H \\
    &= \frac{1}{u}(F_{ss} - \sigma_k^{ijml}h_{ijls}h_{mls}) + k\Phi|A|^2 - \frac{1}{u}\sigma_k^{ij}(h^2)_{ij} \\
    &= \frac{1}{u}[\Phi_{ss} u + 2\Phi_s u_s + \Phi u_{ss}] - \frac{1}{u}\sigma_k^{ijml}h_{ijls}h_{mls} + k\Phi|A|^2 \\
    &= \frac{1}{u}[\Phi_{ss} u + 2\Phi_s u_s] + \Phi [H_t(X, X_l) + H - |A|^2 u] \\
    &= \frac{1}{u}\sigma_k^{ijml}h_{ijls}h_{mls} + k\Phi|A|^2 - \frac{1}{u}\sigma_k^{ij}(h^2)_{ij} \\
    &= \frac{1}{u}[\Phi_{ss} u + 2\Phi_s u_s] + \Phi [H_t(X, X_l) + (H_u)\Phi] \\
    &= \frac{1}{u}\sigma_k^{ijml}h_{ijls}h_{mls} + (k-1)\phi|A|^2 - \frac{1}{u}\sigma_k^{ij}(h^2)_{ij}.
\end{align*}
\]

We also compute

\[
\begin{align*}
    -\frac{1}{u}(H_u)\sigma_k^{ij} u_{ij} &= -\frac{1}{u}(H_u)\sigma_k^{ij} [h_{ij,l}\langle X, X_l \rangle + h_{ij} - (h^2)_{ij}u] \\
    &= -\frac{1}{u}(H_u) F_l(X, X_l) - k\phi(H_u) + (H_u)\sigma_k^{ij}(h^2)_{ij} \\
    &= -\frac{1}{u}(H_u) F_l(X, X_l) - (\frac{H_u}{u})\Phi_t(X, X_l) - k\Phi(H_u) + (H_u)\sigma_k^{ij}(h^2)_{ij},
\end{align*}
\]

where $(h^2)_{ij} = h_{ik}h_{kj}$.

Adding up (3.21) and (3.22), and using the critical point condition, we obtain
\[
\sigma_k^{ij}(\frac{H}{u})_{ij} = \frac{1}{u} [\Phi_{ss} u + 2\Phi_s u_s] + \phi(\frac{H}{u})_t (X, X_t) - (\frac{H}{u}) \Phi_t (X, X_t) - (k - 1) \left( \frac{H}{u} \right) \Phi_{jl} h_{ij} h_{lm,s} + (k - 1) \Phi |A|^2
\]
\[
= \frac{1}{u} [\Phi_{ss} u + 2\Phi_s u_s] - (\frac{H}{u}) \Phi_t (X, X_t) - (k - 1) \left( \frac{H}{u} \right) \Phi_{jl} h_{ij} h_{ml,s} + (k - 1) \Phi |A|^2,
\]
(3.23)

(3.17) is verified.  

We now need the following lemma, which explores the concavity of \( \sigma \) on \( \Gamma_k \).

**Lemma 3.7.** Let \( \alpha = \frac{1}{k-1} \), if \( W \in \Gamma_k \) is a symmetric tensor on a Riemannian manifold \( M \). For any local orthonormal frame \( \{ e_1, \cdots, e_n \} \), denote \( W_{ij,s} = \nabla_{e_s} W_{ij} \). Then

\[
(3.24) (\sigma_k)^{ij,lm} W_{ij,s} W_{lm,s} \leq -\sigma_k \left[ \frac{(\sigma_k)_s}{\sigma_k} - \frac{(\sigma_1)_s}{\sigma_1} \right] \left[ (\alpha - 1) \frac{(\sigma_k)_s}{\sigma_k} - (\alpha + 1) \frac{(\sigma_1)_s}{\sigma_1} \right].
\]

**Proof.** By the concavity of \( \left( \frac{\sigma_k}{\sigma_1} \right)^{1/\alpha - 1} (W) \), we have

\[
(3.25) 0 \geq \frac{\partial^2}{\partial W_{ij} \partial W_{lm}} \left( \left( \frac{\sigma_k}{\sigma_1} \right)^{1/\alpha - 1} \right) W_{ij,s} W_{lm,s}.
\]

Denote \( \alpha = \frac{1}{k-1} \). Direct computations yield,

\[
0 \geq \frac{\partial^2}{\partial W_{ij} \partial W_{lm}} \left( \frac{\sigma_k}{\sigma_1} \right)^{\alpha} \cdot W_{ij,s} W_{lm,s}
\]
\[
= \alpha \left( \frac{\sigma_k}{\sigma_1} \right)^{\alpha} \left[ \frac{(\sigma_k)^{ij,lm}}{\sigma_k} + \frac{(\alpha - 1)(\sigma_k)^{ij}(\sigma_k)^{lm}}{\sigma_k^2} - \frac{2\alpha(\sigma_k)^{ij}(\sigma_1)^{lm}}{\sigma_k^2} + \frac{(\alpha + 1)(\sigma_1)^{ij}(\sigma_1)^{lm}}{\sigma_1^2} \right] W_{ij,s} W_{lm,s}
\]

Equivalently,

\[
\frac{(\sigma_k)^{ij,lm} W_{ij,s} W_{lm,s}}{\sigma_k} \leq - \left[ \frac{(\alpha - 1)(\sigma_k)^{ij}(\sigma_k)^{lm}}{\sigma_k^2} - \frac{2\alpha(\sigma_k)^{ij}(\sigma_1)^{lm}}{\sigma_k^2} + \frac{(\alpha + 1)(\sigma_1)^{ij}(\sigma_1)^{lm}}{\sigma_1^2} \right] W_{ij,s} W_{lm,s}
\]
\[
\leq - \left[ \frac{(\sigma_k)_s}{\sigma_k} - \frac{(\sigma_1)_s}{\sigma_1} \right] \left[ (\alpha - 1) \frac{(\sigma_k)_s}{\sigma_k} - (\alpha + 1) \frac{(\sigma_1)_s}{\sigma_1} \right].
\]

(3.27)

The following is a corollary of Lemma 3.7.
Proof. We have already obtained the estimation of mean curvature $H_2 \leq C$ PDE. 

Lemma 3.9. If $M$ satisfies equation (3.16) for some $1 \leq k \leq n$, then there exists a constant $C$ depending only on $n$, $k$, $\min_{S^k} f$, $|f|_{C^1}$, and $|f|_{C^2}$, such that

$$\max_M \sigma_1 \leq C, \quad |\nabla^2 \rho| \leq C.$$ (3.29)

Proof. We have already obtained the $C^0$ and $C^1$ estimates for $\rho$. For the case of $k = 1$, equation (3.16) is a mean curvature type equation which is of divergent form of quasilinear PDE. $C^2$ estimates follows from the classical quasilinear elliptic PDE theory. We work on $2 \leq k \leq n - 1$ cases. When $k > 1$, the estimation of the curvature bound is equivalent to the estimation of mean curvature $H$ (which yields $C^2$ bound on $\rho$). To see this, suppose mean curvature $H \leq C$ is bounded from above. Since $\kappa \in \Gamma_k \subset \Gamma_2$, $(\kappa|i) \in \Gamma_1$. Hence, for each $i$,

$$C \geq H = \sigma_1(\kappa) = \kappa_i + \sigma_1(\kappa|i) \geq \kappa_i.$$ (3.30)

This gives an upper bound of curvature. A lower bound follows from the fact $\sigma_1(\kappa) > 0$ and $\kappa_i \leq C$ for each $i$.

As $u$ is bounded from below and above, we only need to get an upper bound of $\frac{H}{u}$. Suppose $P \in M$ where $\frac{H}{u}$ achieves its maximum, it follows from (3.17)

$$0 \geq \sigma_k^{ij}\left(\frac{H}{u}\right)_{ij} = \frac{1}{u}[\Phi_{ss}u + 2\phi_su_s] - \left(\frac{H}{u}\right)\Phi_t(X, X_i) - (k - 1)(\frac{H}{u})\Phi - \frac{1}{2}\sigma_k^{ij;ml}h_{ij;sl}h_{ml;st} + (k - 1)\Phi|A|^2.$$ (3.31)

Recall $\Phi(X) = |X|^{-(n+1)}f(\frac{X}{|X|})$ and with $C^0$, $C^1$ estimates of $\rho = |X|$, we have the following estimates.

$$|\Phi_t((P)| \leq C(n, k, \min_{S^k} f, |f|_{C^1})$$

$$|\Phi_t((P)| \leq C(n, k, \min_{S^k} f, |f|_{C^1}, |f|_{C^2})(1 + |A|(P)).$$

On the other hand, $|u_i| = |h^i_j\rho \phi_j| \leq c_3|A|$. By equation (3.16),

$$\frac{\sigma_1}{u} = \frac{\sigma_1 \phi}{\sigma_k}.$$ (3.32)

At a maximum point $P$ of the test function $\frac{\phi_s}{\phi}$, one has

$$\frac{(\sigma_1)_s}{\sigma_1} = \frac{(\sigma_k)_s}{\sigma_k} - \frac{\sigma_s}{\phi}.$$ (3.33)

In Corollary 3.8, set $r = \frac{\phi_s}{\phi}(P)$, then

$$\sigma_k^{ij;ml}h_{ij;st}h_{ml;st} \leq 2r(\phi)_{ss} - \frac{k}{k - 1}r^2(\phi)$$

$$\leq C_1(n, k, \min_{S^k} f, |f|_{C^1})|A| + C_2(n, k, \min_{S^k} f, |f|_{C^1}).$$ (3.34)
With the above estimates, (3.30) can be simplified as
\[
(3.31) \quad |A|^2(P) + c_4 |A|(P) + c_5 \leq 0,
\]
where \( c_4 \) and \( c_5 \) are constants depending only on \( n, k, \min_{S^n} \phi, |f|_{C^1}, \) and \( |f|_{C^2} \). Hence at \( P, \) \( |A|(P) \leq C. \) In turn
\[
\sigma_1(X) \leq u(X) \frac{\sigma_1(P)}{u(P)} \leq C, \quad \text{for any } X \in M.
\]
This implies (3.29).

As before, by the Evans-Krylov Theorem [9, 19] and standard theory of elliptic equations, one obtains higher order regularity estimates of solutions. The existence follows from the method of continuity.

4. Epilogue

The Minkwoski problem was introduced by Minkowski in [20]. The case \( n = 2 \) was solved by Nirenberg [21] and Pogorelov [22] in 1950s, the higher dimensions were settled by Cheng-Yau [8] and Pogorelov [23] in 1970s. The Christoffel-Minkowski problem can be traced back to the origin of the Brunn-Minkowski theory (e.g., [25]). When \( 1 < k < n \), the foundation of this type of PDE was developed by Caffarelli-Nirenberg-Spruck in [6] via Garding’s hyperbolic polynomial theory [10]. The treatment of the Christoffel-Minkowski problem discussed here is extracted from [16, 17, 11]. The issue of the convexity of the solution, the proof in [16] was developed based on ideas independently by Caffarelli-Friedman [4] and Yau [26]. This is often called the Constant Rank Theorem. We refer to [5, 3] for further study, see also [27] for a different approach. The equation for prescribing curvature measure [1] is in different category. Here we follow from [13] (see also [14, 12]). The main issue is the curvature estimate for solutions of equation of the form
\[
\sigma_k(h_{ij}) = f(X, \nu).
\]
If \( f \) is independent of \( \nu \), it is classical. In this case, \( \sigma_k \) can be replaced by general nonlinear operator \( F \) with some nature structural conditions [7] (for \( k = 1 \), see also earlier works of [2, 29]). When \( f \) depends on \( \nu \), the curvature estimate in [7] is not true in general. The explicit function \( u \) in (2.9) provides counter-example to quotient of curvature equations
\[
\frac{\sigma_k}{\sigma_l}(k) = f(X, \nu), \forall 1 \leq l < k \leq n.
\]
We refer [18, 15, 24, 28] for some recent development in this direction. The convexity and concavity properties of the elementary symmetric functions, namely Corollary 2.5 and Lemma 3.7 play crucial roles in our derivation.

References


