ON ADMISSIBLE SQUARE ROOTS OF NON-NEGATIVE $C^{2,2\alpha}$ FUNCTIONS

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Abstract. We establish necessary and sufficient condition for $C^{1,\alpha}$ regularity of the admissible square roots of a non-negative $C^{2,2\alpha}(\mathbb{R})$ functions.

1. Introduction

The paper concerns the following problem: the regularity of square root of $C^{2,2\alpha}$ non-negative functions. Nirenberg-Trèves' gradient estimate for non-negative $C^{1,1}(\mathbb{R}^n)$ functions [14] implies square roots of these functions are Lipschitz. This estimate plays important roles in analysis of linear and nonlinear PDEs (e.g., [9], [1]). The sum of squares theorem of Fefferman and Phong [4,5] stated that any non-negative $C^{3,1}$ function in $\mathbb{R}^n$ can be written as a sum of squares of $C^{1,1}$ functions. A detailed proof was given in [7] which was communicated by Fefferman (see also [3, 16]). This decomposition is crucial to obtain C^2 a priori estimates for degenerate real Monge-Ampère equations in [7] and complex Monge-Ampère equation in [15].

For functions of one variable, Glaeser [6] proved that if $0 \leq f \in C^2(\mathbb{R})$ is 2-flat on its zeroes (i.e., $f(x) = 0$ implies $f''(x) = 0$), then $f^{1/2} \in C^1(\mathbb{R})$. Mandai [13] proved that for any $0 \leq f \in C^2(\mathbb{R})$, f always has an admissible square root $g \in C^1(\mathbb{R})$. In [3], Bony, Broglia, Colombini and Pernazza obtain a necessary and sufficient condition for a non-negative function $f \in C^4(\mathbb{R})$ to have an admissible square root in $C^2(\mathbb{R})$, which is only related to the non-zero local minimum points of f. Korobenko-Sawyer [12] consider higher regularity of square root functions under appropriate sufficient conditions.

The main result of this paper is the necessary and sufficient condition for optimal $C^{1,\alpha}$ regularity of square roots of $C^{2,2\alpha}(\mathbb{R})$ non-negative functions. In the rest of this paper, $C^{2,2\alpha}(\mathbb{R})$ indicates $C^{3,2\alpha-1}(\mathbb{R})$ if $1/2 < \alpha \leq 1$. Below is the statement of the main theorem.

Theorem 1.1. *Let $0 \leq f \in C^{2,2\alpha}(\mathbb{R})$ with $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$. $0 < \alpha \leq 1$. Define the set*

$$\mathcal{A} = \{x_0 \in \mathbb{R} : f(x_0) > 0, f'(x_0) = 0, f''(x_0) > 0\}. \tag{1.1}$$

Then $f = g^2$ for some $g \in C^{1,\alpha}(\mathbb{R})$ if and only if there is a constant $M > 0$ such that

$$f''(x_0) \leq M \cdot (f(x_0))^{\frac{\alpha}{1+\alpha}}, \quad \forall x_0 \in \mathcal{A}. \tag{1.2}$$

Moreover, if (1.2) is satisfied, then $\|g\|_{C^{1,\alpha}(\mathbb{R})} \leq C$ for some universal $C > 0$, depending only on α and M.

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Remark 1.2. *The condition obtained by Bony, Broglia, Colombini and Pernazza in [3] is there is a continuous function γ vanishing at every flat points of f such that*

$$f''(x_0) \le \gamma(x_0) \cdot (f(x_0))^{\frac{1}{2}}, \quad \forall x_0 \in \mathcal{A}. \tag{1.3}$$

Condition (1.2) is a $C^{2,2\alpha}$ version of (1.3).

The main theorem is motivated by regularity problem associated to the isometric embedding problem. Guan and Li [8] showed that if g is a C^4 Riemannian metric on $\mathbb{S}^2$ with Gauss curvature $K_g \ge 0$, then there exists a $C^{1,1}$ isometric embedding $X : (\mathbb{S}^2, g) \to (\mathbb{R}^3, g_{Eucl})$. A natural question is, can the embedding X be improved to $C^{2,1}$? A positive answer was given in Jiang [11] in the graph setting, under the assumption X takes the form $X(x, y) = (x, y, u(x, y))$ in local coordinates. It relies on a square root regularity for square of monotone functions. It is a special case of Theorem 1.1 where $\alpha = 1$ and $\mathcal{A} = \varnothing$, which can be stated as follows.

Corollary 1.3. *Let $I = [-1/2, 1/2]$. Assume $0 \le f \in C^{3,1}(I)$ with $\|f\|_{C^{3,1}(I)} \le 1$. The zero set of f in I is a closed interval $N := [x_0', x_0]$ (possibly $x_0' = x_0$). f is non-increasing in $[-1/2, x_0')$ and f is non-decreasing in $(x_0, 1/2]$. Then $\exists g \in C^{1,1}(I)$ such that $f = g^2$ in I, g is non-decreasing in I and $\|g\|_{C^{1,1}(I)} \le C$ for some universal constant $C > 0$.*

2. Fefferman-Phong's Lemma for $C^{2,2\alpha}$ nonnegative functions

The following lemma is well known (e.g. [16]). We provide a proof here for completeness.

Lemma 2.1 (Even dominate odd, $C^{2,\alpha}$)**.** *Let $0 < \alpha \le 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 non-negative function such that $[f]_{C^{2,\alpha}(\mathbb{R})} \le 1$. Then*

$$|f'(x)| \le \frac{3}{2}|f(x)|^{\frac{1+\alpha}{2+\alpha}} + \frac{1}{2}|f''(x)| \cdot f(x)^{\frac{1}{2+\alpha}} + f(x)^{\frac{\alpha}{2+\alpha}} \cdot |f''(x)|^{\frac{1}{\alpha}} \quad \forall x \in \mathbb{R}. \tag{2.1}$$

Proof. We may assume $f(x) \ne 0$. By Taylor expansion, $\forall x, h \in \mathbb{R}$, $\exists \xi$ between $x, x+h$ such that

$$\begin{aligned} 0 \le f(x+h) &= f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{2}\frac{f''(\xi) - f''(x)}{|\xi - x|^\alpha}|\xi - x|^\alpha h^2 \\ &\le f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{2}|h|^{2+\alpha}. \end{aligned}$$

Replacing h with $\pm h$,

$$|f'(x)h| \le f(x) + \frac{1}{2}|f''(x)|h^2 + \frac{1}{2}|h|^{2+\alpha}. \tag{2.2}$$

Setting $h = \frac{f(x)^{\frac{2}{2+\alpha}}}{f(x)^{\frac{1}{2+\alpha}} + |f''(x)|^{\frac{1}{\alpha}}}$ in (2.2) and using $h \le f(x)^{\frac{1}{2+\alpha}}$, we obtain (2.1). □

Lemma 2.2 (Even dominate odd, $C^{3,\alpha}$)**.** *Let $0 < \alpha \le 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^3 non-negative function such that $[f]_{C^{3,\alpha}(\mathbb{R})} \le 1$. Then*

$$|f'(x)| \le \frac{13}{6}f(x)^{\frac{2+\alpha}{3+\alpha}} + \frac{3}{2}f(x)^{\frac{1+\alpha}{3+\alpha}} \cdot |f''(x)|^{\frac{1}{1+\alpha}} + f(x)^{\frac{1}{3+\alpha}} \cdot |f''(x)|, \quad \forall x \in \mathbb{R}. \tag{2.3}$$

$$|f'''(x)| \le 6f(x)^{\frac{\alpha}{3+\alpha}} + 6|f''(x)|^{\frac{\alpha}{1+\alpha}} \quad \forall x \in \mathbb{R}. \tag{2.4}$$

Proof. By Taylor expansion, $\forall x \in \mathbb{R}$,

$$0 \le f(x+h) \le f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \frac{1}{6}|h|^{3+\alpha}. \tag{2.5}$$

Replacing h with $\pm h$,

$$|f'(x)h + \frac{1}{6}f'''(x)h^3| \le f(x) + \frac{1}{2}|f''(x)|h^2 + \frac{1}{6}|h|^{3+\alpha} =: A. \tag{2.6}$$

Replacing h by $2h$ in (2.6), we have

$$|2\cdot f'(x)h + 8\cdot\frac{1}{6}f'''(x)h^3| \le f(x) + \frac{1}{2}|f''(x)|(2h)^2 + \frac{1}{6}|2h|^{3+\alpha} =: B. \tag{2.7}$$

Combining (2.6) and (2.7),

$$|f'(x)h| \le \frac{8A+B}{6}, \quad |\frac{1}{6}f'''(x)h^3| \le \frac{2A+B}{6}. \tag{2.8}$$

If $f(x)=0$, then $f'(x)=0$ since $f\ge 0$. Otherwise, setting $h = \frac{f(x)^{\frac{2}{3+\alpha}}}{f(x)^{\frac{1}{3+\alpha}} + |f''(x)|^{\frac{1}{1+\alpha}}}$ in (2.8) and using $h \le f(x)^{\frac{1}{3+\alpha}}$, we have

$$\begin{aligned}|f'(x)| &\le \frac{1}{6}\left(9\cdot\frac{f(x)}{h} + 6\cdot|f''(x)|h + 4\cdot|h|^{2+\alpha}\right)\\ &\le \frac{1}{6}\left(9\cdot f(x)^{\frac{1+\alpha}{3+\alpha}}(f(x)^{\frac{1}{3+\alpha}} + |f''(x)|^{\frac{1}{1+\alpha}}) + 6\cdot|f''(x)|\cdot f(x)^{\frac{1}{3+\alpha}} + 4\cdot f(x)^{\frac{2+\alpha}{3+\alpha}}\right).\end{aligned}$$

Thus (2.3) holds.

If $f(x)=f''(x)=0$, then $f'''(x)=0$ by (2.5). Otherwise, let $h=\max\{f(x)^{\frac{1}{3+\alpha}}, |f''(x)|^{\frac{1}{1+\alpha}}\}$ and using $\max\{a,b\}\le a+b$ in (2.8), and as $(a+b)^\alpha \le a^\alpha + b^\alpha$ for $a,b\ge 0$ and $0<\alpha\le 1$,

$$\begin{aligned}|f'''(x)| &\le \frac{3f(x)}{h^3} + \frac{3|f''(x)|}{h} + (\frac{1}{3} + \frac{1}{6}\cdot 2^{3+\alpha})\cdot|h|^\alpha\\ &\le 3f(x)^{\frac{\alpha}{3+\alpha}} + 3|f''(x)|^{\frac{\alpha}{1+\alpha}} + 3\cdot\left|f(x)^{\frac{1}{3+\alpha}} + |f''(x)|^{\frac{1}{1+\alpha}}\right|^\alpha\\ &\le 3f(x)^{\frac{\alpha}{3+\alpha}} + 3|f''(x)|^{\frac{\alpha}{1+\alpha}} + 3\cdot\left(f(x)^{\frac{\alpha}{3+\alpha}} + |f''(x)|^{\frac{\alpha}{1+\alpha}}\right).\end{aligned}$$

Thus (2.4) holds. □

We define some constants which will be used in the rest of the paper.

$$\begin{gathered}c_0 = 1/10, \quad C = 1000;\\ N(\alpha)=2,\ \text{if } 0<\alpha\le 1/2, \quad N(\alpha)=3,\ \text{if } 1/2<\alpha\le 1;\\ \epsilon_0 = (\frac{1}{10^5})^{1/(2\alpha)},\ \text{if } 0<\alpha\le 1/2, \quad \epsilon_0 = (\frac{1}{10^5})^{1/(2\alpha-1)},\ \text{if } 1/2<\alpha\le 1;\\ \tilde{c} = \frac{1}{10^3}\cdot(\frac{1}{10^5})^{3/(2\alpha)},\ \text{if } 0<\alpha\le 1/2, \quad \tilde{c} = \frac{1}{10^4}\cdot(\frac{1}{10^5})^{4/(2\alpha-1)},\ \text{if } 1/2<\alpha\le 1. \end{gathered} \tag{2.9}$$

Denote the set of flat points of f by

$$\mathcal{F} := \{x \in \mathbb{R} : f(x) = f'(x) = f''(x) = 0\}. \tag{2.10}$$

We note that if $f \in C^3$ and $f \geq 0$, $x \in \mathcal{F}$ implies $f'''(x) = 0$.

Next lemma is a $C^{2,2\alpha}$-version of Fefferman-Phong's lemma (see [4] and Lemma 18.6.9 of [10]).

Lemma 2.3 (Fefferman-Phong's Lemma). *Let $I = [-1/2, 1/2]$. If $0 \leq \phi \in C^{2,2\alpha}(I)$ such that*

$$|\phi^{(k)}(t)| \leq C \quad \forall t \in I \text{ for } k = 0, 1, \cdots, N(\alpha), \quad [\phi]_{C^{2,2\alpha}(I)} \leq 1 \tag{2.11}$$

$$\text{and} \quad \max\{\phi(0), |\phi''(0)|\} \geq \tilde{c}, \tag{2.12}$$

where $N(\alpha)$, C, $\tilde{c}$ are defined in (2.9). Then there exist universal constants $r_0 > 0$, $\tilde{A} > 0$, $c_2 > 0$ such that, for $t \in (-r_0, r_0)$, either

$$c_2 \leq \phi(t) \leq C, \quad \|\sqrt{\phi(t)}\|_{C^{1,\alpha}((-r_0,r_0))} \leq \tilde{A}; \tag{2.13}$$

or

$$c_2 \leq \phi''(t) \leq C, \tag{2.14}$$

$$\phi(t) = \phi(T) + (t-T)^2 \int_0^1 \phi''(t + s(T-t))s\, ds, \tag{2.15}$$

where $t = T$ is the unique strict local minimum point of the function ϕ in $(-r_0, r_0)$.

Moreover, the function

$$g(t) := (t-T)\left(\int_0^1 \phi''(t + s(T-t))s\, ds\right)^{1/2} \tag{2.16}$$

is in $C^{1,\alpha}((-r_0, r_0))$.

Proof. Set $\mu := \min\{\frac{\tilde{c}}{3C}, (\frac{\tilde{c}}{3})^{\frac{1}{2\alpha}}\}$, where $\tilde{c}$ and C are defined in (2.9).

(i). If $\phi(0) \geq \tilde{c}$, $\forall |t| < \mu$,

$$\phi(t) \geq \frac{1}{3}\tilde{c}, \quad \text{and} \quad |(\sqrt{\phi})'(t_1)| = \left|\frac{\phi'(t_1)}{2\sqrt{\phi(t_1)}}\right| \leq \frac{C}{2\sqrt{\frac{1}{3}\tilde{c}}} =: b. \tag{2.17}$$

By (2.11), (2.17), and the mean value theorem, for $|t_1| < \mu$ and $|t_2| < \mu$, $t_1 \neq t_2$,

$$\begin{aligned} 2|(\sqrt{\phi})'(t_1) - (\sqrt{\phi})'(t_2)|/|t_1 - t_2|^\alpha &= \left|\frac{\phi'(t_1)}{\sqrt{\phi(t_1)}} - \frac{\phi'(t_2)}{\sqrt{\phi(t_2)}}\right|/|t_1 - t_2|^\alpha \\ \leq \left|\frac{\phi'(t_1)}{\sqrt{\phi(t_1)}} - \frac{\phi'(t_2)}{\sqrt{\phi(t_1)}}\right|/|t_1 - t_2|^\alpha &+ \left|\frac{\phi'(t_2)}{\sqrt{\phi(t_1)}} - \frac{\phi'(t_2)}{\sqrt{\phi(t_2)}}\right|/|t_1 - t_2|^\alpha \\ \leq \frac{1}{\sqrt{\frac{1}{3}\tilde{c}}} \cdot \frac{|\phi''(\xi_1)||t_1 - t_2|}{|t_1 - t_2|^\alpha} &+ C \cdot \frac{|\phi'(\xi_2)||t_1 - t_2|}{2(\frac{1}{3}\tilde{c})^{3/2}|t_1 - t_2|^\alpha} \leq C_1 \end{aligned} \tag{2.18}$$

where $b, C_1 > 0$ are universal constants, and ξ_1, ξ_2 are some points between t_1, t_2.

(ii). Assume $|\phi''(0)| \geq \tilde{c}$.

(a) If $\phi''(0) \le -\tilde{c}$, then for $|t| < \mu$, $\phi''(t) \le -\frac{1}{3}\tilde{c}$. For any $|t_0| < \frac{1}{2}\mu$, expanding ϕ near t_0, we have

$$0 \le \phi(t_0 + h) + \phi(t_0 - h) \le 2 \cdot \left(\phi(t_0) + \frac{1}{2} \cdot (-\frac{1}{3}\tilde{c}) \cdot h^2 + \frac{1}{2}|h|^{2+2\alpha} \right).$$

Letting $h = \frac{1}{2}\mu$, $\forall |t_0| < \frac{1}{2}\mu$, $\phi(t_0) \ge \frac{1}{6}\tilde{c}h^2 - \frac{1}{2}|h|^{2+2\alpha} \ge \frac{1}{24}\mu^2\tilde{c}(1 - 2^{-2\alpha})$.

Similar to case (i), we have $\sqrt{\phi} \in C^{1,\alpha}((-\mu/2, \mu/2))$.

(b) If $\phi''(0) \ge \tilde{c}$ and $\phi(0) < c_1$, where $c_1 > 0$ is a small and universal constant to be determined, then $|\phi'(0)|$ is also small since $\phi \ge 0$. By expansion of $\phi' \in C^{1,2\alpha}(I)$ near 0,

$$\phi'(t) = \phi'(0) + \phi''(0)t + R(t), \quad \text{where} \quad |R(t)| \le C|t|^{1+\alpha}. \tag{2.19}$$

In particular, (2.19) shows that $\phi'(r) > 0$ and $\phi'(-r) < 0$ if

$$\phi''(0)r > |\phi'(0)| + 2Cr^{1+\alpha}. \tag{2.20}$$

Fix $r = \min\{\frac{\tilde{c}}{3C}, (\frac{\tilde{c}}{3})^{\frac{1}{2\alpha}}\}$. As $\phi'' \in C^{2\alpha}(I)$,

$$\phi''(t) \ge \frac{1}{3}\tilde{c}, \quad |t| \le r. \tag{2.21}$$

This implies $\phi'(t)$ is strictly increasing in $[-r, r]$, thus $\phi'(t) = 0$ has a unique solution $t = T$ in $B_r := (-r, r)$. By Taylor expansion of ϕ near $t = T$, we obtain in B_r,

$$\phi(t) = \phi(T) + (t - T)^2 \int_0^1 \phi''(t + s(T - t))s\, ds. \tag{2.22}$$

We note $t = T$ is the unique strict local minimum point of the function ϕ in B_r.

We will estimate Hölder seminorm of g' where g is defined in (2.16). Assume without loss of generality that $\phi(T) = 0$. Then in B_r, $g(t) = \sqrt{\phi(t)}$ if $t \ge T$ and $g(t) = -\sqrt{\phi(t)}$ if $t < T$. By Taylor expansion,

$$\lim_{t \to T^+} \frac{g(t) - g(T)}{t - T} = \lim_{t \to T^+} \frac{\sqrt{\frac{1}{2}\phi''(T)(t-T)^2 + O(|t - T|^{2+\min\{1,2\alpha\}})} - \sqrt{0}}{t - T} = \sqrt{\frac{1}{2}\phi''(T)}.$$

We obtain the same value for the left limit and hence $g'(T) = \sqrt{\frac{1}{2}\phi''(T)}$.

If $t \ne T$, then by Taylor expansion,

$$\phi(t) = \frac{1}{2}\phi''(T)(t - T)^2 + A_1, \tag{2.23}$$

$$\phi'(t) = \phi''(T)(t - T) + A_2, \tag{2.24}$$

$$\phi''(t) = \phi''(T) + A_3. \tag{2.25}$$

By (2.21), (2.11) and $|t - T| \le 2r$,

$$\begin{aligned} |A_1| &\le C_1 \cdot |t - T|^{2+\min\{1,2\alpha\}} \le \frac{1}{3}\phi''(T)(t - T)^2, \\ |A_2| &\le C_2 \cdot |t - T|^{1+\min\{1,2\alpha\}} \le |\phi''(T)(t - T)|, \\ |A_3| &\le C_3 \cdot |t - T|^{\min\{1,2\alpha\}}. \end{aligned} \tag{2.26}$$

Suppose $t > T$. By (2.23), (2.24), (2.26) and $\phi''(t) \sim 1$ in B_r, $\forall t \in B_r$,

$$\begin{aligned}
|g'(T) - g'(t)| &= \left| \sqrt{\frac{1}{2}\phi''(T)} - \frac{\phi'(t)}{2\sqrt{\phi(t)}} \right| \\
&\leq \left| \sqrt{\frac{1}{2}\phi''(T)} - \frac{\phi'(t)}{2\sqrt{\frac{1}{2}\phi''(T)(t-T)^2}} \right| + \left| \frac{\phi'(t)}{2\sqrt{\frac{1}{2}\phi''(T)(t-T)^2}} - \frac{\phi'(t)}{2\sqrt{\phi(t)}} \right| \\
&= \left| \frac{A_2}{\sqrt{2\phi''(T)(t-T)^2}} \right| + \frac{1}{2}|\phi'(t)| \left| \frac{A_1}{\sqrt{\frac{1}{2}\phi''(T)(t-T)^2} \cdot \sqrt{\phi(t)} \cdot (\sqrt{\phi(t)} + \sqrt{\frac{1}{2}\phi''(T)(t-T)^2})} \right| \\
&\leq b \cdot |T - t|^{\alpha}, \qquad (2.27)
\end{aligned}$$

where $b > 0$ is a universal constant. Proof is the same for $t < T$.

By (2.23), (2.24), (2.25), and $\phi''(t) \sim 1$ in B_r, there exists a universal $c > 0$ such that, $\forall t \in B_r$ with $t \neq T$,

$$|\phi''(t) \cdot \phi(t) - \frac{1}{2}\phi'(t)^2| = O(|t - T|^{2+\min\{1,2\alpha\}}),$$

$$|g''(t)| = \frac{1}{2} \left| \frac{\phi''(t) \cdot \phi(t) - \frac{1}{2}\phi'(t)^2}{\phi(t)^{3/2}} \right| \leq c \cdot |t - T|^{\min\{0,2\alpha-1\}}. \qquad (2.28)$$

$\forall t_1, t_2 \in B_r$, we want to estimate $|g'(t_1) - g'(t_2)|$. By (2.27), we only need to deal with

$$T < t_1 < t_2 \quad \text{or} \quad t_2 < t_1 < T, \quad \text{with}$$

$$|t_1 - t_2| \leq |t_1 - T|. \qquad (2.29)$$

We only consider the case $T < t_1 < t_2$ ($t_1 < t_2 < T$ is similar). By assumption (2.29),

$$|\xi - T| \geq |t_1 - T| \geq |t_1 - t_2|, \quad \forall \xi \in (t_1, t_2).$$

By mean value theorem, $\exists \xi \in (t_1, t_2)$ such that

$$|g'(t_1) - g'(t_2)| = |g''(\xi)||t_1 - t_2| \leq c \cdot |\xi - T|^{\min\{0,2\alpha-1\}} \cdot |t_1 - t_2| \leq c \cdot |t_1 - t_2|^{\alpha}.$$

(c) If $c_1 \leq \phi(0) < \tilde{c}$, then similar to case (i), we have $\sqrt{\phi} \in C^{1,\alpha}((-\frac{c_1}{3C}, \frac{c_1}{3C}))$.

To summarize, case (i), (ii)(a) and (ii)(c) lead to (2.13). Case (ii)(b) leads to (2.14). □

3. A Calderón-Zygmund decomposition

We use the Calderón-Zygmund decomposition, which was originally suggested by Fefferman in [7].

Lemma 3.1. *Let $0 \leq f \in C^{2,2\alpha}(\mathbb{R})$ with $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$. There is a countable collection of intervals $\{Q_\nu\}_{\nu \geq 1}$ taking the form of $(a, b]$, whose interiors are disjoint, such that*

(1) *$\mathbb{R} = \mathcal{F} \cup (\cup_\nu Q_\nu)$ and $\mathcal{F} \cap (\cup_\nu Q_\nu) = \varnothing$, where $\mathcal{F}$ is defined in (2.10).*
(2) *Let $\delta_\nu = diam(Q_\nu)$. Then for any ν, $\delta_\nu \leq 1$. With $N(\alpha)$ defined in (2.9),*

$$\inf_{x \in Q_\nu} \left(\sum_{k=0}^{N(\alpha)} \delta_\nu^{k-(2+2\alpha)} |\nabla^k f(x)| \right) > N(\alpha) + 1. \qquad (3.1)$$

Proof. We decompose $\mathbb{R}$ into a mesh of equal intervals $(a_n, b_n]$, whose interiors are disjoint, and whose common diameter is so large that

$$\inf_{x\in Q'}\left(\sum_{k=0}^{N(\alpha)}(diam(Q'))^{k-(2+2\alpha)}|\nabla^k f(x)|\right)\leq N(\alpha)+1$$

for every intervalQ' in this mesh. As $\|f\|_{C^{2,2\alpha}(\mathbb{R})}\leq 1$, the common diameter can be chosen to be 1.

Let Q' be a fixed intervalin this mesh. By bisecting each of the sides of Q', we divide Q' into 2 congruent intervals. Let Q'' be one of these new intervals.

(i) If

$$\inf_{x\in Q''}\left(\sum_{k=0}^{N(\alpha)}(diam(Q''))^{k-(2+2\alpha)}|\nabla^k f(x)|\right)> N(\alpha)+1,$$

then we don't sub-divide Q'' any further, and Q'' is selected as one of the intervals Q_ν.

(ii) If

$$\inf_{x\in Q''}\left(\sum_{k=0}^{N(\alpha)}(diam(Q''))^{k-(2+2\alpha)}|\nabla^k f(x)|\right)\leq N(\alpha)+1,$$

then we proceed with the sub-division of Q'', and repeat this process until we are forced to the case (i).

□

Lemma 3.2. *Let $3Q_\nu$ be the intervalof diameter $3\delta_\nu$, with the same center at Q_ν, then*

$$\sum_{k=0}^{N(\alpha)}\delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x)|\leq C\quad\forall x\in 3Q_\nu, \tag{3.2}$$

where C is defined in (2.9).

Proof. We prove the case where $1/2<\alpha\leq 1$. $0<\alpha\leq 1/2$ is similar.

Let $\tilde{Q}_\nu$ be the step before we get Q_ν. Then $Q_\nu\subset\tilde{Q}_\nu$ and diameter of $\tilde{Q}_\nu$ is $2\delta_\nu$. Since we didn't stop at $\tilde{Q}_\nu$, there is $x_0\in\tilde{Q}_\nu\subset 3Q_\nu$ such that $\sum_{k=0}^{3}(2\delta_\nu)^{k-(2+2\alpha)}|\nabla^k f(x_0)|\leq 4$. That is

$$|\nabla^k f(x_0)|\leq 4(2\delta_\nu)^{(2+2\alpha)-k},\quad k=0,1,2,3. \tag{3.3}$$

Using $\|f\|_{C^{2,2\alpha}(\mathbb{R})}\leq 1$ and $dist(x,x_0)\leq 3\delta_\nu$, we get

$$|\nabla^3 f(x)|\leq|\nabla^3 f(x_0)|+1\cdot|x-x_0|^{2\alpha-1}\leq 4(2\delta_\nu)^{2+2\alpha-3}+(3\delta_\nu)^{2\alpha-1}\leq 11\delta_\nu^{2\alpha-1}\quad\forall x\in 3Q_\nu. \tag{3.4}$$

Using (3.3) and (3.4), we get

$$|\nabla^2 f(x)|\leq\sup_{3Q_\nu}|\nabla^3 f|\cdot|x-x_0|+|\nabla^2 f(x_0)|\leq 11\delta_\nu^{2\alpha-1}\cdot 3\delta_\nu+4(2\delta_\nu)^{(2+2\alpha)-2}\leq 49\delta_\nu^{2\alpha}\quad\forall x\in 3Q_\nu.$$

Going backwards, we get $|\nabla f(x)|\leq 179\delta_\nu^{1+2\alpha}$ and $|f(x)|\leq 601\delta_\nu^{2+2\alpha}\quad\forall x\in 3Q_\nu$.

□

Lemma 3.3. *Let Q_ν^* be the intervalof diameter $(1+\epsilon_0)\delta_\nu$, with the same center at Q_ν, then*

$$\inf_{x\in Q_\nu^*}\Big(\sum_{k=0}^{N(\alpha)}\delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x)|\Big)\geq c_0, \tag{3.5}$$

where c_0, ϵ_0 are defined in (2.9).

Proof. Let $B := \{x \in \mathbb{R} : dist(x, x_0) \le \epsilon_0\delta_\nu\}$.

We prove the case where $1/2 < \alpha \le 1$. $0 < \alpha \le 1/2$ is similar.

Assume not, then $\exists x_0 \in Q_\nu^*$ such that $\sum_{k=0}^{3} \delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x_0)| < c_0$.

Using $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \le 1$ and the mean value theorem, we get

$$|\nabla^3 f(x)| \le |\nabla^3 f(x_0)| + 1 \cdot |x - x_0|^{2\alpha-1} \le (c_0 + 1)\delta_\nu^{2\alpha-1} \quad \forall x \in B.$$

$$|\nabla^2 f(x)| \le \sup_B |\nabla^3 f| \cdot |x - x_0| + |\nabla^2 f(x_0)| \le (2c_0 + 1)\delta_\nu^{2\alpha} \quad \forall x \in B.$$

Going backwards, we get $|\nabla f(x)| \le (3c_0 + 1)\delta_\nu^{1+2\alpha}$ and $|f(x)| \le [(3c_0 + 1)\epsilon_0 + c_0]\delta_\nu^{2+2\alpha}$. Note $\epsilon_0 < \frac{1}{10^5}$, so for any $x \in B$, $\sum_{k=0}^{3} \delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x)| < 4$, contradicting with (3.1).

□

Lemma 3.4. *Let $\lambda = \epsilon_0/2$. Let Q_ν^+ be the intervalof diameter of $(1+\lambda)\delta_\nu$, with the same center at Q_ν. Then for $z \in Q_\nu^+$, either*

$$f(z) \ge \tilde{c}\delta_\nu^{2+2\alpha}, \tag{3.6}$$

or

$$f(z) < \tilde{c}\delta_\nu^{2+2\alpha} \text{ and } |\nabla^2 f(z)| \ge \tilde{c}\delta_\nu^{2\alpha}, \tag{3.7}$$

where $\tilde{c}$ is defined in (2.9).

Proof. By translation we assume $z = 0$, with

$$f(0) < \tilde{c}\delta_\nu^{2+2\alpha} \quad \text{and} \quad |\nabla^2 f(0)| < \tilde{c}\delta_\nu^{2\alpha}. \tag{3.8}$$

First we assume $1/2 < \alpha \le 1$. Let $c > 0$ small such that $2c\delta_\nu < (diam(Q_\nu^*) - diam(Q_\nu^+))/2$. By Taylor expansion, (3.8) and $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \le 1$, for any $|x| < 2c\delta_\nu$,

$$0 \le f(x) \le \tilde{c}\delta_\nu^{2+2\alpha} + f'(0)x + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}x^2 + \frac{1}{6}f'''(0)x^3 + \frac{1}{6}|x|^{2+2\alpha}. \tag{3.9}$$

Taking x and $-x$ in (3.9), for any $|x| < 2c\delta_\nu$,

$$|f'(0)x + \frac{1}{6}f'''(0)x^3| \le \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}x^2 + \frac{1}{6}|x|^{2+2\alpha}. \tag{3.10}$$

In particular, for any $|x| < c\delta_\nu$,

$$|f'(0)x + \frac{1}{6}f'''(0)x^3| \le \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}(c\delta_\nu)^2 + \frac{1}{6}|c\delta_\nu|^{2+2\alpha} =: A\delta_\nu^{2+2\alpha}. \tag{3.11}$$

On the other hand, by substituting x with $2x$ in (3.10), for any $|x| < c\delta_\nu$,

$$\begin{aligned}|f'(0)(2x) + \frac{1}{6}f'''(0)(2x)^3| &\le \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}(2x)^2 + \frac{1}{6}|2x|^{2+2\alpha} \\ &\le \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}(2c\delta_\nu)^2 + \frac{1}{6}|2c\delta_\nu|^{2+2\alpha} =: B\delta_\nu^{2+2\alpha}. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we obtain for any $|x| < c\delta_\nu$,

$$|f'(0)x| \le \frac{1}{6}(8A + B)\delta_\nu^{2+2\alpha}, \quad |\frac{1}{6}f'''(0)x^3| \le \frac{1}{6}(2A + B)\delta_\nu^{2+2\alpha}.$$

Thus $|f'(0)| \le \frac{8A+B}{6c}$ and $|f'''(0)| \le \frac{2A+B}{c^3}$. If $c = \epsilon_0/10$, $\tilde{c} = c^4$, then

$$\sum_{k=0}^{3} \delta_\nu^{k-(2+2\alpha)} |\nabla^k f(0)| \le \tilde{c} + \frac{8A+B}{6c} + \tilde{c} + \frac{2A+B}{c^3} < 0.01^4 + 0.01 + 0.01^4 + 0.07 < c_0,$$

contradicting with (3.5).

Next we deal with the case $0 < \alpha \le 1/2$.

Let $c > 0$ small such that $2c\delta_\nu < (diam(Q_\nu^*) - diam(Q_\nu^+))/2$. By Taylor expansion, (3.8) and $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \le 1$, for any $|x| < 2c\delta_\nu$,

$$0 \le f(x) \le \tilde{c}\delta_\nu^{2+2\alpha} + f'(0)x + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}x^2 + \frac{1}{2}|x|^{2+2\alpha}. \tag{3.13}$$

If $c = \epsilon_0/10$, $\tilde{c} = c^3$, setting $x = \pm c\delta_\nu$ in (3.13) yields

$$|f'(0)| \le (c^2 + \frac{1}{2}c^4 + \frac{1}{2}c^{1+2\alpha})\delta_\nu^{1+2\alpha} < 0.01\delta_\nu^{1+2\alpha}.$$

Hence

$$\sum_{k=0}^{2} \delta_\nu^{k-(2+2\alpha)} |\nabla^k f(0)| \le \tilde{c} + 0.01 + \tilde{c} < 0.01^3 + 0.01 + 0.01^3 < c_0,$$

contradicting with (3.5). □

For any $z \in Q_\nu^+$, we apply Fefferman-Phong Lemma 2.3 to the function $\phi(t) := \delta_\nu^{-(2+2\alpha)} \cdot f(z+t\delta_\nu)$.

Corollary 3.5. *Let $C = 1000$. For $z \in Q_\nu^+$, there exist universal constants $r_0 > 0, \tilde{A} > 0,\ c_2 > 0$ such that, for $x \in (z - r_0\delta_\nu, z + r_0\delta_\nu)$, either*

$$c_2\delta_\nu^{2+2\alpha} \le f(x) \le C\delta_\nu^{2+2\alpha},\ \|\sqrt{f(x)}\|_{C^1((z-r_0\delta_\nu,z+r_0\delta_\nu))} \le \tilde{A}\delta_\nu^{\alpha},\ \|\sqrt{f(x)}\|_{C^{1,\alpha}((z-r_0\delta_\nu,z+r_0\delta_\nu))} \le \tilde{A}; \tag{3.14}$$

or

$$c_2\delta_\nu^{2\alpha} \le f''(x) \le C\delta_\nu^{2\alpha}, \tag{3.15}$$

$$f(x) = f(X) + (x-X)^2 \int_0^1 f''(x + t(X-x))t\,dt, \tag{3.16}$$

where $x = X$ is the unique strict local minimum point of the function f in $(z - r_0\delta_\nu, z + r_0\delta_\nu)$.

Moreover, $g(x) := (x-X)(\int_0^1 f''(x+t(X-x))t\,dt)^{1/2}$ is in $C^{1,\alpha}((z-r_0\delta_\nu, z+r_0\delta_\nu))$ with $C^{1,\alpha}$ norm under control.

4. Proof of Theorem 1.1

Let $0 \le f \in C^{2,2\alpha}(\mathbb{R})$ with $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \le 1$.

4.1. **Proof of sufficiency.**

4.1.1. *Construction of g.* We write $\mathbb{R} \setminus \mathcal{F}$ (where $\mathcal{F}$ is defined in (2.10)) as a countable union of disjoint open intervals, so that $\mathbb{R} \setminus \mathcal{F} = \cup_{k=1}^{\infty} I_k$. Note if $\exists x_0 \in I_k$ with $f(x_0) = 0$, then $f''(x_0) \ne 0$. (If $0 < \alpha \le 1/2$, by Lemma 2.1, $|f(x_0)|$ and $|f''(x_0)|$ dominate $|f'(x_0)|$. If $1/2 < \alpha \le 1$, by Lemma

2.2, $|f(x_0)|$ and $|f''(x_0)|$ dominate $|f'(x_0)|$ and $|f'''(x_0)|$.) For each $m, k \in \mathbb{N}$, define

$$I_{k,m} = \{x \in I_k : dist(x, \mathcal{F}) > \frac{1}{m}\}, \quad B = \{x \in \mathbb{R} : f(x) = 0, f''(x) \neq 0\}.$$

Lemma 4.1. *$I_k \cap B$ is at most countable for each k, and*

$$I_k \cap B = \{\cdots x_{-2} < x_{-1} < x_0 < x_1 < x_2 \cdots\}.$$

Proof. $\forall N > 0$, we claim that $I_{k,m} \cap B \cap [-N, N]$ is finite for each $m, k \in \mathbb{N}$. Assume $I_{k,m} \cap B \cap [-N, N]$ is infinite, then $\exists x_0 \in \mathbb{R}$ such that x_0 is an accumulation point of $I_{k,m} \cap B$. So there is a sequence $\{x_n\}$ in B such that $\lim_{n\to\infty} x_n = x_0$, and $f(x_0) = \lim_{n\to\infty} f(x_n) = 0$. Note $f \geq 0$, so $f'(x_0) = 0$.

If $f''(x_0) \neq 0$, then $x = x_0$ is a strict local minimum point of f. However, by construction, near x_0 there is a point $x_1 \in B$, so that $f(x_1) = 0$, contradicting with strict local minimality.

If $f''(x_0) = 0$, then $x_0 \in \mathcal{F}$. However, $(x_0 - \frac{1}{2m}, x_0 + \frac{1}{2m}) \cap I_{k,m} = \varnothing$, contradiction.

Now since I_k is an interval and $I_{k,m} \subset I_{k,m+1}$. Points in $I_{k,m+1} \setminus I_{k,m}$ is either on the left or right of $I_{k,m}$. The points in $I_k \cap B \cap [-N, N]$ can be ordered. The lemma follows by letting $N \to \infty$. □

We define the function g as follows. If $x \in \mathcal{F}$, set $g(x) := 0$. For each k, if $I_k \cap B = \varnothing$ in I_k, then define $g(x) := \sqrt{f(x)}$ for $x \in I_k$. Otherwise,

$$I_k \cap B = \{\cdots x_{-2} < x_{-1} < x_0 < x_1 < x_2 \cdots\}.$$

Define $g(x) := (-1)^i \sqrt{f(x)}$ for $x \in [x_{i-1}, x_i]$. Note that g changes sign when crossing each x_i in I_k.

4.1.2. *C^1 regularity of g.* g is continuous in each $I_k = (a_k, b_k)$. It suffices to discuss the continuity at $x_0 \in \mathcal{F}$. By Taylor expansion of f near x_0, $f(x) = O(|x - x_0|^{2+2\alpha})$, so that $|\pm \sqrt{f(x)}| = O(|x - x_0|^{1+\alpha}) \to 0$ as $x \to x_0$ and $\lim_{x\to x_0} g(x) = 0$.

Lemma 4.2. *$g \in C^1(I_k)$ for each k.*

Proof. If $I_k \cap B = \varnothing$, then $g' = \frac{f'}{2\sqrt{f}} \in C^0(I_k)$. If $I_k \cap B \neq \varnothing$, then for each $x_i \in I_k \cap B$, $x_i \in Q_\nu$ for some $\nu = \nu(x_i)$. By Corollary 3.5, only (3.15) holds and near x_i, f can locally be written as

$$f(x) = (x - x_i)^2 \int_0^1 f''(x + t(x_i - x))t\, dt,$$

with $\int_0^1 f''(x + t(x_i - x))t\, dt \sim \delta_\nu^{2\alpha}$. By definition of g, near x_i, $g(x) = \pm(x - x_i)(\int_0^1 f''(x + t(x_i - x))t\, dt)^{1/2}$(the sign depends only on the choice of sign of g near x_0), so that g changes sign when crossing x_i. By Corollary 3.5, g' is continuous at x_i. □

The next is a key lemma to obtain uniform estimate for g' under (1.2).

Lemma 4.3. *Assume condition (1.2) is satisfied. There exists a universal constant $C_2 > 0$ such that, for any $x_0 \in I_k$ with $x_0 \in Q_\nu$ for some $\nu = \nu(x_0)$, then*

$$|g'(x_0)| \leq C_2 \delta_\nu^\alpha. \tag{4.1}$$

Proof. By Corollary 3.5, either (3.14) holds which implies (4.1); or

$$f(x) = f(X) + (x - X)^2 \int_0^1 f''(x + t(X - x))t\, dt, \tag{4.2}$$

where $x = X$ is the unique strict local minimum point of the function f in $(x_0 - r_0\delta_\nu, x_0 + r_0\delta_\nu)$.

If $f(X) = 0$, then $g(x) = \pm(x-X)(\int_0^1 f''(x+t(X-x))t\,dt)^{1/2}$. By (3.15), local Hölder continuity of g', and $g'(X) = \sqrt{\frac{1}{2}f''(X)}$, there is universal $b > 0$ such that,

$$|g'(x)| \le |g'(X)| + b|x-X|^\alpha \le \sqrt{\frac{1}{2}C\delta_\nu^{2\alpha}} + b\delta_\nu^\alpha \le C_2\delta_\nu^\alpha, \quad \forall x \in (x_0 - r_0\delta_\nu, x_0 + r_0\delta_\nu).$$

If $f(X) \neq 0$, then by (1.2) and (3.15),

$$M \cdot (f(X))^{\frac{\alpha}{1+\alpha}} \ge f''(X) \ge c_2\delta_\nu^{2\alpha}.$$

So that (4.2) reads

$$f(x) \ge f(X) \ge (\frac{c_2}{M})^{\frac{1+\alpha}{\alpha}} \cdot \delta_\nu^{2+2\alpha}.$$

By (3.2), $f(x) \sim \delta_\nu^{2+2\alpha}$ and the computation is reduced to case (2.13). □

Corollary 4.4. *Assume $I_k = (a_k, b_k)$, where $b_k < +\infty$. Then*

$$\lim_{x\to b_k^-} g'(x) = 0.$$

Similarly, if $a_k > -\infty$, then $\lim_{x\to a_k^+} g'(x) = 0$.

Proof. By Corollary 3.5, for each $x \in I_k$, $(x - r_0\delta_{\nu(x)}, x + r_0\delta_{\nu(x)}) \subset I_k$. Hence $\lim_{x\to b_k^-}\delta_{\nu(x)} = 0$. By (4.1), $|g'(x)| \le C_2\delta_{\nu(x)}^\alpha \to 0$ as $x \to b_k^-$. □

Corollary 4.5. *For any $x_0 \in \mathcal{F}$, $g'(x)$ is continuous at x_0, with*

$$\lim_{x\to x_0} g'(x) = g'(x_0) = 0.$$

Proof. By Taylor expansion of f near x_0, $f(x) = O(|x-x_0|^{2+2\alpha})$, so that

$$\left|\frac{g(x)-g(x_0)}{x-x_0}\right| = \left|\frac{\pm\sqrt{f(x)}}{x-x_0}\right| = O(|x-x_0|^\alpha) \to 0 \text{ as } x \to x_0.$$

If x_0 has a neighbourhood which is contained in $\mathcal{F}$, then the result is trivial. Otherwise, x_0 is the boundary point of some interval $I_k = (a_k, b_k)$. Without loss of generality we assume $x_0 = b_k < +\infty$.

If x_0 is discrete, then x_0 is the boundary point of two consecutive intervals I_k and I_{k+1}, with $a_k < b_k = x_0 = a_{k+1} < b_{k+1}$. By Corollary 4.4,

$$\lim_{x\to b_k^-} g'(x) = \lim_{x\to a_{k+1}^+} g'(x) = 0.$$

Otherwise, $x_0 \in [x_0, a_{k+1}] \subset \mathcal{F}$ for some a_{k+1}. By Corollary 4.4 again,

$$\lim_{x\to b_k^-} g'(x) = \lim_{x\to x_0^+} g'(x) = 0.$$

□

To summarize, $g \in C^1(\mathbb{R})$, with $|g'(x)| \le C_2$, $\forall x \in \mathbb{R}$, since $\delta_\nu \le 1$.

4.1.3. *Global Hölder estimate.* Let $x, y \in \mathbb{R}$ with $x \neq y$.

(1) If $\exists z \in \mathbb{R} \setminus \mathcal{F}$ such that x and y are both contained in $(z - r_0\delta_{\nu(z)}, z + r_0\delta_{\nu(z)})$, then by Corollary 3.5, the Hölder estimate is trivial if (3.14) holds or (3.15) holds with $f(X) = 0$. If case (3.15) holds with $f(X) \neq 0$, then by (1.2),

$$M \cdot (f(X))^{\frac{\alpha}{1+\alpha}} \geq f''(X) \geq c_2\delta_\nu^{2\alpha}.$$

So that (3.16) reads

$$f(x) \geq f(X) \geq (\frac{c_2}{M})^{\frac{1+\alpha}{\alpha}} \cdot \delta_\nu^{2+2\alpha} \text{ and } f(y) \geq (\frac{c_2}{M})^{\frac{1+\alpha}{\alpha}} \cdot \delta_\nu^{2+2\alpha}. \tag{4.3}$$

The computation is reduced to case (2.13), and $|g'(x) - g'(y)|/|x-y|^\alpha$ is bounded by a constant depending only on M and α.

(2) Assume $\nexists z \in \mathbb{R} \setminus \mathcal{F}$ such that x and y are both contained in $(z - r_0\delta_{\nu(z)}, z + r_0\delta_{\nu(z)})$.
   (a) If $x \in \mathcal{F}$ and $y \in \mathcal{F}$, then by Corollary 4.5, $|g'(x) - g'(y)| = |0 - 0| = 0$.
   (b) If $x \notin \mathcal{F}$ and $y \in \mathcal{F}$, then $x \in Q_\nu$ for some $\nu = \nu(x)$ and $|x - y| \geq r_0\delta_\nu$. By (4.1) and Corollary 4.5,

$$|g'(x) - g'(y)| = |g'(x)| \leq C_2\delta_\nu^\alpha \leq \frac{C_2}{r_0^\alpha} \cdot |x-y|^\alpha.$$

   (c) If $x \notin \mathcal{F}$ and $y \notin \mathcal{F}$, then $x \in Q_{\nu(x)}$ and $x \in Q_{\nu(y)}$, with $|x - y| \geq r_0\delta_{\nu(x)}$ and $|x - y| \geq r_0\delta_{\nu(y)}$. By (4.1),

$$|g'(x) - g'(y)| \leq |g'(x)| + |g'(y)| \leq C_2\delta_{\nu(x)}^\alpha + C_2\delta_{\nu(y)}^\alpha \leq \frac{2C_2}{r_0^\alpha} \cdot |x-y|^\alpha.$$

Remark 4.6. *$C^{1,\alpha}$ estimate of g doesn't depend on the choice of sign of g in each interval I_k.*

4.2. **Proof of necessity.** Assume (1.2) doesn't hold and $f = g^2$ for some $g \in C^{1,\alpha}(\mathbb{R})$, then there is a sequence x_n in $\mathcal{A}$ such that

$$f''(x_n) \geq n f^{\frac{\alpha}{1+\alpha}}(x_n), \quad \forall n \in \mathbb{N}. \tag{4.4}$$

$f(x_n) > 0$, so $x_n \in Q_\nu$ for some $\nu = \nu(n)$.

In case (i) of Lemma 3.4, $f(x_n) \geq \tilde{c}\delta_\nu^{2+2\alpha}$ and $f''(x_n) < C\delta_\nu^{2\alpha}$. By (4.4),

$$C\delta_\nu^{2\alpha} \geq n(\tilde{c}\delta_\nu^{2+2\alpha})^{\frac{\alpha}{1+\alpha}}, \tag{4.5}$$

so that δ_ν get cancelled. Letting $n \to \infty$ in (4.5), contradiction.

In case (ii) of Lemma 3.4, $f(x_n) < \tilde{c}\delta_\nu^{2+2\alpha}$ and $f''(x_n) \geq \tilde{c}\delta_\nu^{2\alpha}$. Define $s_n = \sqrt{\frac{f(x_n)}{f''(x_n)}}$. By (4.4) and $\delta_\nu \leq 1$,

$$s_n = \sqrt{\frac{f(x_n)}{f''(x_n)}} \leq \sqrt{\frac{f(x_n)}{n f^{\frac{\alpha}{1+\alpha}}(x_n)}} = \frac{f^{\frac{1}{2+2\alpha}}(x_n)}{\sqrt{n}} \leq \tilde{c}^{\frac{1}{2+2\alpha}} \cdot \frac{\delta_{\nu(n)}}{\sqrt{n}} \to 0 \text{ as } n \to \infty. \tag{4.6}$$

If $1/2 < \alpha \leq 1$, by Taylor expansion and $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$,

$$f(x_n + s_n) \geq f(x_n) + \frac{1}{2}f''(x_n)s_n^2 - \frac{1}{6}|f'''(x_n)|s_n^3 - \frac{1}{6}s_n^{2+2\alpha}.$$

$$f(x_n + s_n) \leq f(x_n) + \frac{1}{2}f''(x_n)s_n^2 + \frac{1}{6}|f'''(x_n)|s_n^3 + \frac{1}{6}s_n^{2+2\alpha}.$$

By (3.2) and (4.6),

$$|f'''(x_n)|s_n^3 = |f'''(x_n)|s_n \cdot \frac{f(x_n)}{f''(x_n)} \leq C\delta_\nu^{2+2\alpha-3} \cdot \tilde{c}^{\frac{1}{2+2\alpha}} \cdot \frac{\delta_\nu}{\sqrt{n}} \cdot \frac{f(x_n)}{\tilde{c}\delta_\nu^{2\alpha}} \leq \frac{1}{2}f(x_n) \text{ for large } n,$$

$$s_n^{2+2\alpha} = s_n^{2\alpha} \cdot \frac{f(x_n)}{f''(x_n)} \leq (\tilde{c}^{\frac{1}{2+2\alpha}} \cdot \frac{\delta_\nu}{\sqrt{n}})^{2\alpha} \cdot \frac{f(x_n)}{\tilde{c}\delta_\nu^{2\alpha}} \leq \frac{1}{2}f(x_n) \text{ for large } n.$$

So $4f(x_n) \geq f(x_n+s_n) \geq f(x_n) > 0$ for large n. By mean value theorem,

$$\begin{aligned} 2|g'(x_n+s_n) - g'(x_n)| &= |\pm\frac{f'(x_n+s_n)}{\sqrt{f(x_n+s_n)}} - (\pm\frac{f'(x_n)}{\sqrt{f(x_n)}})| \\ &= |\frac{f'(x_n+s_n) - f'(x_n)}{\sqrt{f(x_n+s_n)}}| = \frac{|f''(\xi_n)| \cdot s_n}{\sqrt{f(x_n+s_n)}}, \end{aligned}$$

where $\xi_n \in (x_n, x_n+s_n)$. By Taylor expansion of f'', for large n,

$$f''(\xi_n) \geq f''(x_n) - |f'''(x_n)|s_n - s_n^{2\alpha} \geq \frac{1}{2}f''(x_n).$$

If $0 < \alpha \leq 1/2$, by expansion to the second order, we also have $4f(x_n) \geq f(x_n+s_n) \geq f(x_n) > 0$ and $f''(\xi_n) \geq \frac{1}{2}f''(x_n)$ for large n.

Therefore, for any $0 < \alpha \leq 1$, by (4.4),

$$\begin{aligned} 2|g'(x_n+s_n) - g'(x_n)| &= \frac{|f''(\xi_n)| \cdot s_n}{\sqrt{f(x_n+s_n)}} \geq \frac{\frac{1}{2}f''(x_n) \cdot s_n}{\sqrt{4f(x_n)}} = \frac{1}{4}\sqrt{f''(x_n)} \\ &= \frac{1}{4}s_n^\alpha \cdot \frac{f''(x_n)^{1/2}}{s_n^\alpha} = \frac{1}{4}s_n^\alpha \cdot \left(\frac{f''(x_n)}{f(x_n)^\alpha} \cdot f''(x_n)^\alpha\right)^{1/2} \\ &= \frac{1}{4}s_n^\alpha \cdot \left(\frac{f''(x_n)^{1+\alpha}}{f(x_n)^\alpha}\right)^{1/2} \geq \frac{1}{4}s_n^\alpha \cdot \sqrt{n}. \end{aligned}$$

Hence

$$|g'(x_n+s_n) - g'(x_n)|/s_n^\alpha \geq \frac{1}{8}\sqrt{n} \to \infty \text{ as } n \to \infty.$$

Contradiction.

References

[1] B. Bian and P. Guan, *A microscopic convexity principle for nonlinear partial differential Equations*, Inventiones Mathematicae, Vol. 177, No. 2, (2009), 307-335.

[2] J.-M. Bony, *Sommes de carrés de fonctions dérivables*, Bulletin de la Société Mathématique de France, Vol. 133, No. 4, (2005), 619-639.

[3] J.-M. Bony and F. Broglia and F. Colombini and L. Pernazza. *Nonnegative functions as squares or sums of squares*, Journal of Functional Analysis, Vol. 232, No. 1 (2006), 137-147.

[4] C. Fefferman and D. H. Phong, *On positivity of pseudo-differential operators*, Proceedings of the National Academy of Sciences, Vol. 75, No. 10, (1978), 4673-4674.

[5] C. Fefferman and D. H. Phong, *The uncertainty principle and sharp Garding inequalities*, Commu. Pure & Appl. Math., Vol. 34, No. 3, (1981), 285-331.

[6] G. Glaeser, *Racine carrée d'une fonction différentiable*, Annales de l'Institut Fourier, Vol. 13, No. 2, (1963), 203-210.

[7] P. Guan, C^2 *a priori estimates for degenerate Monge-Ampère equations*, Duke Mathematical Journal, Vol. 86, No. 2, (1997), 323-346.

[8] P. Guan and Y.Y. Li, *The Weyl problem with nonnegative Gauss curvature*, Journal of Differential Geometry, Vol. 39, No. 2, (1994), 331-342.
[9] P. Guan and E. Sawyer, *Regularity Estimates for the Oblique Derivative Problem.* Annals of Mathematics, Vol. 137, No. 1, (1993), 1-70.
[10] L. Hörmander, *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*, Springer Berlin Heidelberg, (2007).
[11] X. Jiang, *Isometric embedding with nonnegative Gauss curvature under the graph setting*, Discrete and Continuous Dynamical Systems, Vol. 39, No. 6, (2019), 3463-3477.
[12] L. Korobenko and E. Sawyer, *Sum of squares I: scalar functions*, https://arxiv.org/pdf/2107.12840.pdf
[13] T. Mandai, *Smoothness of roots of hyperbolic polynomials with respect to one-dimensional parameter*, Bull. Fac. Gen. Ed. Gifu Univ, No. 21, (1985), 115-118.
[14] L. Nirenberg and F. Trèves, *On local solvability of linear partial differential equations part I: Necessary conditions*, Communications on Pure and Applied Mathematics, Vol. 23, No. 1, (1970), 1-38.
[15] S. Picard, *A priori estimates of the degenerate Monge-Ampère equation on Kähler manifolds of nonnegative bisectional curvature*, Math. Res. Lett. 20 (2013), 1145-1156.
[16] D. Tataru, *On the Fefferman-Phong inequality and related problems*, Communications in Partial Differential Equations, Vol. 27, No. 11-12, (2002), 2101-2138.

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