Monge-Ampére Equations and Related Topics

P. Guan

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One of the important connection between Monge-Ampére equations and differential geometry is exhibited by the following simple fact: if $M = \{(x, u(x)) \mid x \in \Omega \subset \mathbf{R}^n\}$ is a graph of a function $u(x)$, the Gauss curvature of the (graph) surface $k(x, u(x))$ satisfies

$$\det(D^2u) = k(x, u(x))(1 + |Du|^2)^{\frac{n+2}{2}}.$$

Historically, the study of MA is very much motivated by the following two problems: Minkowski problem and Weyl problem. One is of prescribing (Gauss) curvature type, another is of embedding type. The development of MA theory in PDE is closely related to that of Fully Nonlinear Equations (elliptic). The main contributors are: Alexandrov, Lewy, Nirenberg, Pogolerov, Cheng-Yau, Krylov, Caffarelli, Trudinger and others.

In these notes, we treat some geometrical and analytical problems related to Monge-Ampére equations. Basically, we look Monge- Ampére type equations in two kinds of spaces: (i), on S^n; (ii), in bounded convex domains in R^n. The problems in the case (i) are usually connected to geometry, like Minkowski problem and reflection problem related to geometric optics. While for the problems in case (ii), we consider Dirichlet problem and second boundary value problem (a problem arising from mass transport problem). The main discussions are concerned with exsistence and regularity of the solutions to the problems. Generally speeking, there are two ways to tackle the problems: one is via continuity mothed which involving some appropriate a priori estimates, the other is weak solution theory. In the first four sections, we employ continuity mothed to get classical solutions. We treat the weak solution theory in the last section.

These notes were compiled from materials of four-week lectures given by author at Morningside Mathematics Institute at Academy of Science in Beijing in May 1998. Due to the limitation of the time, many important topics related to Monge-Ampére equations were not covered. The most important missing is complex Monge-Ampére equations. And some topics in real Monge-Ampeŕe equations originally intended to be included were also left over. We hope to address these topics in other place. I would like take this opportunity to thank S.T. Yau and L. Yang for the invitation and friends Liqun Zhang, Yuefeng Wang, Youde Wang and others at the Institute for the warm hospitality. The notes would never have been completed without forceful prodding by Weiyue Ding.

1 Minkowski Problem

Suppose M is a strongly convex (closed) hypersurface in $\mathbf{R}^{n+1}$. $\forall\, x \in M$, there is a Gauss map $\vec{n}_M(x)$ (the unit outer normal at x): $M \to S^n$ is a diffeomorphism. If $k(x)$ is the Gauss curvature at $x \in M$, the function $k(\vec{n}_M^{-1}(y))$ is a positive function on S^n. The Minkowski problem is: given a positive function $f(y)$ on S^n, find a convex surface M, such that $k(\vec{n}_M^{-1}(y)) = f(y)$.

If $k(\vec{n}_M^{-1}(y)) = f(y)$, we have

$$\int_{S^n} \frac{y_i}{f(y)} = \int_M \vec{n} \cdot \vec{x}_i = 0 \quad \text{by Divergence Theorem.}$$

Therefore, a necessary condition is:

$$\int_{S^n} \frac{y_i}{f(y)} = 0 \quad \forall\, i = 1, \ldots, n+1. \tag{1}$$

It turns out this is also sufficient.

Theorem 1 *Suppose $f \in C^2(S^n)$, $f(x) > 0$, $\forall\, x \in S^n$, and f satisfies (1). Then, there is a $C^{3,\alpha}$ ($\forall\, 0 < \alpha < 1$) strongly convex surface M in $\mathbf{R}^{n+1}$, such that $k(\vec{n}_M^{-1}(y)) = f(y)$ $\forall\, y \in S^n$. And such M is unique up to rigidity Euclidean motions.*

The above result is due to Nirenberg, Pogolerov for $n = 2$, and Cheng-Yau, Pogolerov for general $n \geq 2$. For f continuous, Minkowski himself obtained weak solutions, Alexandrov later deals with more general f and developed general theory of weak solutions. Here, we will only concentrate on regular solutions. The uniqueness result follows from Minkowski mixed volume inequalities, we will not deal with it here.

Supporting function $H(x)$ of convex surfaces: Let M be a convex surface, we define

$$H(x) = \vec{n}^{-1}(x) \cdot x, \quad \forall\, x \in S^n.$$

We extend H to $R^{n+1}\{0\}$ as a homogeneous function of degree one, i.e., $H(x) = |x| H\left(\frac{x}{|x|}\right)$, $\forall\, x \in R^{n+1}\{0\}$. If $y \in \vec{n}^{-1}(x)$, $H(x) = \sum_{i=1}^{n+1} x_i y_i$. We may also express $H(x) = \sup_{y \in M}\left(\sum_{i=1}^{n+1} x_i y_i\right)$. H is a convex function. On the other if such H is given, we may recover M from H:

$$y_i = \frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n+1,$$

they are completely determined by their values on $x_i = -1$, $i = 1, \ldots, n+1$ by homogeneity.

A straight forward computation yields:

$$\det(H_{ij} + H\delta_{ij}) = \frac{1}{f} \quad \text{on } S^n \tag{2}$$

where H_{ij} covariant derivatives with respect to ortho normal frames in X^n. Our main goal is to solve (2) for $(H_{ij} + H\delta_{ij}) > 0$.

If $f_0 \equiv 1$, we know $H \equiv 1$ is a solution. We set $f_t = (1-t)f_0 + tf$, and we consider

$$\det(H_{ij}^{(t)} + H^{(t)}\delta_{ij}) = \frac{1}{f_t} \quad \text{on } S^n \tag{3}$$

for $0 \le t \le 1$.

If we can show that the solution set $E = \{0 \le t \le 1 \mid (3) \text{ is solvable}\}$ is open and closed, we are at home, since $[0,1]$ is connected, for $0 \in E$ (3).

Openness: Suppose for some $t_0 \in [0,1]$. (3) is solvable. We denote H be a solution of (3), $H \in C^{3,\alpha}$. We want to show $\exists\ \varepsilon > 0$, such that if $\|g - f_{t_0}\|_{C^{1,1}} < \varepsilon$, and $\int_{S^n} x_i g(x) = 0$, $i = 1, \ldots, n+1$, then $\exists\ \tilde{H} \in C^{3,\alpha}$ solve (2) for g (g in place of f). This will be done using Implicit Function Theorem.

Let L_H be the linearized operator of

$$F(H) = \det(H_{ij} + H\delta_{ij}).$$

Set $w_{ij} = H_{ij} + H\delta_{ij}$, we have $L_H(u) = \sum_{i,j} c(w_{ij})(u_{ij} + u\delta_{ij})$, where $(c(w,j))$ is a cofactor of (w_{ij}). It's easy to show L_H is self-adjoint. Since $\forall\ \delta > 0$ small, $\tilde{H} = H + \delta x_i$ is a supporting function of $\tilde{M}$ which is a translation of M. So, $\tilde{H}$ is also a solution of (2) (with the same g). Since δ is arbitrary. We conclude that $L_H(x_i) = 0$, $\forall\ i = 1, 2, \ldots, n+1$.

Fact: $\forall\ u \in C^2$, if $\delta > 0$ small, $H + \delta u$ is a supporting function of some convex surface.

Therefore $\int_{S^n} x_i F(H + \delta u) = 0$. Now,

$$\delta \text{ arbitrary } \Rightarrow \int_{S^n} x_i L_H(u) = 0, \quad i = 1, \ldots, n+1.$$

That is Range$(L_H) \perp$ Span$\{x_1, \ldots, x_{n+1}\} \Rightarrow$ Ker$(L_H^*) \supseteq$ Span$\{x_1, \ldots, x_{n+1}\}$.

Lemma 1 *Ker(L_H^*) = Span$\{x_1, \ldots, x_{n+1}\}$. (Note that $L_H^* = L_H$.)*

Assuming the Lemma, we get $L_H : C^{3,\alpha}(S^n) \to S$, surjective, where $S \stackrel{\text{def}}{\equiv} \{v \in C^{1,\alpha}(S^n) \mid \int_{S^n} x_i v = 0\}$. By Implicit Function Theorem, F is invertible near H. This gives the openness of the problem (3).

The following proof of the lemma is due to Cheng-Yau.

Proof of Lemma. Suppose $u \in$ Ker(L_H^*) = Ker(L_H). Set $\vec{Z} = \sum_{i=1}^n u_i \vec{e}_i + u\vec{e}_{n+1}$, where $\vec{e}_1, \ldots, \vec{e}_n$ a orthonormal frame on S^n (local). We will show $\overline{Z} \equiv$ const. This will give $u = \sum_{j=1}^{n+1} a_j x_j$ (since $u = \vec{Z}_1 \cdot \vec{e}_n = \vec{Z} \cdot x$, $\vec{e}_{n+1} = (x_1, \ldots, x_{n+1}) \in S^n$). Let w_j be 1-forms dual to e_j ($j = 1, \ldots, n$). We have

$$d\vec{Z} = \sum_{j=1}^n \left(\sum_{i=1}^n (u_{ij} + u\delta_{ij})\vec{e}_i \right) w_j.$$

Let $\vec{X} = \sum_{i=1}^{n} H_i\vec{e}_i + H\vec{e}_{n+1} \Rightarrow d\vec{X} = \sum_j^i \left(\sum_i^n (H_{ij} + H\delta_{ij})\vec{e}_i\right) w_j$. Set $\Omega = \vec{X} \wedge \overline{Z} \wedge d\vec{Z} \wedge \overbrace{d\vec{X} \wedge \cdots \wedge d\vec{X}}^{n-2}$.

$$\begin{aligned} d\Omega &= d\vec{X} \wedge \vec{Z} \wedge d\vec{Z} \wedge d\vec{X} \wedge \cdots \wedge d\vec{X} + \vec{X} \wedge d\vec{Z} \wedge d\vec{Z} \wedge d\vec{X} \wedge \cdots \wedge d\vec{X} \\ &= \sum_{ij} [c(w_{ij})(u_{ij} + u\delta_{ij})](\vec{e}_1 \wedge \cdots \wedge \vec{e}_n) \otimes w_1 \wedge \cdots \wedge w_n \\ &\quad + \vec{X} \wedge d\vec{Z} \wedge d\vec{Z} \wedge d\vec{X} \wedge \cdots \wedge d\vec{X} \\ &= \vec{X} \wedge d\vec{Z} \wedge d\vec{Z} \wedge d\vec{X} \wedge \cdots \wedge d\vec{X}. \end{aligned}$$

Since $L_H(u) = 0$. Integrate over S^n, we get

$$0 = \int_{S^n} d\Omega = \int_{S^n} \vec{X} \wedge d\vec{Z} \wedge d\vec{Z} \wedge d\vec{X} \wedge \cdots \wedge d\vec{X}.$$

Let $V_{kj} = \sum_i (u_{ki} + u\delta_{ki})c(w_{ij})$, $\vec{e}_j^* = \sum_i^n (H_{ij} + H\delta_{ij})\vec{e}_i$. If necessary, we diagonalize $(H_{ij} + H\delta_{ij})$, we may assume $V_{ij}V_{ji} \geq 0$, $\forall\, i, j$. Now

$$\begin{aligned} d\vec{Z} &= \det(H_{ij} + H\delta_{ij})^{-1} \sum_{i,j} V_{ij}\vec{e}_j \otimes w_i \\ &\Rightarrow \int_{S^n} \langle \vec{X}, \vec{e}_{n+1} \rangle \sum_{i \neq j} (V_{ii}V_{jj} - V_{ij}V_{ji}) \det(H_{ij} + H\delta_{ij})^{-1} = 0 \\ &\quad L_H(u) = 0 \Rightarrow \sum_i V_{ii} = 0 \Rightarrow \sum_{i \neq j} (V_{ii}V_{jj} - V_{ij}V_{ji}) \\ &= \frac{1}{2}\left[\left(\sum_i V_{ii}\right)^2 - \sum_i V_{ii}^2\right] - \sum_{i \neq j} V_{ij}V_{ji} \leq 0 \\ &\quad \langle \vec{X}, \vec{e}_{n+1} \rangle = H > 0, \end{aligned}$$

we conclude that $V_{ij} = 0$, $\forall\, i, j \leq n$. In turn, we get $u_{ij} + u\delta_{ij} = 0 \Rightarrow d\vec{Z} = 0 \Rightarrow \equiv$ const. □

Closeness. To show the closeness, we want to establish a priori estimates for problem (3): $\|H\|_{C^{3,\alpha}(S^n)} \leq C$.

Since $(H_{ij} + \delta_{ij}H) > 0$, $\det^{\frac{1}{n}}$ is concave, we only need to obtain $\|H\|_{C^{1,1}(S^n)} \leq C$ by Evans-Krylov Theorem.

1. C^0-estimates:

Cheng-Yau's Lemma. *Suppose $M \in C^2$, $M \subset \mathbf{R}^{n+1}$, strongly convex, and K be the Gauss curvature of M. Let L, r be the extrinsic diameter and inner radius of M, then*

$$\begin{aligned} L &\leq c_n \left(\int_{S^n} \frac{1}{K}\right)^{\frac{n}{n-1}} \left\{\int_{y \in S^n} \int_{S^n} \max(0, \langle y, x \rangle)/K(x)\right\}^{-1}, \\ r &\geq \tilde{c}_n \left(\int_{S^n} \frac{1}{K}\right)^{-n} \left\{\int_{y \in S^n} \max(0, (y, x))/K(x)\right\}^n \end{aligned}$$

where $c_n, \tilde{c}_n$ are constants that depend only on n.

Proof. Let $p, q \in M$, the line segment joint p and q has length L. We may assume 0 is in the middle of the line segment. Let $\vec{y}$ be a limit vector in the direction of this line. $\forall\, x \in S^n$.

$$H(x) = \sup_{Z \in M} \langle Z, x\rangle \geq \frac{1}{2} L \max(0, \langle y, x\rangle).$$

multiply by $\frac{1}{K}$ and integrate over S^n, we get

$$L \;\leq\; \frac{1}{2}\left(\int_{S^n} \frac{H}{K}\right)\left(\int_{S^n} \max(0, \langle y, x\rangle)/K(x)\right)^{-1}$$

Since

$$\int_{S^n} \frac{H}{K} = \int_M H,$$

and

$$\Delta \sum_{i=1}^{n+1} x_i^2 = 2(n+1),$$

by the divergence theorem,

$$\int_M (n+1) = \int_M \vec{X} \cdot \vec{N} = \int_M H$$

(where $\overline{M}$ is the convex body bounded by M). By isoperimetric inequality,

$$Vol(\overline{M}) \leq c_n \left(\int_{S^n} \frac{1}{K}\right)^{\frac{n}{n-1}},$$

$$\Rightarrow L \leq c_n \left(\int_{S^n} \frac{1}{K}\right)^{\frac{n}{n-1}} \left(\inf_{y \in S^n} \int_{S^n} \max(0, (y, x))/K(x)\right)^{-1}.$$

As for a lower bound of r, Cheng-Yau has an elementary (but highly nontrivial) proof. It can also be deduced from John's Lemma. Since we will need John's Lemma in the discussion of weak solution theory of Monge-Ampére equations. We now state (and prove):

John's Lemma. *Let $\overline{M}$ be a convex (bounded) body in $\mathbf{R}^n$, then there is an ellipsoid E, (after a proper translation), $\frac{1}{n}\, E \subset \overline{M} \subset E$.*

Proof of John's Lemma. Let E be an ellipsoid of smallest volume containing $\overline{M}$. We assume 0 be the center of E. We claim $\frac{1}{n}\, E \subset \overline{M}$. By affine transform, we may assume

$E = B_1$. Now, suppose $\operatorname{dist}(\partial\overline{M}, 0) = \lambda < \frac{1}{n}$. We may assume $-\lambda\vec{e}_n$ is the closest point of $\partial\overline{M}$ to 0. Consider now the ellipsoid E_δ:

$$(1+\delta)\left[(x_n-1)+\frac{1}{(1+\delta)^{1/2}}\right]^2+\frac{1}{(1+\delta)^{\frac{1}{n-1}}}\,|x'|^2\le 1 \qquad (x'=(x_1,\ldots,x_{n-1})).$$

$\operatorname{Vol}(E_\delta) = \operatorname{Vol}(B_1)$. ∂E_δ intersect ∂B_1 at $x_n = 1$, $x' = 0$ and where

$$(1+\delta)\left[(x_n-1)^2+\frac{2(x_n-1)}{(1+\delta)^{1/2}}\right]+\frac{(1-x_n)(1+x_n)}{(1+\delta)^{1/n-1}}=0,$$

i.e., at where $x_n = 1$ and

$$(1+\delta)\left[(x_n-1)+\frac{2}{(1+\delta)^{1/2}}\right]+\frac{1+x_n}{(1+\delta)^{1/n-1}}=0.$$

Develop in δ, we get

$$0=\frac{n}{n-1}\left(x_n+\frac{1}{n}\right)+O(\delta)\Rightarrow x_n=\frac{-1}{n}+O(\delta).$$

If δ small enough, $\partial(E_\delta + \frac{\delta}{2}\vec{e}_n)$ strictly contains $\partial\overline{M}$. So we may obtain a smaller ellipsoid $\tilde{E}$ which contains M. Contradiction. □

Now, back to the lower bond of r. Let E be an ellipsoid in $\mathbf{R}^{n+1}$, such that

$$\frac{1}{(1+n)}\,E\subset\overline{M}\subset E.$$

Let $0 < a_1 \le a_2 \le \cdots \le a_{n+1}$ be the principal axies of E. We have $r \ge \frac{1}{n+1}\,a_1$, and $L \ge \frac{1}{n+1}\,a_i$, $i = 1, \ldots, n+1$. Since

$$\left(\frac{1}{n+1}\right)^{n+1}a_1\ldots a_{n+1}\le\operatorname{Vol}(\overline{M})\le a_1\ldots a_{n+1}\le(n+1)^{n+1}rL^n.$$

On the other hand, $\operatorname{Vol}(\overline{M}) = \frac{1}{n+1}\int_{S^n}\frac{H}{K} \ge \frac{2L}{n+1}\left(\inf_{y\in S^n}\int_{S^n}\max(0,(y,x))/k(x)\right)$ (by the 1st inequality in the Lemma). This yields

$$\begin{aligned}\frac{2L}{n+1}\left(\inf_{y\in S^n}\int_{S^n}\max(0,(y,x))/k(x)\right)&\le(n+1)^{n+1}rL^n\\ \Rightarrow r\ &\ge\ \frac{2L^{-(n-1)}}{(n+1)^{n+2}}\left(\inf_{y\in S^n}\int_{S^n}\max(0,(y,x))/k(x)\right)\\ &\ge\ \tilde{c}_n\left(\int_{S^n}\frac{1}{K}\right)^{-n}\left(\inf_{y\in S^n}\int_{S^n}\max(0,(y,x))/k(x)\right)^n.\end{aligned}$$

□

C^1-estimates: by C^0-estimates (Cheng-Yau's Lemma), $0 < \frac{1}{c} < H < c < \infty$. Since $\max_{S^n} |\nabla H| \leq \max_{S^n}(H^2 + |\nabla H|^2)^{\frac{1}{2}}$. Let $x_0 \in S^n$, $\max_{S^n}(H^2 + |\nabla H|^2) = H^2(x_0) + |\nabla H(x_0)|^2$. At x_0,

$$HH_i + \sum_j H_j H_{ij} = 0 \Rightarrow (H_{ij} + H\delta_{ij}) \cdot \begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix} = 0$$

since $(H_{ij} + H\delta_{ij})$ invertible

$$\Rightarrow \nabla H = 0$$

at x_0,

$$\Rightarrow \max_{S^n}(H^2 + |\nabla H|^2) \leq \max_{S^n} H^2 \Rightarrow \max_{S^n} |\nabla H| \leq \max_{S^n} H \leq c$$

.

C^2-estimates: Since $(H_{ij} + H\delta_{ij}) > 0$ we only need to get upper bound of $\mathrm{Tr}(H_{ij})$. Let $x_0 \in S^n$, such that $\mathrm{Tr}(H_{ij})(x_0) = \max_{S^n} \mathrm{Tr}(H_{ij})$. We may assume $\mathrm{Tr}(H_{ij})(x_0) \geq 1$, and $(H_{ij}(x_0))$ diagonal at x_0. At x_0,

$$(\nabla(\mathrm{Tr}(H_{ij})))_{x_0} = 0,$$

and

$$\left(\nabla^2(\mathrm{Tr}(H_{ij}))\right)_{x_0} \leq 0 \Rightarrow$$

at x_0,

$$\begin{aligned} 0 \geq w^{ij}(\mathrm{Tr}(\nabla^2 H))_{ij} &= w^{ii}(\mathrm{Tr}(\nabla^2 H))_{ii} \\ &= w^{ii}\{\Delta(H_{ii}) + 2\Delta H - 2nh_{ii}\} \\ &= w^{ii}\Delta(H_{ii}) + 2(\mathrm{Tr}(\nabla^2 H))(\sum w^{ii}) - 2n^2 + 2nH(\sum w^{ii}). \end{aligned}$$

We compute $w^{ii}\Delta(H_{ii})$. Apply Δ to $\det(w_{ij})^{\frac{1}{n}} = \left(\frac{1}{K}\right)^{\frac{1}{n}}$ and by concavity of $\det^{\frac{1}{n}}$, we get

$$\begin{aligned} w^{ii}\Delta(H_{ii}) &= w^{ij}\Delta(H_{ij}) = w^{ij}\{\Delta(w_{ij}) - \delta_{ij}\Delta H\} \\ &\geq \Delta\left(\frac{1}{K}\right) - \left(\sum w^{ii}\right)\mathrm{Tr}(\nabla^2 H) \\ \Rightarrow 0 &\geq \mathrm{Tr}(\nabla^2 H)\left(\sum w^{ii}\right) - 2n^2 + \Delta\left(\frac{1}{K}\right) + 2nH\left(\sum w^{ii}\right) \\ & \sum w^{ii} \geq n(\Pi w^{ii})^{\frac{1}{n}} = n\left(\frac{1}{K}\right)^{\frac{1}{n}} \geq c > 0 \\ &\Rightarrow \mathrm{Tr}(\nabla^2 H)|_{x_0} \leq c \Rightarrow \max_{S^n} |\nabla^2 H| \leq c. \end{aligned}$$

□

This will conclude the closeness of (3), and gives the solution for Minkowski problem.

Remark: In above proof, we in fact need $H \in C^k$ for large k, since the final bounds depend only on $\|K\|_{C^2}$, so the Theorem holds for $K \in C^{1,1}(S^n)$.

2 Alexanderov Problem

For $n \geq 2$, Let M^n be a finite convex, not necessarily smooth, hypersurface in Euclidean space R^{n+1} containing the origin. More precisely, M^n is the boundary of some convex domain in R^{n+1} containing a neighborhood of the origin. We write $M^n = \{R(x) = \rho(x)x | \, x \in S^n\}$, where ρ is a function from S^n to R^+. Let $\nu : M^n \to S^n$ denote the generalized Gauss map, namely, $\nu(Y)$ is the *set* of outward unit normals to supporting hyperplanes of M^n at Y. The integral Gaussian curvature of M^n is defined by

$$\mu(F) = |\nu(R(F))|,$$

for all Borel set $F \subset S^n$. It is clear that μ is a nonnegative, completely additive function on the Borel sets of S^n. For any set $F \subset S^n$, let $Cone(F) = \{tX \mid X \in F, t \geq 0\}$ be the cone generated by F. For any cone $C \subset R^{n+1}$, let $C^* = \{X \in R^{n+1} \mid X \cdot Y \leq 0, \forall\, Y \in C\}$ be the dual cone. $F^* = (Cone(F))^* \cap S^n$ is the dual angle.

Alexandrov problem is: for a given measure μ on S^n, find a convex surface M as a graph over S^n, such that the Gauss measure of M is the pull-back μ by radial mapping. The problem is similar to Minkowski problem. Alexandrov established the following result [[?]].

Theorem 2 *A necessary and sufficient condition for a nonnegative, completely additive function μ on the Borel sets of S^n to be the integral Gaussian curvature of some finite convex hypersurface in Euclidean space R^{n+1} containing the origin is:*

(A1) $\mu(S^n) = |S^n|$,

(A2) For every convex subset F of S^n, $\mu(F) < |S^n| - |F^*|$.

Such hypersurface is unique up to a homothetic transformation.

When M^n is C^2, it is clear that $|\nu(R(F))| = \int_{R(F)} \kappa$, where κ is the Gauss-Kronecker curvature of M^n. Therefore, there is similar differential geometric question for the problem as in Minkowski problem case. When the density of μ is a smooth positive function on S^n, the solution to the Alexandrov problem is smooth, see [?] and [?]. We will establish a $C^{1,1}$ regularity result for nonegative μ. Of course when μ is positive, higher regularity will follow from Evans-Krylov theorem.

A priori C^0 estimates of solutions to the Alexandrov problem are studied in [?], where a necessary and sufficient condition was given.

Let k be some nonnegative function defined on S^n. We set

$$\mu(F) = \int_F k, \tag{4}$$

for all Borel sets F of S^n. It is clear that μ is a nonnegative, completely additive function on the Borel sets of S^n.

Theorem 3 *(a) Let $k \in C^{1,1}(S^2)$ be a nonnegative function, and μ be given by 4. Suppose that μ satisfies (A1) and (A2), then there exists some $C^{1,1}$ finite convex surface M^2 having μ as its integral Gaussian curvature.*

(b) Let $k \in C^{3,1}(S^3)$ be a nonnegative function, and μ be given by 4. Suppose that μ satisfies (A1) and (A2), then there exists some $C^{1,1}$ finite convex surface M^3 having μ as its integral Gaussian curvature.
Such hypersurface is unique up to a homothetic transformation.

For higher dimensions, we need some additional hypothesis to conclude the $C^{1,1}$ regularity. We introduce the following condition for $n \geq 2$.

Condition (I): $k \in C^{0,1}(S^n)$ is nonnegative, and for some constant $A > 0$, satisfies that
(i) $\Delta(k^{1/(n-1)}) \geq -A$, on S^n,
(ii) $|\nabla(k^{1/(n-1)})| \leq A$, on S^n.

It is clear that for nonnegative function $k \in C^{1,1}(S^n)$, part (i) of Condition (I) is equivalent to

$$(n-1)k(x)\Delta k(x) - (n-2)|\nabla k(x)|^2 \geq -(n-1)^2 A k(x)^{2-1/(n-1)} \qquad \forall\, x \in S^n,$$

and part (ii) of Condition (I) is equivalent to

$$|\nabla k(x)| \leq (n-1)Ak(x)^{(n-2)/(n-1)} \qquad \forall\, x \in S^n.$$

Theorem 4 *For $n \geq 2$, let k satisfy Condition (I), and μ be given by 4. Suppose that μ satisfies (A1) and (A2), then there exists some $C^{1,1}$ finite convex surface M^n having μ as its integral Gaussian curvature. Such hypersurface is unique up to a homothetic transformation.*

We also introduce another condition
Condition (II): $k \in C^{1,1}(S^n)$ is nonnegative, and for some constant $A > 0$, satisfies that

$$2k(x)\Delta k(x) - 3|\nabla k(x)|^2 \geq -Ak(x)^{2-1/(n-1)} \qquad \forall\, x \in S^n.$$

Theorem 5 *For $n \geq 2$, let $k \in C^{2,\alpha}(S^n)$ $(0 < \alpha < 1)$ satisfy Condition (II), and μ be given by 4. Suppose that μ satisfies (A1) and (A2), then there exists some $C^{1,1}$ finite convex surface M^n having μ as its integral Gaussian curvature. Such hypersurface is unique up to a homothetic transformation.*

The above theorems are the consequence of the following proposition.

Proposition 1 *(a) For $n \geq 2$, let $k \in C^{2,\alpha}(S^n)(0 < \alpha < 1)$ satisfy Condition (I) and μ, given by 4, be the integral Gaussian curvature of $M^n = \{R(x) = \rho(x)|x \in S^n\}$. Then*

$$\|\rho\|_{C^2(S^n)} \leq C,$$

where C depends only on $n, \max_{S^n} \rho / \min_{S^n} \rho, \max_{S^n} |\nabla\rho| / \min_{S^n} \rho, A,$ and $\max_{S^n} k$.
(b) For $n \geq 2$, let k satisfy Condition (II), then

$$\|\rho\|_{C^2(S^n)} \leq C,$$

where C depends only on $n, \max_{S^n} \rho / \min_{S^n} \rho, \max_{S^n} |\nabla\rho| / \min_{S^n} \rho, A,$ and $\max_{S^n} k$.

Let k be some nonnegative function on S^n, and $\mu(F)$ be given in 4. Let $e_1, \cdots, e_n$ be some smooth local frame field on S^n and let ∇ denote the covariant differentiation. Let $\sigma_{ij} =< e_i, e_j >$ denote the metric on S^n, σ^{ij} denote its inverse, and $\sigma = \det(\sigma_{ij})$. For $M^n = \{R(x) = \rho(x)x \mid x \in S^n\}$ to have μ as its integral Gaussian curvature if and only if ρ satisfies

$$\det(\rho^2\sigma_{ij} + 2\nabla_i\rho\nabla_j\rho - \rho\nabla_{ij}\rho) = k\sigma\rho^{n-1}(\rho^2 + |\nabla\rho|^2)^{(n+1)/2}, \qquad S^n. \tag{5}$$

Set $u = 1/\rho$, $K = k(u^2 + |\nabla u|^2)^{(n+1)/2}/u$. Equation 5 is equivalent to

$$\det(u\sigma_{ij} + \nabla_{ij}u) = \sigma K. \tag{6}$$

As we are looking for convex solutions, $(u\sigma_{ij} + \nabla_{ij}u) \geq 0$ as a matrix. Let $H(x) = nu(x) + \Delta u(x) \geq 0$. A C^2-bounds of u will follow from an upper bound of H. Let $P \in S^n$ be a maximum point of H, i.e.

$$H(P) = \max_{x \in S^n} H(x).$$

It follows from Evans-Krylov theorem that $\rho \in C^{4,\alpha}(S^n)$. We choose an orthonormal frame field $e_1, \cdots, e_n$ near P such that $(\nabla_{ij}u(P))$ is diagonal (so will $(u\sigma_{ij} + \nabla_{ij}u)$ at P). Now at P, we have

$$\nabla_\beta H = n\nabla_\beta u + \nabla_\beta(\Delta u) = 0, \qquad \beta = 1, \cdots, n, \tag{7}$$

and

$$(H_{\beta\gamma}) = (n\nabla_{\gamma\beta}u + \nabla_{\gamma\beta}\Delta u) \leq 0. \tag{8}$$

Throughout this section, we use notation $u_{ij} = \nabla_j\nabla_j u = \nabla_{ji}u$, and repeated upper and lower indices denote summation over the indices. Set $w_{ij} = u\sigma_{ij} + \nabla_{ij}u, (w^{ij}) = (w_{ij})^{-1}$. We may assume without loss of generality that

$$w_{11}(P) \leq w_{22}(P) \leq \cdots \leq w_{nn}(P), \qquad w_{nn}(P) \geq 1. \tag{9}$$

Since $(w_{ij}) > 0$, we know at P,

$$w^{ij}H_{ij} \leq 0.$$

The following formula for commuting covariant derivatives are elementary.

$$\nabla_{ii}\nabla_{jj} = \nabla_{jj}\nabla_{ii} + 2\nabla_{jj} - 2\nabla_{ii},$$

and

$$\nabla_{ii}\Delta = \Delta\nabla_{ii} + 2\Delta - 2n\nabla_{ii}. \tag{10}$$

Using 8, 10, and $\sigma_{ij} = \delta_{ij}, w_{ij}(P) = w_{ii}(P)\delta_{ij}$, we have at at P that,

$$
\begin{aligned}
0 &\geq w^{ij}H_{ij} = w^{ii}H_{ii} = nw^{ii}u_{ii} + w^{ii}\nabla_{ii}\Delta u &(11)\\
&= nw^{ii}u_{ii} + w^{ii}\{\Delta\nabla_{ii}u + 2\Delta u - 2nu_{ii}\} &(12)\\
&= -nw^{ii}u_{ii} + 2(\textstyle\sum_i w^{ii})\Delta u + w^{ii}\Delta\nabla_{ii}u &(13)\\
&= -nw^{ii}u_{ii} + 2(\textstyle\sum_i w^{ii})\Delta u + w^{ii}\{\Delta(u\sigma_{ii}) + \Delta u_{ii}\} - w^{ii}(\Delta(u\sigma_{ii})) &(14)\\
&= -nw^{ii}u_{ii} + (\textstyle\sum_i w^{ii})\Delta u + w^{ii}(\Delta w_{ii}) &(15)\\
&= -nw^{ii}(w_{ii} - u\sigma_{ii}) + (\textstyle\sum_i w^{ii})(H - nu) + w^{ii}(\Delta w_{ii}) &(16)\\
&= -n^2 + (\textstyle\sum_i w^{ii})H + w^{ii}(\Delta w_{ii}). &(17)
\end{aligned}
$$

We will compute $w^{ii}(\Delta w_{ii})$. By the chain rule, we have

$$\nabla_\beta \det(w_{ij}) = \det(w_{ij})w^{ij}\nabla_\beta w_{ij}, \quad \nabla_\beta w^{ij} = -w^{ik}w^{lj}\nabla_\beta w_{kl}.$$

Applying Δ to the equation $\det(w_{ij})^{1/(n-1)} = K^{1/(n-1)}$, we obtain, by using the above formula, that

$$
\begin{aligned}
w^{ij}(\Delta w_{ij}) = \; & w^{ik}w^{lj}\{\textstyle\sum_\beta \nabla_\beta w_{ij}\nabla_\beta w_{lk}\} - \frac{1}{n-1}w^{ij}w^{lk}\{\textstyle\sum_\beta \nabla_\beta w_{ij}\nabla_\beta w_{lk}\}\\
& +K^{-2}\{K\Delta K - \tfrac{n-2}{n-1}|\nabla K|^2\}.
\end{aligned}
$$

At P, $w^{ij} = \delta_{ij}w^{ii}$, so we have

$$
\begin{aligned}
w^{ii}(\Delta w_{ii}) = \; & w^{ii}w^{jj}\textstyle\sum_\beta(\nabla_\beta w_{ij})^2 - \frac{1}{n-1}w^{ii}w^{jj}\{\textstyle\sum_\beta \nabla_\beta w_{ii}\nabla_\beta w_{jj}\}\\
& +K^{-2}\{K\Delta K - \tfrac{n-2}{n-1}|\nabla K|^2\}.
\end{aligned}
$$

It follows from 7 that $\sum_i \nabla_\beta w_{ii} = 0$ at P for each β. So for each fixed β, we can put $\{\nabla_\beta w_{11}, \nabla_\beta w_{22}, \cdots, \nabla_\beta w_{nn}\}$ into two groups $\{\nabla_\beta w_{ii}\}_{i\in I}$ and $\{\nabla_\beta w_{jj}\}_{j\in J}$ with $I \cup J = \{1, 2, \cdots, n\}$, $I \cap J = \varphi$, $|I| \leq n-1$, $|J| \leq n-1$, $\nabla_\beta w_{ii} \geq 0$ for $i \in I$, and $\nabla_\beta w_{jj} \leq 0$ for $j \in J$. Now

$$
\begin{aligned}
& w^{ii}w^{jj}(\nabla_\beta w_{ij})^2 - \tfrac{1}{n-1}w^{ii}w^{jj}\{\nabla_\beta w_{ii}\nabla_\beta w_{jj}\}\\
\geq\; & (w^{ii})^2(\nabla_\beta w_{ii})^2 - \tfrac{1}{n-1}\{\textstyle\sum_{i\in I} w^{ii}\nabla_\beta w_{ii} + \sum_{i\in J} w^{ii}\nabla_\beta w_{ii}\}^2\\
\geq\; & (w^{ii})^2(\nabla_\beta w_{ii})^2 - \tfrac{1}{n-1}(\textstyle\sum_{i\in I} w^{ii}\nabla_\beta w_{ii})^2 - \tfrac{1}{n-1}(\sum_{i\in J} w^{ii}\nabla_\beta w_{ii})^2\\
=\; & \{\textstyle\sum_{i\in I}(w^{ii})^2(\nabla_\beta w_{ii})^2 - \tfrac{1}{n-1}(\sum_{i\in I} w^{ii}\nabla_\beta w_{ii})^2\}\\
& +\{\textstyle\sum_{i\in J}(w^{ii})^2(\nabla_\beta w_{ii})^2 - \tfrac{1}{n-1}(\sum_{i\in J} w^{ii}\nabla_\beta w_{ii})^2\}\\
\geq\; & 0.
\end{aligned}
$$

In the last inequality above, we have used $|I| \leq n-1, |J| \leq n-1$, and the Cauchy-Schwarz inequality. In turn, we have at P that

$$w^{ii}(\Delta w_{ii}) \geq K^{-2}\{K\Delta K - \frac{n-2}{n-1}|\nabla K|^2\}. \tag{18}$$

Putting 18 into 11, we obtain at P that

$$-n^2 + (\sum_i w^{ii})H + K^{-2}\{K\Delta K - \frac{n-2}{n-1}|\nabla K|^2\} \leq 0. \quad (19)$$

Set $g = (u^2 + |\nabla u|^2)^{(n+1)/2}/u$, we have $K = kg$. It follows that

$$K^{-2}\{K\Delta K - \tfrac{n-2}{n-1}|\nabla K|^2\} \quad (20)$$

$$= k^{-2}\{k\Delta k - \tfrac{n-2}{n-1}|\nabla k|^2\} + \tfrac{2}{n-1}k^{-1}g^{-1}\nabla k\nabla g + g^{-2}\{g\Delta g - \tfrac{n-2}{n-1}|\nabla g|^2\}. \quad (21)$$

Using 7 and $u_{ij}(P) = \delta_{ij}u_{ii}(P)$, we have at P that

$$\nabla_\beta g = (n+1)\frac{(u^2 + |\nabla u|^2)^{(n-1)/2}}{u}u_{\beta\beta}u_\beta + C_0(u, u^{-1}, \nabla u), \quad (22)$$

$$|\nabla g|^2 = (n+1)^2\frac{(u^2 + |\nabla u|^2)^{n-1}}{u^2}\sum_\beta u^2_{\beta\beta}u^2_\beta + C_1(u, u^{-1}, \nabla u)(\nabla^2 w), \quad (23)$$

$$g\Delta g = (n+1)\tfrac{(u^2+|\nabla u|^2)^{n-1}}{u^2}\{(u^2 + |\nabla u|^2)\textstyle\sum_i u^2_{ii} + (n-1)\sum_i u^2_{ii}u^2_i\} \quad (24)$$

$$+C_2(u, u^{-1}, \nabla u)(\nabla^2 w), \quad (25)$$

here and in the following, $C_j(u, u^{-1}, \nabla u)$ denotes some quantity depending on $u, u^{-1}, \nabla u$, and $C_j(u, u^{-1}, \nabla u)(\nabla^2 w)$ denotes some quantity linear in $\nabla^2 w$ with coefficients depending on u, u^{-1} and∇u.

We first assume that k satisfies Condition (II). Using the Cauchy-Schwartz inequality, we have

$$\frac{2}{n-1}k^{-1}g^{-1}|\nabla k \cdot \nabla g| \leq \frac{n+1}{2(n-1)}k^{-2}|\nabla k|^2 + \frac{2}{n^2-1}g^{-2}|\nabla g|^2.$$

Putting this into 20, and by 23, 24 and Condition (II), we have

$$\begin{aligned} &K^{-2}\{K\Delta K - \tfrac{n-2}{n-1}|\nabla K|^2\} \\ \geq\ & k^{-2}\{k\Delta k - \tfrac{3}{2}|\nabla k|^2\} + g^{-2}\{g\Delta g - \tfrac{n}{n+1}|\nabla g|^2\} \\ \geq\ & -\tfrac{A}{2}k^{-1/(n-1)} - C_3(u, u^{-1}, \nabla u)H. \end{aligned}$$

Back to 19, we have

$$(\sum_i w^{ii})H - \frac{A}{2}k^{-1/(n-1)} - C_3(u, u^{-1}, \nabla u)H \leq n^2.$$

Due to 9, we have $w_{nn}(P) \geq H/n$. It follows that

$$\sum_i w^{ii} \geq \textstyle\sum_{i=1}^{n-1} w^{ii} \geq \frac{1}{n-1}(\Pi_{i=1}^{n-1}w^{ii})^{1/(n-1)} \quad (26)$$

$$= \tfrac{1}{n-1}(w_{nn}/K)^{1/(n-1)} \geq \tfrac{1}{n-1}(\tfrac{1}{n})^{1/(n-1)}H^{1/(n-1)}K^{-1/(n-1)}. \quad (27)$$

Therefore

$$\frac{1}{n-1}(\frac{1}{n})^{1/(n-1)}K^{-1/(n-1)}H^{1+1/(n-1)} - \frac{A}{2}k^{-1/(n-1)} - C_3(u,u^{-1},\nabla u)H \le n^2,$$

which yields

$$H\{\frac{1}{n-1}(\frac{1}{n})^{1/(n-1)}H^{1/(n-1)} - k^{1/(n-1)}C_4(u,u^{-1},\nabla u)\} \le \frac{A}{2} + n^2 k^{1/(n-1)}.$$

We conclude from the above that

$$H(P) \le C_5(u,u^{-1},\nabla u,k,A).$$

This gives an upper bound for $\max_{S^n} H$.

When k satisfies Condition (I), it follows from 22 that

$$\frac{2}{n-1}|\nabla k\cdot\nabla g| \le \frac{2}{n-1}A_2 k^{1-1/(n-1)}C_6(u,u^{-1},\nabla u)H.$$

Putting the above into 20, using the above, 23, 24 and Condition (I), we have

$$K^{-2}\{K\Delta K - \tfrac{n-2}{n-1}|\nabla K|^2\} \tag{28}$$

$$\ge \quad k^{-2}\{k\Delta k - \tfrac{n-2}{n-1}|\nabla k|^2\} - \tfrac{2}{n-1}A_2 k^{-1/(n-1)}C_6(u,u^{-1},\nabla u)H \tag{29}$$

$$+g^{-2}\{g\Delta g - \tfrac{n-2}{n-1}|\nabla g|^2\} \tag{30}$$

$$\ge \quad -\tfrac{A_1}{n-1}k^{-1/(n-1)} - \tfrac{2}{n-1}A_2 k^{-1/(n-1)}C_6(u,u^{-1},\nabla u)H - C_7(u,u^{-1},\nabla u)H. \tag{31}$$

Putting 28 into 19, we obtain (using $H(P)\ge 1$) that

$$\sum_i w^{ii}H - C_8(u,u^{-1},\nabla u,A_1,A_2)k^{-1/(n-1)}H - C_9(u,u^{-1},\nabla u)H \le 0,$$

namely,

$$\sum_i w^{ii} \le C_8(u,u^{-1},\nabla u,A_1,A_2)k^{-1/(n-1)} + C_9(u,u^{-1},\nabla u).$$

By 26, we conclude

$$H^{1/(n-1)} \le C_{10}(u,u^{-1},\nabla u,A_1,A_2) + C_{11}(u,u^{-1},\nabla u)k^{1/(n-1)}.$$

This provides an upper bound for $\max_{S^n} H$. We have thus proved the proposition.

For the discussion of Condition (I), we refer to section 4.

3 A Reflection Problem Related to MA

Let M be a strongly convex (bounded) surface in $\mathbf{R}^{n+1}$, suppose 0 is inside of M. For $\forall\ x \in S^n$, we issue a ray of light from 0, the light will reflect at M in the direction of $y \equiv T(x) = x - 2 < x,\ \gamma > \gamma$, where γ is the unit normal of M at the reflection point. Therefore, we may view T as a mapping from $S^n \to S^n$. This is a diffeomorphism. S.T. Yau asked what information can we get from T. The question can be proceeded as follows: We write M as a graph of $S^n : M = \{\rho(x)x \mid x \in S^n\}$. Suppose $f(x)$ be the density of the light in the direction of $x \in S^n$, and $g(y)$ be the density of the reflected light at y. If we assume there is no loss of energy, we have $\int_E f(x) = \int_{T(E)} g(y)\ \forall$ Borel $E \in S^n$. If M is smooth, $f(x) = g(T(x))(J_T(x))$, i.e., $J_T(x) = \frac{f(x)}{g(T(x))}$. The Jacobi of T can be calculated using the formula $T(x) = x - 2 < x, \gamma > \gamma$ (where $\gamma = \gamma(\rho(x)x)$. ρ the graph function):

$$J_T(x) \equiv \frac{\det(\Delta_{ij}u + (u-\eta)e_{ij})}{\eta^n \det(e_{ij})} = \frac{f(x)}{g(x, u(x), \Delta u(x))} \tag{32}$$

where $u = \frac{1}{\rho}$, $\eta = (|\nabla u|^2 + u^2)/2u$. If M is smooth, $\forall\ x_0 \in M$, there is a unique paraboloid $F = \{x \cdot \psi(x) \mid x \in S^n\}$ which is tangent at x_0, with focus point at 0 and reflection direction y, where

$$\psi(x) = \frac{c}{1 - \langle x, y \rangle}, \quad x \in S^n,\ x \neq y.$$

In fact, M must be inside of F. Note that, if $v = \frac{1}{\psi}$, $\{\Delta_{ij}v + (v - \eta)e_{ij}\} \equiv 0$, i.e., and the lights reflect in the direction of y.

Admissible surface: Suppose M is a convex surface (no smoothness assumption), o is inside of M. We say M is an admissible surface of reflection, if $\forall\ x \in M$, there is a paraboloid F which is tangent at x, focused at o, and M is on one side of F. (F is called a supporting paraboloid at x.)

Remark: If M is closed, M must be inside of each such paraboloid. If M is not smooth, at some point $x \in M$, there may be many supporting paraboloids.

For any admissible surface, we define $T(x)$ to be the collection of all reflect directions of supporting paraboloids at $\rho(x)x$. T is a multi-valued mapping (we may compare T with the gradient mapping in Alexandrov sense for MA.)

For (S^n, f), (S^n, g) with $\int_{S^n} f = \int_{S^n} g$. We want to find an admissible reflection surface M, s.t.

$$\int_E f = \int_{T(E)} g \quad \forall\ E \subset S^n,\ E \text{ Borel}.$$

(We assume f, g are positive measures.) Note that, if M is a solution, AM is also a solution for any $A > 0$.

Theorem 6 *If f and g are positive functions on S^n, $f, g \in C^{1,1}(S^n)$ and $\int_{S^n} f = \int_{S^n} g$. Then, there is a unique (up to a dilation) reflection surface $M \subset \mathbf{R}^{n+1}$, with $M \in C^{3,\alpha}$ $\forall\ 0 < \alpha < 1$.*

Like in the previous section (Minkowski problem), we will employ continuity method to prove the theorem.

We write $M = \{\rho(x)x \mid x \in S^n\}$. We normalize $\int_{S^n} f = 1$.

Closeness:

C^0-estimates: We may assume $\inf_{x\in S^n} \rho(x) = 1$ (after a dilation if necessary). Since $f, g \in L^1(S^n)$, $\exists\ r_0 > 0$, such that $\forall\ y \in S^n$,

$$\int_{B_{r_0}(y)} g \le \int_{S^n} g \le 2\int_{B_{2\pi - r_0}(y)} g$$
$$\int_{B_{r_0}(y)} f \le \int_{S^n} f \le 2\int_{B_{2\pi - r_0}(y)} f.$$

Lemma 2 *There is a constant $c(r_0)$, such that* $\sup_{x\in S^n} \rho(x) \le c(r_0)$.

Proof. Suppose $\int_{S^n} \rho(x) = \rho(x_0)$, we may assume $x_0 = -\vec{e}_{n+1}$. Let $\psi_0(x) = \frac{c_0}{1-\langle x, y_0\rangle}$ be a supporting paraboloid at $x_0\rho(x_0)$. Since x_0 is the minimum point (to 0), the paraboloid is tangent at x_0 (with M), we get $y_0 = \vec{e}_{n+1}$. Again, since $\rho(x_0) = 1$, we have $c_0 = 2$.

$\forall\ \tilde{c} > 0$, $\forall\ p \in M \cap \{x_{n+1} \le \tilde{c}\}$, let $\psi_p(x) = \frac{c_1}{1-\langle x, y(p)\rangle}$ be a supporting paraboloid of M at p. If $\exists\ p \in M \cap \{x_{n+1} \le \tilde{c}\}$, such that $y(p) \notin B_{r_0}(e_{n+1})$, then the constant c_1 is bounded by $c(r_0, \tilde{c})$. In turn, M is bounded by the two paraboloid ψ_0, ψ_1.

Suppose this is not the case, that is, $\forall\ \tilde{c} > 0$, $\forall\ p \in M \cap \{x_{n+1} \le \tilde{c}\}$, $y(p) \in B_{r_0}(e_{n+1})$. If $\tilde{c}$ large enough, $T(x) \in B_{r_0}(e_{n+1})$, $\forall\ x \in S^n_{B_{r_0}(e_{n+1})}$. Now

$$\frac{1}{2}\int_{S^n} g > \int_{B_{r_0}(e_{n+1})} g \ge \int_{T(S^n - B_{r_0}(e_{n+1}))} g = \int_{S^n - B_{r_0}(e_{n+1})} f > \frac{1}{2}\int_{S^n} f = \frac{1}{2}\int_{S^n} g.$$

Contradiction. End the proof of Lemma 2. We obtain a L^∞ bound for M.

C^1-estimates: Since M is convex, $(\rho_{ij} + \rho e_{ij})$ is positive definite. By C^0 estimate ρ is bounded below and above, the same argument as in the proof of C^1 estimates in Minkowski problem yields the boundnedss of $|\nabla\rho|$.

C^2-estimates:

Lemma 3 *Let $\lambda \ge 0$, $\lambda \in C^2(S^n)$, $\Omega_\lambda = \{x \in S^n \mid \lambda > 0\}$. Suppose $f, g \in C^2(\overline{\Omega}_\lambda)$ (i.e., $\|f\|_{C^2(\overline{\Omega}_\lambda)} \le C_1$, $\|g\|_{C^2(\overline{\Omega}_\lambda)} \le C_1$) and $u \in C^4(\Omega_\lambda)$, u satisfies equation (32). Then, $\exists\ c > 0$ (depends only on $\|\lambda\|_{C^2}$, c_1 and c_2) such that*

$$|\nabla^2 u(x)| \le c/\lambda(x), \quad \forall\ x \in \Omega_\lambda.$$

Proof. Let $H(x) = \mathrm{tr}(u_{ij}(x))$, $(w_{ij}) = (\nabla_{ij}u + (u-\eta)e_{ij})$.
Set

$$E(x) = \lambda(x)h(x).$$

We only need to obtain an upper bound of E. Let $E(x_0) = \max_{x\in\Omega_\lambda} E(x)$, $x_0 \in \Omega_\lambda$. We may assume

$$H(x_0) \geq 1 + c\sup_{S^n}(u-\eta),$$

and $(u_{ij}(x_0))$ diagonal. At x_0,

$$\nabla E = 0, \; and \; \{\nabla^2 E\} \leq 0,$$

$$\Rightarrow H_i = \frac{-\lambda_i}{\lambda} H$$

, and

$$\lambda H_{ij} \leq -\left(\lambda_{ij} - 2\,\frac{\lambda_i\lambda_j}{\lambda}\right) H - w^{ij}\left(\lambda_{ij} - 2\,\frac{\lambda_i\lambda_j}{\lambda}\right) H.$$

$$\begin{aligned} 0 \geq \lambda w^{ij}H_{ij} = \lambda w^{ii}(\Delta u)_{ii} & \\ &= \lambda w^{ii}[\Delta(u_{ii}) + 2\Delta u - 2nu_{ii}] \\ &= \lambda\{w^{ii}\Delta(u_{ii}) + 2H(\textstyle\sum w^{ii}) - 2nw^{ii}w_{ii} + 2n(u-\eta)(\sum w^{ii})\} \\ &= \lambda w^{ii}\Delta(u_{ii}) + \lambda O(1 + H(\textstyle\sum w^{ii})). \end{aligned}$$

Apply Δ to $\det^{\frac{1}{n}}(w_{ij}) = \eta\left[\frac{f(x)}{g(T(x))}\det(e_{ij})\right]^{\frac{1}{n}} \stackrel{def}{=} K(x,u,\nabla u)$. At x_0, by the concavity of $\det^{\frac{1}{n}}$,

$$\begin{aligned} w^{ii}\Delta(u_{ii}) &= w^{ij}\Delta(u_{ij}) = w^{ij}\{\Delta(w_{ij}) - \Delta[(u-\eta)e_{ij}]\} \\ &= w^{ij}\Delta(w_{ij}) + (\textstyle\sum w^{ii})\Delta\eta + O(H\sum w^{ii}) \\ &\geq \Delta K + \Delta\eta(\textstyle\sum w^{ii}) + O(H\sum w^{ii}) \\ &\Rightarrow w^{ij}\left(\lambda_{ij} - 2\,\frac{\lambda_i\lambda_j}{\lambda}\right) H \geq \lambda\Delta K + \lambda\Delta\eta(\textstyle\sum w^{ii}) + \lambda O(H\sum w^{ii}). \end{aligned}$$

While, at x_0

$$\begin{aligned} \Delta\eta &= \frac{1}{u}\sum u_{ii}^2 + \frac{1}{u}\nabla u\cdot\nabla(\Delta u) + \Delta(H) \\ &= \frac{1}{u}\sum u_{ii}^2 - H\,\frac{\nabla\lambda}{\lambda}\cdot\frac{\nabla u}{u} + O(H) \\ &\geq \bar{c}H^2 - c\left(H\,\frac{|\nabla\lambda|}{\lambda} + 1\right) H \quad (\text{since } \tfrac{1}{u} \geq C > 0 \text{ by } C^0\text{-estimates}). \end{aligned}$$

And

$$
\begin{aligned}
\Delta K &= K_p \cdot \nabla(\Delta u) + \sum_{ij} K_{p_i p_j} \nabla u_i \cdot \nabla u_j + O(H) \\
&\geq -c(H \frac{|\nabla \lambda|}{\lambda}) H - cH^2 \\
\Rightarrow &- w^{ij}\left(\lambda_{ij} - 2\,\frac{\lambda_i \lambda_j}{\lambda}\,H\right) \geq \lambda \bar{c} H^2 (\sum w^{ii}) - c\lambda H^2 - c\lambda (H\,\frac{|\nabla \lambda|}{\lambda}) H (\sum w^{ii}).
\end{aligned}
$$

Since

$$
\begin{aligned}
-w^{ij}\left(\lambda_{ij} - 2\,\frac{\lambda_i \lambda_j}{\lambda}\,H\right) &\leq c\,\frac{|\nabla \lambda|^2}{\lambda^2}\,H(\sum w^{ii}) \\
\Rightarrow \lambda \bar{c} H^2 (\sum w^{ii}) &\leq c\,\frac{|\nabla \lambda|^2}{\lambda}\,H(\sum w^{ii}) + c\lambda H^2 + c\lambda (H\,\frac{|\nabla \lambda|}{\lambda}) H (\sum w^{ii}) \\
&\leq cH(\sum w^{ii}) + c\lambda H^2 \quad (\text{therefore } |\nabla \lambda|^2 \leq c\lambda).
\end{aligned}
$$

We estimate $\sum w^{ii}$: At x_0, we may assume $w^{nn} \leq w^{n-1,n-1} \leq \cdots \leq w^{11}$,

$$
\begin{aligned}
\sum^n w^{ii} &\geq \sum^{n-1} w^{ii} \geq (n-1)(\prod_i^{n-1} w^{ii})^{\frac{1}{n-1}} \\
&= (n-1)\,(\frac{\prod^n w^{ii}}{w^{nn}})^{\frac{1}{n-1}} = (n-1)\,\frac{K^{-\frac{1}{n-1}}}{(w^{nn})^{\frac{1}{n-1}}} \\
&= (n-1)\,\frac{w^{\frac{1}{n-1}}}{K^{\frac{1}{n-1}}} \geq (n-1)\,\frac{(\frac{1}{n}\,H)^{\frac{1}{n-1}}}{K^{\frac{1}{n-1}}} \\
&= (n-1)(\frac{1}{n})^{\frac{1}{n-1}}\,\frac{H^{\frac{1}{n-1}}}{K^{\frac{1}{n-1}}}
\end{aligned}
$$

(therefore $w_{11} \leq \cdots \leq w_{nn}$ and $\sum w_{ii} = H$.) Insert it into previous inequality,

$$
H \leq c/\lambda, \quad \text{i.e., } E \leq c \text{ at } x_0.
$$

□

With the C^2 boundedness and positivity of f and g by Evans-Krylov Theorem, we obtain $\|\rho\|_{C^{3,\alpha}} \leq c_\alpha$, $\forall\ 0 < \alpha < 1$.

Openness: Set $F(u) = g(T_u(x))\ \frac{\det(\nabla_{ij} u + (u-\eta) e_{ij})}{\eta^n \det(e_{ij})}$.

Let $f_0(x) = g(-x)$, $f_t(x) = tf(x) + (t-1) f_0(x)$, $u \equiv 1$ is a solution of (32) for f_0, g. Suppose for some t_0 (32) is solvable (should be $(32)_t$, if there is no confusion, we still write (32)), and u is a solution $\forall\ v \in C^2(S^n)$, $\frac{1}{u+\delta v}$ is admissible if δ small enough. If $F(u+\delta v) = h_\delta$, we must have

$$
\int_{S^n} h_\delta = \int_{S^n} F(u+\delta v) = \int_{S^n} g = 1 = \int_{S^n} h_0.
$$

(Remember we normalize $\int_{S^n} g = \int_{S^n} f = 1$.) If L_u is a linearized operator of F at u, we get

$$\int_{S^n} L_u(v)dx = \lim_{\delta\to 0}\frac{1}{\delta}\int_{S^n}(h_\delta(x) - h_0(x)) = 0. \tag{33}$$

We want to show L_u is surjective to $\mathcal{S} = \{h \mid \int_{S^n} h = 0\}$. By Implicit Function Theorem, F_u is invertible near $f_t \in \mathcal{S}$. L_u is surjective $\Leftrightarrow$ $\mathrm{Ker}(L_u^*) = \{0\}$ in $\mathcal{S}$. In any local coordinate chart, we can express

$$L_u^* = \sum_{i,j} a_{ij}(x)\delta_{ij} + \sum_i b_i(x)\delta_i + c(x).$$

Though a_{ij}, b_i depends on the choice of the chart, $c(x)$ is globally defined.

Claim: $c(x) \equiv 0$.

If the claim is true, since S^n is compact, by maximum principle, $\mathrm{Ker}(L_u^*) = \mathrm{Span}\{1\}$. Restricted to $\mathcal{S}$, $\mathrm{Ker}(L_u^*) = \{0\}$ in $\mathcal{S}$. So L_u is surjective to $\mathcal{S}$, and F is invertible.

Proof of the claim: For any smooth function v, we have

$$\int_{S^n} v(x)c(x) = \int_{S^n} v(x)L_u^*(1) = \int_{S^n} L_u(v)\cdot 1 = \int_{S^n} L_u(v) = 0.$$

Since we have shown $\int_{S^n} L_u(v) = 0\ \forall\ v \in C^2(S^n)$ in (33). Finally,

Uniqueness: We assume $\int_E g > 0\ \forall$ open $E \subset S^n$, $E \neq \emptyset$. Suppose we have two solutions M_1, M_2. Let ρ_1 and ρ_2 be the corresponding graph functions respectively. Suppose $\rho_1/\rho_2 \not\equiv$ constant. Dilating if necessary, we may assume $\Omega_1 = \{\frac{\rho_1}{\rho_2} > 1\}$ and $\Omega_2 = \{\frac{\rho_1}{\rho_2} < 1\}$ are both nonempty.

Fact: $T_{\rho_1}(\Omega_1) \supset T_{\rho_2}(\Omega_1)$ (similarly $T_{\rho_1}(\Omega_1) \subset T_{\rho_2}(\Omega_2)$).

Proof. Since $\forall\ y \in T_{\rho_2}(\Omega_1)$, the family of paraboloids $\{\psi_c = \frac{c}{1-\langle x,y\rangle},\ x \in S^n\}$ with focus point at 0 will touch ρ_1 first before touching ρ_2 at $T^{-1}(y) \in \Omega_1$. Let $G = \{x|\rho_1(x), \rho_2(x)$ are both different

Lemma 4 *$T_{\rho_1}(x) = T_{\rho_2}(x)\ \forall\ x \in G$. Of course, it is easy to deduce from the lemma $\rho_1 \equiv \rho_2$ since G is dense in S^n.*

Proof. Suppose $\exists\ x_0$, $T_{\rho_1}(x_0) \neq T_{\rho_2}(x_0)$, we may suppose $\rho_1(x_0) = \rho_2(x_0)$ and $\Omega_1 = \{x|\rho_1/\rho_2 > 1\} \neq \emptyset$. Let $y_0 \in T_{\rho_2}(x_0)$, $\psi_{y_0} = \frac{c_0}{1-\langle x,y_0\rangle}$ be the supporting paraboloid of ρ_2 at x_0. Since $T_{\rho_1}(x_0) \neq T_{\rho_2}(x_0)$, $\exists\ \psi_{y_\varepsilon} = \frac{c_\varepsilon}{1-\langle x,y_\varepsilon\rangle}$, a small perturbation of ψ_{y_0}, such that ψ_{y_ε} is a supporting paraboloid of ρ_2 at some point $x_\varepsilon \notin \Omega_1$. ψ_{y_ε} cuts off a cap from $\rho_1 \Rightarrow y_\varepsilon \in T_{\rho_1}(\Omega_1)^{\mathrm{int}}$, $y_\varepsilon \notin T_{\rho_2}(\Omega_1)$. But

$$\int_{T_{\rho_1}(\Omega_1)} g = \int_{\Omega_1} f = \int_{T_{\rho_2}(\Omega_1)} g \Rightarrow \int_{T_{\rho_1}(\Omega_1)\setminus T_{\rho_2}(\Omega_1)} g = 0$$

therefore, $T_{\rho_1}(\Omega_1)\backslash T_{\rho_2}(\Omega_1) \neq \emptyset$ and open. Contradiction to the assumption $\int_E g > 0 \; \forall \; E$ open, $(E \neq \emptyset)$.

Remark: If Ω_1, Ω_2 are two domains in S^n, f, g are positive functions on Ω_1 and Ω_2 respectively, and $\int_{\Omega_1} f = \int_{\Omega_2} g$. We may consider the following problem: Find M, such that $T_M : \Omega_1 \to \Omega_2$ and $\int_E f = \int_{T(E)} g \; \forall \; E \subset \Omega_1$, E Borel. If we extend f and g to S^n:

$$f(x) = \begin{cases} f(x) & x \in \Omega_1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} g(x) & x \in \Omega_2 \\ 0 & \text{otherwise} \end{cases}$$

Our C^0 estimate still holds for f, g (we only need $f, g \in L^1(S^n)$ there). So, M is automatically $C^{0,1}$. In general M fail to be $C^{1,1}$. But, if know that $T(\Omega_1) \subset \Omega_2$, the localized C^2-estimates in Lemma 2 gives $M \in C^{3,\alpha}(\Omega_1)$ if $f \in C^2(\Omega_1)$, $g \in C^2(\Omega_2)$.

4 Monge-Ampére Equations: Homegeneous Dirichlet Problem

We consider the following Dirichlet problem for Monge-Ampére equations:

$$\begin{cases} \det(u_{ij}) = f & \Omega \subset \mathbf{R}^n \\ u|_{\partial\Omega} = g \end{cases}$$

$$\partial\Omega \in C^{3,1}, \qquad g \in C^{3,1}.$$

The fundamental result in elliptic case is due to Krylov, Caffarelli-Nirenberg-Spruck, they proved:

$$f > 0,\ f \in C^2, \Rightarrow \exists\ u \in C^{3,\alpha},\ \forall\ 0 < \alpha < 1.$$

To handle the problem, again, we employ continuity method. we assume Ω strongly convex, $\partial\Omega \in C^{3,1}$, $\Omega = \{x \in \mathbf{R}^n | \rho(x) > 0\}$, ρ defining function of Ω. We may assume $\rho \in C^{3,1}$, $\|\nabla\rho\|_{\partial\Omega} = 1$, and $-\left(\frac{\partial^2\rho}{\partial x_i \partial x_j}\right)(x) \geq CI$, for some $C > 0$, $\forall\ x \in \Omega$.

We also assume $g \in C^{3,1}(\overline{\Omega})$.

For the openness, take c large, set

$$v = -c\rho + g$$

, we have $(v_{ij}) > 0$,

$$\det(v_{ij}) = f_0 > 0,$$
$$v|_{\partial\Omega} = g.$$

Let $f_t = tf + (1-t)f_0$, $f_1 = f$.

Consider $M(u) = \log\det(u_{ij})$. The linearized operator of M at u is

$$L_u(w) = \sum u^{ij} w_{ij},$$

since $(u_{ij}) > 0 \Rightarrow \begin{cases} L_u(w) = h, & \Omega \\ v|_{\partial\Omega} = 0 \end{cases}$ is solvable, this give the openness of the problem.

Closeness:

C^0-est: u convex $\max_\Omega u = \max_{\partial\Omega} u = \max_{\partial\Omega} g \leq C$. For the lower bound of u, we first state

Comparison lemma: u, v $\det(u_{ij}) = f_1$, $\det(v_{ij}) = f_2$ and $u|_{\partial\Omega} = v|_{\partial\Omega}$, if $f_2 \geq f_1 \Rightarrow v(x) \leq u(x)\ \forall\ x \in \Omega$.

Proof. $w = v - u$, $\det(v_{ij}) - \det(u_{ij}) = f_2 - f_1 \geq 0$. $\sum_{i,j} A^{ij}(x) w_{ij}$, $(A^{ij}(x)) > 0$. Max. principle $\max_\Omega w - \max_{\partial\Omega} w = \max_{\partial\Omega}(v - u) \leq 0$. □

Since u is convex, $\max_\Omega u = \max_{\partial\Omega} u$, so

$$u \leq \max_{\partial\Omega} g \leq C.$$

With c large enough, let $v = -c\rho + g \Rightarrow v(x) \leq u(x) \ \forall \ x \in \Omega$,

$$\Rightarrow \inf_{x\in\Omega} u(x) \geq -c_0.$$

C^1**-est.** Since u is convex, $\max_\Omega |\nabla u|$ is attained on $\partial\Omega$. $\max_{\partial\Omega} |\nabla u| \leq \max_{\partial\Omega} |u_\nu| + \max_{\partial\Omega} |Tu|$, where T is taken to all the unit tangential vectors. Since $Tu = Tg$ on $\partial\Omega$, we only need to control u_ν. For $v = -c\rho + g$, c large enough, we have $w = v - u \leq 0$, $w = 0$,

$$\Rightarrow w_\nu|_{\partial\Omega} \geq 0$$

$$\Rightarrow v_\nu - u_\nu \leq 0 \Rightarrow u_\nu \leq v_\nu \leq c.$$

To estimate u_ν from below we again make use of the convexity. Consider any point on $\partial\Omega$. We may suppose it is the origin and that thex_n-axis is interior normal to $\partial\Omega$. Let $y \in \partial\Omega$ be the point where the positive x_n-axis exists from Ω. By convexity and our previous estimate (from above),

$$-u_\nu(o) = u_n(0) \leq u_n(y) \leq 2|\nabla v(y)|.$$

□

Now we turn into C^2-estimates.

Since we will deal with degenerate case late on, we state a general lemma here.

Lemma 5 *$u \in C^4$, suppose $f > 0$, $f^{\frac{1}{n-1}}$ is pseudo-subharmonic, i.e., $\Delta f^{\frac{1}{n-1}}(x) \geq -A$, $\forall \ x \in \Omega$, in Ω for x near $\partial\Omega \Rightarrow C(A, diam(\Omega))$, such that* $\sup_\Omega |\nabla^2 u| \leq C + \sup_{\partial\Omega} |\nabla^2 u|$.

Proof. Let $H(x) = \Delta u(x)$, define

$$G(x) = H(x)e^{\alpha \frac{|x|^2}{2}} \tag{34}$$

where

$$\alpha = \frac{1}{2(2n^2+1)\text{diam}^2(\Omega)} \, . \tag{35}$$

By the convexity of u, it is sufficient to obtain an upper bound for G. Let $p \in \bar{\Omega}$, $G(p) = \max_\Omega G(x)$. If $p \in \partial\Omega$, the Lemma is trivial. We may assume $p \in \Omega$ and $0 \in \Omega$. Then at p, $\nabla G = 0$, and $\nabla^2 G \leq 0$. That is, at p

$$H_\beta = -\alpha x_\beta H, \qquad \beta = 1, 2, \ldots, n \tag{36}$$

and

$$(H_{ij}) \leq (\alpha^2 x_i x_j - \alpha\delta_{ij})H \ . \tag{37}$$

At p, making an orthonormal linear transformation if necessary, we may assume (u_{ij}) is diagonal, and $u_{11} \leq u_{22} \leq \cdots \leq u_{nn}$. Since (u_{ij}) is positive definite at p, by (37)

$$\begin{aligned} L(H) \equiv u^{ij} H_{ij} &\leq \left(\alpha^2 u^{ij} x_i x_j - \alpha \sum_{i=1}^{n} u^{ii} \right) H \\ &\leq -\alpha(1 - \alpha \operatorname{diam}^2(\Omega)) \sum_{i=1}^{n} u^{ii} H \ . \end{aligned} \tag{38}$$

By the chain rule,

$$\frac{\partial u^{ij}}{\partial u_{k\ell}} = -u^{ik} u^{\ell j} \ . \tag{39}$$

We have

$$\begin{aligned} &D^2 \left(\det(u_{ij})^{\frac{1}{n-1}} \right) \\ &= \frac{1}{n-1} \det(u_{ij})^{\frac{1}{n-1}} \left\{ u^{ij} D^2 u_{ij} + \frac{1}{n-1} u^{\ell k} D u_{\ell k} u^{ij} D u_{ij} - u^{ik} u^{\ell j} D u_{ij} D u_{\ell k} \right\} \end{aligned} \tag{40}$$

Since $\det(u_{ij})^{\frac{1}{n-1}} = f^{\frac{1}{n-1}}$, and

$$D^2 \left(f^{\frac{1}{n-1}} \right) = \frac{1}{n-1} f^{-2+\frac{1}{n-1}} \left\{ f D^2 f - \left(1 - \frac{1}{n-1} \right) |Df|^2 \right\} \ . \tag{41}$$

We get the identity

$$\begin{aligned} &f^{-2} \left\{ f D^2 f - \left(1 - \frac{1}{n-1} \right) |Df|^2 \right\} \\ &= \left\{ u^{ij} D^2 u_{ij} + \frac{1}{n-1} u^{\ell k} u^{ij} D u_{\ell k} D u_{ij} - u^{ik} u^{\ell j} D u_{ij} D u_{\ell k} \right\} \ . \end{aligned} \tag{42}$$

Summing up,

$$\begin{aligned} &f^{-2} \left\{ f \Delta f - \left(1 - \frac{1}{n-1} \right) |\nabla f|^2 \right\} \\ &= \left\{ L(H) + \frac{1}{n-1} \sum_{\beta=1}^{n} u^{\ell k} u^{ij} u_{\beta \ell k} u_{\beta ij} - \sum_{\beta=1}^{n} u^{ik} u^{\ell j} u_{\beta ij} u_{\beta \ell k} \right\} \ . \end{aligned} \tag{43}$$

Using the fact that (u^{ij}) is diagonal at p,

$$\begin{aligned} &f^{-2} \left\{ f \Delta f - \left(1 - \frac{1}{n-1} \right) |\nabla f|^2 \right\} \\ &= \left\{ L(H) + \frac{1}{n-1} \sum_{\beta=1}^{n} u^{ii} u^{jj} u_{\beta ii} u_{\beta jj} - \sum_{\beta=1}^{n} u^{ii} u^{jj} u_{\beta ij}^2 \right\} \ . \end{aligned} \tag{44}$$

For each fixed β, we wish to compute $\frac{1}{n-1}\, u^{ii}u^{jj}u_{\beta ii}u_{\beta jj} - u^{ii}u^{jj}u^2_{\beta ij}$. Let $I_\beta = \{1 \le i \le n \mid u_{\beta ii} > 0\}$, $J_\beta = \{1 \le i \le n \mid u_{\beta ii} < 0\}$. We divide into two cases:

(i) $|I_\beta| \le n-1,\ |J_\beta| \le n-1$

(ii) $I_\beta \cup J_\beta = \{1, 2, \ldots, \}$, one of I_β, J_β is empty.

In case (i)

$$\begin{aligned} \frac{1}{n-1}\, u^{ii}u^{jj}u_{\beta ii}u_{\beta jj} - u^{ii}u^{jj}u^2_{\beta ij} &\le \frac{1}{n-1}\left(\sum_{i=1}^{n} u^{ii}u_{\beta ii}\right)^2 - \sum_{i=1}^{n}(u^{ii}u_{\beta ii})^2 \\ \le\ & \frac{1}{n-1}\left(\sum_{i\in I_\beta} u^{ii}u_{\beta ii}\right)^2 + \frac{1}{n-1}\left(\sum_{i\in J_\beta} u^{ii}u_{\beta ii}\right)^2 - \sum_{i\in I_\beta}(u^{ii}u_{\beta ii})^2 - \sum_{i\in J_\beta}(u^{ii}u_{\beta ii})^2 \\ \le\ & 0, \qquad \text{since } |I_\beta| \le n-1,\ |J_\beta| \le n-1\,. \end{aligned} \tag{45}$$

In case (ii), we may assume $J_\beta = \varphi$. That is at p, $u_{\beta ii} > 0$, $i = 1, 2, \ldots, n$. By (36),

$$\begin{aligned} u_{\beta 11} + u_{\beta 22} + \cdots + u_{\beta nn} &= -\alpha x_\beta H, \\ |u_{\beta ii}| &\le -\alpha x_\beta H\,. \end{aligned} \tag{46}$$

We deduce

$$\begin{aligned} \frac{1}{n-1}\, u^{ii}u^{jj}u_{\beta ii}u_{\beta jj} - u^{ii}u^{jj}u^2_{\beta ij} &\le \frac{1}{n-1}\left(u^{nn}u_{\beta nn} + \sum_{i=1}^{n-1} u^{ii}u_{\beta ii}\right)^2 - \sum_{i=1}^{n}(u^{ii}u_{\beta ii})^2 \\ =\ & \frac{1}{n-1}(u^{nn}u_{\beta nn})^2 + \frac{1}{n-1}\left(\sum_{i=1}^{n} u^{ii}u_{\beta ii}\right)^2 - \sum_{i=1}^{n}(u^{ii}u_{\beta ii})^2 + \frac{2}{n-1}\, u^{nn}u_{\beta nn}\sum_{i=1}^{n-1} u^{ii}u_{\beta ii} \\ \le\ & \frac{2}{n-1}\, u^{nn}u_{\beta nn}\sum_{i=1}^{n-1} u^{ii}u_{\beta ii} \\ \le\ & \frac{2\alpha^2}{n-1}\, u^{nn}|x|^2H^2\sum_{i=1}^{n-1} u^{ii} \\ \le\ & \frac{2\alpha^2|x|^2}{(n-1)u_{nn}}\, H\sum_{i=1}^{n} u^{ii} \\ \le\ & \frac{2n\alpha^2|x|^2}{n-1}\, H\sum_{i=1}^{n} u^{ii} \\ \le\ & \frac{2n}{n-1}\,\alpha^2\, \mathrm{diam}^2(\Omega) H\sum_{i=1}^{n} u^{ii} \end{aligned} \tag{47}$$

(since $H = \sum u_{ii}$, $u_{nn} \ge u_{n-1;n-1} \ge \cdots \ge u_{11}$). Put (45) and (47) into (44),

$$f^{-2}\left\{f\Delta f - \frac{n-2}{n-1}\,|\nabla f|^2\right\} \le \left\{L(H) + \frac{2n^2}{n-1}\,\alpha^2\,\mathrm{diam}^2(\Omega)\sum_{i=1}^{n} u^{ii}H\right\}\,. \tag{48}$$

In view of (38), we have at p

$$f^{-2}\left\{f\Delta f-\frac{n-2}{n-1}\,|\nabla f|^{2}\right\}\leq-\alpha\{1-(2n^{2}+1)\alpha\ \mathrm{diam}^{2}(\Omega)\}\left(\sum_{i=1}^{n}u^{ii}\right)H\ . \tag{49}$$

While at p

$$\begin{aligned}\sum_{i=1}^{n}u^{ii} &\geq \sum_{i=1}^{n-1}u^{ii}\\ &\geq (n-1)\left(\prod_{i=1}^{n-1}u^{ii}\right)^{\frac{1}{n-1}}\\ &= \frac{\left(\prod_{i=1}^{n-1}u^{ii}\right)^{\frac{1}{n-1}}}{(u^{nn})^{\frac{1}{n-1}}}\\ &= (n-1)f^{-\frac{1}{n-1}}\ u_{nn}^{\frac{1}{n-1}}\\ &\geq \frac{n-1}{n^{\frac{1}{n-1}}}\,f^{-\frac{1}{n-1}}\ H^{\frac{1}{n-1}}\ .\end{aligned} \tag{50}$$

Insert (50) into (49), we get at p, by Condition (C′),

$$\begin{aligned}\frac{n-1}{n^{\frac{1}{n-1}}}\,\alpha\{1-(2n^{2}+1)\alpha\ \mathrm{diam}^{2}(\Omega)\}H^{1+\frac{1}{n-1}} &\leq f^{-2+\frac{1}{n-1}}\left\{f\Delta f-\frac{n-2}{n-1}\,|\nabla f|^{2}\right\}\\ &\leq \frac{A}{n-1}\ .\end{aligned} \tag{51}$$

Since $\alpha=\frac{1}{2(2n^{2}+1)\mathrm{diam}^{2}(\Omega)}$, we get the desired estimate. The proof of the lemma is complete. □

Second derivative est. We may write $u=g+\sigma\rho$ near $\partial\Omega$. $\forall\ p\in\partial\Omega$, we want to estimate $|\nabla^{2}u(p)|$. We may assume $p=0$ and $\rho_{i}(0)=0\ \forall\ i\leq n-1$, $\rho_{n}(0)=1$ near $p=0$, $u=g+\sigma\rho\Rightarrow u_{n}(0)=g_{n}(0)+\sigma(0)\Rightarrow|\sigma(0)|\leq c$. Hence, $u_{ij}(0)=g_{ij}(0)+\sigma(0)\rho_{ij}(0)$, $i,j\leq n-1\Rightarrow|u_{ij}(0)|\leq c(\Omega,\|u\|_{C^{1}},\|g\|_{C^{2}})$, $i,j\leq n-1$.

Lemma 6 *If* $|\nabla(f^{\frac{1}{n-1}}(x))|\leq A$ *near* $dist(x,\partial\Omega)\leq\delta$.

$$\Rightarrow|u_{in}(0)|\leq c(\Omega,A,\|g\|_{C^{3}},\|f\|_{\infty},\delta).$$

Proof. $\forall\ \varepsilon>0$, set $S_{\varepsilon}=\{x\in U|\rho(x)\geq0,x_{n}\leq\varepsilon\}$ U nbhd of 0, near 0,

$$\rho(x)=x_{n}+\sum^{n-1}a_{ij}x_{i}x_{j}+O(|x|^{3}).$$

For $T_\alpha = \frac{\partial}{\partial x_\alpha} - \frac{\rho_\alpha(x)}{\rho_n(x)}\frac{\partial}{\partial x_n} \Rightarrow T$ tangential on $\partial\Omega$, $T - \alpha(\rho) = 0$. Now, we want to construct a barrier function (to control $u_{in}(0)$). $w = \pm T_\alpha(u-g) - \tilde{B}x_n + B|x|^2$.

Claim: $\exists\, \varepsilon > 0$, $B, \tilde{B}$ (depends on $A, \Omega, \|f\|_{L^\alpha}, \|u\|_{C^1}$), such that $\max_{x\in S_\varepsilon} w = w(0) = 0$.

Proof. $L = u^{ij}(x)\partial_{x_i}\partial_{x_j}$, at T_α: $\log\det(u_{ij}) = \log f$.

$$\begin{aligned} L(T_\alpha u) &= \frac{T_\alpha f}{f} - u^{ij}(\frac{\rho_\alpha}{\rho_n})_i u_{ij} - u^{ij}(\frac{\rho_\alpha}{\rho_n})_j u_{ni} + u^{ij}(\frac{\rho_\alpha}{\rho_n})_{ij} u_i \\ &= \frac{T_\alpha f}{f} - 2(\frac{\rho_\alpha}{\rho_n})_n + u^{ij}(\frac{\rho_\alpha}{\rho_n})_{ij} u_n \end{aligned}$$

$\Rightarrow Lw(x) \geq B(\sum^n u^{ii}) - Af^{-\frac{1}{n-1}} - c(\sum^n u^{ii} + 1)$, $\forall$ $\mathrm{dist}(x, \partial\Omega) \leq \delta$.

Pick $0 < \varepsilon \leq \delta$, such that $2x_n - |\sum_{ij}^{n-1} a_{ij}x_ix_j| \geq 0$ and $|\frac{\rho_\alpha}{\rho_n}| \leq 1$ in S_ε. We decompose $S_\varepsilon = S_\varepsilon^1 \cup S_\varepsilon^2$, where $S_\varepsilon^1 = S_\varepsilon \cap \{\Delta u \geq 1\}$, $S_\varepsilon^2 = S_\varepsilon \cap \{\Delta u < 1\}$.

1. $Lw > 0$ in S_ε^1 if B large.
2. For $\tilde{B} >> B$, if $\max_{S_\varepsilon} w = w(x_0) \Rightarrow x_0 \in \delta S_\varepsilon$ by (1) $x_0 \notin S_\varepsilon^1$. If $x_0 \in S_\varepsilon^2 \Rightarrow w_n(x_0) = 0$ (interior pt)

$$\begin{aligned} \Rightarrow 0 &= w_n(x_0) = \pm u_{\alpha n} \mp \frac{\rho_\alpha}{\rho_n} u_{nn} \mp (\frac{\rho_\alpha}{\rho_n})_n u_n - \tilde{B} + 2x_n^0 B \\ &\quad \mp g_{\alpha n} \pm \frac{\rho_\alpha}{\rho_n} g_{nn} \pm (\frac{\rho_\alpha}{\rho_n})_n g_n \\ |u_{\alpha n} - \frac{\rho_\alpha}{\rho_n} u_{nn}| &\leq \frac{u_{\alpha\alpha} - u_{nn}}{2} + u_{nn} \leq \frac{3}{2}\Delta u \leq \frac{3}{2}. \end{aligned}$$

$$\Rightarrow w|_{\partial S_\varepsilon} \leq 0 \Rightarrow \max_{S_\varepsilon} w = w(0) \Rightarrow w_n(0) \leq 0$$

$$\Rightarrow |(T_\alpha u)_n(0)| \leq c \Rightarrow |u_{in}(0)| \leq c, \quad i \leq n-1.$$

To estimate u_{nn}, we want to control lower bound of u_{ii}, $i = 1, \ldots, n-1$. Pick at principal coordinates at 0 (x_n fixed) $(-\rho_{ij}(0)) = \begin{bmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_{n-1} \end{bmatrix}$, k_i principal curvatures of $\partial\Omega$ at 0. If necessary, add change $g \to g + c\rho$. We may assume $(g_{ij}) < 0$ in Ω, therefore, $u_{ij}(0) = g_{ij}(0) = \sigma(0)\rho_{ij}(0)$, $i, j \leq n-1$.

$$\nabla^2 u - \nabla^2 g = \begin{bmatrix} -[u_n(0) - g_n(0)]k_1 & & 0 & (u-g)_{n1} \\ 0 & \ddots & & \\ 0 & & -[u_n(0) - g_n(0)]k_{n-1} & \vdots \\ (u-g)_{n1} & & & (u-g)_{nn} \end{bmatrix}$$

therefore $u - g$ convex in Ω. $|(u-g)_n| \geq \sup_\Omega |u-g|/\mathrm{diam}(\Omega)$.

If f is positive, $\|u\|_{C^{2,1}(\overline{\Omega})} \leq c$ by Krylov-Evans Theorem. In conclusion, we have proved.

Theorem 7 *Let Ω be a strongly convex bounded domain in $\mathbf{R}^n$, $\partial\Omega \in C^{2,1}$. Suppose $f \in C^{1,1}(\overline{\Omega})$, f positive, then the Dirichlet problem (homogeneous)*

$$\begin{cases} \det(D^2u) = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{$*$}$$

has an unique $C^{2,1}$ convex solution u. If $\partial\Omega \in C^{3,1}$, then $u \in C^{3,\alpha}(\overline{\Omega})$ $\forall$ $0 < \alpha < 1$ and $\|u\|_{C^{2,1}(\overline{\Omega})} \leq c$.

Since the a priori C^2-boundedness of u is independent of $\inf_{x\in\Omega} f(x)$, if $f^{\frac{1}{n-1}}$ is pseudo-subharmonic in Ω and $f^{\frac{1}{n-1}}$ is Liptshitz near $\partial\Omega$. We can find solution u_ε for $(*)$ with f_ε in place of f, taking the limit, we get

Theorem 8 *Ω as in the previous theorem, suppose $f \geq 0$ in Ω, $f^{\frac{1}{n-1}}$ pseudo-subharmonic in Ω, and $f^{\frac{1}{n-1}}$ Liptshitz near $\partial\Omega$, then the Dirichlet problem (homogeneous) for degenerate MA:*

$$\begin{cases} \det(D^2u) = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

has an unique $C^{1,1}(\overline{\Omega})$ convex solution u, with $\|u\|_{C^{1,1}(\overline{\Omega})} \leq c$.

Remark 1: Theorem 3 is due to Krylov, Caffarelli-Nirenberg-Spruck. Theorem 4 is due to Caffarelli-Kohn-Nirenberg-Spruck under the stronger assumption that $f^{\frac{1}{n}} \in C^{1,1}(\overline{\Omega})$. Theorem 2 in the present form is due to P. Guan.

Remark 2. There is a corresponding version of Theorem 2 for prescribing Gauss curvature equation

$$\begin{cases} \det(D^2u) = K(x,u(x))(1+|Du|^2)^{\frac{n+2}{2}} & \text{in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

If K satisfies the conditions: $\int_\Omega K < w_n$, $K|_{\partial\Omega} = 0$ (these are necessary), and $K^{\frac{1}{n-1}}$ pseudo-subharmonic in Ω, $K^{\frac{1}{n-1}}$ Liptshitz near $\partial\Omega$, then $u \in C^{1,1}(\overline{\Omega})$.

The following are some discussions about pseudosubharmonicity of $f^{\frac{1}{n-1}}$. If $0 \notin \partial\Omega$, the function $f(x) = |x|^\alpha$ $\forall$ $\alpha > 0$ is pseudo-subharmonic. The condition for $n = 2$ is quite clear. For $n = 3$, if $f \in C^{3,1}(\mathbf{R}^3)$, $f \geq 0$, then f is pseudo-subharmonic. This follows from the next lemma:

Lemma 7 *If $f \in C^{3,1}(\mathbf{R})$, $f \geq 0$, then*

$$ff'' - \frac{1}{2}\,|f'|^2 \geq -Af^{3/2} \qquad \forall\, x \in \mathbf{R}.$$

Proof. Suppose $\|f\|_{C^{3,1}(\mathbf{R})} \leq M$ for some M. $\forall\, x_0 \in \mathbf{R}$,

$$\begin{aligned} 0 \leq f(x) &\leq f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 \\ &\quad + \frac{f'''(x_0)}{3!}(x-x_0)^3 + M(x-x_0)^4 \\ &= (a+bt+ct^2)(A+Bt+Ct^2) \qquad (t = x - x_0). \end{aligned}$$

with $a, b, c, A, B, C \in \mathbf{R}$. If the quadratic polynomy $a+bt+ct^2$ changes sign at some where, then $A+Bt+Ct^2$ will also changes sign at the same place, in this case $(a+bt+ct^2)(A+Bt+Ct^2) = (\alpha+\beta t+\gamma t^2)^2$ therefore $\alpha^2 = f(x_0)$, $2\alpha\beta = f'(x_0)$, $2\alpha\gamma+\beta^2 = f''(x_0)/2$, $|\gamma| \leq M \Rightarrow f(x_0)f''(x_0) - \frac{1}{2}\,|f'(x_0)|^2 = 4\alpha^2\gamma = 4f(x_0)^{3/2}\gamma \leq -4M^{\frac{1}{2}} f^{3/2}(x_0)$. If none of the above two quadratic polynomials change sing, we may assume $a+bt+ct^2 \geq 0$, $A+Bt+Ct^2 \geq 0$, $\Rightarrow b^2 \leq 4ac$, $B^2 \leq 4AC$. By the relations $aA = f(x_0)$, $aB + Ab = f'(x_0)$, $bB + aC + Ac = \frac{f''(x_0)}{2}$

$$\begin{aligned} f(x_0)f''(x_0) - \frac{1}{2}(f'(x_0))^2 &= 2aA(bB+aC+Ac) - \frac{1}{2}(a^2B^2 + A^2b^2 + 2aAbB) \\ &= aAbB + 2a^2(AC - B^2/4) + 2A^2(ac - b^2/4) \\ &\geq aAbB \geq -4aA\sqrt{aAcC} \\ &= -4f(x_0)^{3/2}\sqrt{cC} = -4Mf^{3/2}(x_0). \end{aligned}$$

□

For general n, if $f = f^{\alpha_1} + \cdots + f_k^{\alpha_k}$, with $\alpha_j \geq n-1$ and $f_j \in C^{1,1} \Rightarrow f^{\frac{1}{n-1}}$ pseudo-subharmonic.

5 Monge-Ampére Equations: General Dirichlet Problem

The following theorem is due to Caffarelli-Nirenberg-Spruck and Krylov.

Theorem 9 *If Ω is strongly convex, $\partial\Omega \in C^{3,1}$, f positive $f, \in C^{1,1}(\overline{\Omega})$, $g \in C^{3,1}(\partial\Omega)$ the following problem*

$$\begin{cases} \det(D^2u) = f \\ u|\partial\Omega = g \end{cases}$$

has an unique solution $u \in C^{3,\alpha}(\overline{\Omega})$, with $\|u\|_{C^{3,\alpha}(\overline{\Omega})} \leq C$.

One may ask if $\partial\Omega \in C^{2,1}$, $g \in C^{2,1}(\partial\Omega)$, can we conclude $u \in C^{2,1}(\overline{\Omega})$. In general, this is not true, $\exists\ f > 0$, $f \in C^{1,1}(\overline{\Omega})$, $\partial\Omega \in C^{2,1}$, $g \in C^{2,1}(\partial\Omega)$, Ω strongly convex, but $u \notin C^{1,1}(\overline{\Omega})$. Nevertheless, if $\partial\Omega \in C^3$, $g \in C^3$, the $u \in C^{2,1}(\overline{\Omega})$. This improved result is due to X.J. Wong.

Lemma 8 *Suppose Ω is strongly convex, $f \in C^{1,1}(\overline{\Omega})$, positive, $g \in C^3(\partial\Omega)$, then $\|u\|_{C^{1,1}(\overline{\Omega})} \leq C$.*

Note once we have $\|u\|_{C^{1,1}(\overline{\Omega})} \leq C$, by Krylov-Evans Theorem, $\|u\|_{C^{2,1}(\overline{\Omega})} \leq C$.

Proof of Lemma. We have proved $\max_\Omega |\nabla^2 u| \leq C(H \max_{\partial\Omega} |\nabla^2 u|)$, and $|u_{\tau\tau}| \leq C$ on $\partial\Omega$, $|u_{\tau n}| \leq C$ on $\partial\Omega$ for τ tangential to $\partial\Omega$. We only need to control u_{nn}. $\forall\ x_0 \in \partial\Omega$, we may take $x_0 = 0$ near x_0, $\partial\Omega$ is expressed as:

$$\begin{aligned} x_n &= \frac{1}{2}\,|x'|^2 + \text{ cubic of } x' + o(|x'|^3), \quad x' = (x_1, \ldots, x_{n-1}) \\ u_{ij} &= g_{ij} + (g_n - u_n)\delta_{ij} \text{ at } 0 \\ u(0) &= u_i(0) = 0 \qquad \forall\ i \leq n-1, \text{ on } \partial\Omega \\ u = \varphi &= \frac{1}{2} A_{ij} x_i x_j + \text{ cubic of } x' + o(|x'|^3),\ (A_{ij}) \text{ diagonal.} \end{aligned}$$

We wish to show $u_{ii} \geq \theta_0 > 0$, $\forall\ i \leq n-1$. We may take $i = 1$. Let $\tilde{u} = u - A_{11}x_n$, on $\partial\Omega$.

$$\begin{aligned} \tilde{u} &= \tilde{A}_2 x_2^2 + \cdots + \tilde{A}_{n-1} x_{n-1}^2 + \tilde{A} x_1^3 + \sum_{\alpha+\beta+\gamma>3} A_{\alpha\beta\gamma} x_\alpha x_\beta x_\gamma + o(|x'|^3) \\ &\leq \tilde{A} x_1^3 + \frac{1}{2}\,|\tilde{x}|^2 + o(|x'|^3), \end{aligned}$$

where $\tilde{x} = (x_2, \ldots, x_{n-1})$. Set

$$D = \left\{ |x_1| < \frac{1}{f}\,\sqrt{\delta}, |\tilde{x}| < \sigma\delta^{3/4}, |x_n - \delta| < \frac{1}{2}\,\delta \right\}$$

δ, σ small to be chosen $\Rightarrow D \subset \Omega$.

We want to show that

$$\tilde{u}(x) \le \frac{1}{2}\, M|\tilde{X}|^2 + o(\delta^{3/2}) \text{ on } \partial D. \qquad (*)$$

By convexity, we only need to prove $(*)$ on $\partial D \cap \{x_1 = \pm\frac{1}{f}\,\sqrt{\delta}\}$. We may assume $\tilde{A} > 0$. Let

$$\ell = \{x_1 = t,\ \tilde{x} = 0,\ x_n = -K\delta^{1/2}A\}, \quad K > 1 \text{ to be determined.}$$

$\forall\, p = (x_1, \tilde{x}, x_n) \in \partial D \cap \{x_1 = \pm\frac{1}{f}\,\sqrt{\delta}\}$, let $p^\pm = (x_1^\pm, \tilde{x}^\pm, x_n^\pm)$ denote the two intersection points of $\partial\Omega$ with the line $p + \ell$. Such that $x_1^- < x_1^+$. Then we have

$$\begin{aligned} -4K\delta^{1/2} - o(\delta^{1/2}) &\le x_1^- \le -K\delta^{1/2} + o(\delta^{1/2}) \\ -\frac{1}{f}\,\delta^{1/2} &\le x_1^+ \le 4\delta^{1/2} + o(\delta^{1/2}), \quad \text{and} \\ x_1^+ - x_1 &> \frac{1}{4K}\,\delta^{1/2} + o(\delta^{1/2}) \quad \text{if } K >> 1 \text{ large.} \end{aligned}$$

These give us the following estimates for $\tilde{u}$ at p^-, p^+:

$$\begin{aligned} \tilde{u}(p^-) &\le -\tilde{A}K^3\delta^{3/2} + o(\delta^{3/2}) + \frac{1}{2}\, M|\tilde{x}|^2 \\ \tilde{u}(p^+) &\le 64\tilde{A}\delta^{3/2} + o(\delta^{3/2}) + \frac{1}{2}\, M|\tilde{x}|^2 \\ \Rightarrow\ \tilde{u}(p) &\le \frac{x_1^+ - x_1}{x_1^+ - x_1^-}\, u(p^-) + \frac{x_1 - x_1^-}{x_1^+ - x_1^-}\, u(p^+) \\ &\le \frac{1}{2}\, M|\tilde{X}|^2 + o(\delta^{3/2}) \end{aligned}$$

(K large). Now, we construct a barrier function: $w = k\delta^{1/2}x_1^2 + M\delta^{-\frac{1}{2}}|x_n - \delta|^2 + M|\tilde{x}|^2 - \frac{1}{2}\,\sigma^2\delta^{3/2}$, with $k = 2^{-n}M^{1-n}\min f(x)$.

$$w \text{ satisfies: } \det(D^2 w) = \min f$$

on $\partial\Omega$,

$$w(x) \ge \min\left\{\frac{k}{64}\,\delta^{3/2}, \frac{M}{4}\,\delta^{3/2}, M\sigma^2\delta^{3/2}\right\} - \frac{1}{2}\,\sigma^2\delta^{3/2}$$

with suitable choices of $\sigma, \delta \Rightarrow w(x) \ge \tilde{u}(x)$ on ∂D.

$$\begin{aligned} &\Rightarrow\ w \ge \tilde{u} \text{ in } D \Rightarrow \tilde{u}(0,\delta) \le w(0,\delta) = -\frac{1}{2}\,\sigma\delta^{3/2} \\ &\Rightarrow\ \tilde{u}_n(0) \le -\frac{1}{2}\,\sigma\delta^{1/2}. \end{aligned}$$

On the other hand, $\frac{\partial^2}{\partial x_1^2}\,(\tilde{u}(x', \rho(x'))) = 0$ at 0,

$$\Rightarrow \tilde{u}_{11} + \tilde{u}_n\rho_H = 0 \Rightarrow u_{11}(0) = \tilde{u}_{11}(0) = -\tilde{u}_n(0)\rho_{11}(0) \ge \frac{1}{2}\,\sigma^2\delta^{1/2}.$$

□

Finally, we consider the $C^{1,1}(\overline{\Omega})$ regularity of degenerate MA with general boundary value.

Theorem 10 *Let Ω be a strongly convex domain in $\mathbf{R}^n$ with $\partial\Omega \in C^{3,1}$, $g \in C^{3,1}(\partial\Omega)$, and let f be a non-negative function in Ω such that $f^{1/(n-1)} \in C^{1,1}(\overline{\Omega})$, then there is an unique convex solution $u \in C^{1,1}(\overline{\Omega})$ of the Dirichlet problem*

$$\begin{cases} \det(D^2u) = f & \text{in } \Omega \\ u|_{\partial\Omega} = g. \end{cases}$$

Consequently, any generalized solution of the above problem in $C^0(\overline{\Omega})$ must belong to $C^{1,1}(\overline{\Omega})$. And

$$\|u\|_{C^{1,1}(\overline{\Omega})} \le C$$

where C depends only on Ω, $\|g\|_{C^{3,1}}$, $\|f^{\frac{1}{n-1}}\|_{C^{1,1}}$.

Remark. Above result was obtained by Krylov under assumption that $f^{\frac{1}{n}} \in C^{1,1}(\overline{\Omega})$. The result in the present form is due to P. Guan–N. Trudinger–X.J. Wang. Examples show that the above result (with exponent $\frac{1}{n-1}$) is optimal.

Proof of Theorem. (We assume $f > 0$, but our estimates will be independent of $\inf_\Omega f$.) We already have

$$\max_{\Omega} |D^2u| \le C(H \max_{\partial\Omega} |D^2u|), \quad \text{and } |u_{TT}| \le C,$$

$|u_{Tn}| \le C$ for τ tangential at $\partial\Omega$. To control u_{nn}, we estimate lower bounds of u_{TT}. $\forall$ $x_0 \in \partial\Omega$, we may assume $x_0 = 0$, and $(u_{ij})_{n-1,n-1}$ diagonal at 0. Now, assume $f = h^{n-1}$, $h \ge 0$, $h \in C^{1,1}(\overline{\Omega})$. We state

Lemma 9 $\prod_{i=1}^{n-1} u_{ii}(0) \ge \gamma_0 f(0)$ *(γ_0 constant depending only on Ω, $\|g\|_{3,1}$, $\|h\|_{1,1}$.)*

Lemma 10 $\|u_{in}(0)| \le C_0\sqrt{u_{ii}(0)}$ *(C_0 depending only on Ω, $\|g\|_{C^{3,1}}$, $\|h\|_{C^{1,1}}$.)*

Assuming the lemmas, we have

$$\begin{aligned} &\left(\prod_{i=1}^{n-1} u_{ii}(0)\right) u_{nn}(0) \\ &\quad = \sum_{j=1}^{n-1} \left(\prod_{i\neq j} u_{ii}(0)\right) u_{jn}^2(0) + f(0) \quad \text{(by Lemma 10)} \\ &\quad \le (n-1)C_0 \prod_{i=1}^{n-1} u_{ii}(0) + f(0) \\ \Rightarrow u_{nn}(0) &\quad \le (n-1)C_0 + \frac{f(0)}{\prod_{i=1}^{n-1} u_{ii}(0)} \le (n-1)C_0 + \frac{1}{\gamma_0} \end{aligned}$$

The Theorem is proved. $\square$

We now turn to the proofs of the lemmas. First, we normalize the boundary near 0.

$$x_n = \rho(x) = \frac{1}{2}\,|x'|^2 + \text{ cubic of } x' + O(|x'|^4).$$

By subtracting a linear function, we may assume $\nabla u(0) = 0$, $\inf_\Omega u(x) = u(0) = 0$, and $u_{ij}(0) = 0$, $i \neq j$, $i \leq n-1$, $j \leq n-1$.

$$g(z', \rho(x')) = \frac{1}{2}\sum_{i=1}^{n-1} b_i x_i^2 + R(x') + O(|x'|^4)$$

$$b_i = u_{ii}(0) > 0,\ i = 1, \ldots, n-1, \quad 0 \leq b_1 \leq b_2 \leq \cdots \leq b_{n-1},\ (b_{n-1} \leq C)$$

Fact: If $u = g = \alpha x_1^2 + \beta x_1^3 + R_4(x_1)$ on $\partial\Omega$, $\alpha \geq 0$ and $|R_4| \leq A|x_1|^4$, then $|\beta| \leq (1+A)\sqrt{\alpha}$.

Proof. Since $u \geq 0$ at $x_1 = \pm\sqrt{\alpha}$.

Now, we provide of Lemma 9.

Proof. We make a transformation $X \to Y = T(X)$, where $Y_i = M_i X_i$, $i = 1, \ldots, n$, $\begin{cases} M_i = b_i^{1/2} M \\ M_n = M = \frac{1}{b_1}. \end{cases}$ Set $V(Y) = M^2 U(X)$, we have

$$\det(D_Y^2 V) = \tilde{f}(Y) \overset{\det}{\equiv} f\left(\frac{Y_1}{M_1}, \cdots, \frac{Y_n}{M_n}\right) M^{2n} \Big/ \prod_{i=1}^{n} M_i^2,$$

$\tilde{\Omega} = T(\Omega)$, near 0, $\partial\tilde{\Omega}$ is expressed:

$$Y_n = \tilde{\rho}(Y') = \frac{1}{2}\sum_{i=1}^{n-1} d_i y_i^2 + O(|y'\beta|),\ d_i = \frac{M_n}{M_i^2} = \frac{b_1}{b_i} \leq 1,$$

and $|D_{y'}^K \tilde{\rho}| \leq |D_{x'}^K \rho| \leq C$ for $k = 3, 4$. Also, on $\partial\Omega$,

$$V(y) = \tilde{g}(y') = \frac{1}{2}\sum_{i=1}^{n-1} \tilde{b}_i y_i^2 + R(y') + \text{ 4th orders} \tag{52}$$

where $\tilde{b}_i = \frac{M^2}{M_i^2}\, b_i = 1$. And

$$|D_{y_i}^4 \tilde{g}(y)| = \frac{M^2}{M_i^4} |D_{x_i}^4 g(x)| \leq C.$$

The 4th orders in (52) are bounded, by **Fact**, Coeff. of R in (52) are uniformly bounded (i.e., independent of M_i). Therefore, $\partial\tilde{\Omega}$, $\tilde{g} \in C^{3,1}$ with norms independent of M_i. Let

$\omega = \{y \in \tilde{\Omega} | y_n < 1, |y_i| < 1, i = 1, \ldots, n-1\}$ by convexity of V, $V = \tilde{g} \leq C$ on $\partial\omega \cap \partial\tilde{\Omega} \Rightarrow$ $V \leq C$ in CV.

Claim: $\sup\{\tilde{f}(y) | y \in \omega\} \leq C$. If the claim is true, $C \geq \tilde{f}(y) = f(y_i/m_i)/\prod_{i=1}^{n-1} b_i$. So, Lemma 9 follows.

Proof of the Claim.

$$\begin{aligned} \frac{\partial}{\partial y_i} \tilde{f}^{\frac{1}{n-1}}(y) &= \frac{\partial}{\partial y_i} \left(f^{\frac{1}{n-1}} \left(\frac{y_i}{m_i} \right) \right) \Big/ \left(\prod_{j=1}^{n-1} b_j \right)^{\frac{1}{n-1}} \\ &= \frac{\partial}{\partial x_i} \left(f^{\frac{1}{n-1}} \right) \Big/ \left[M_i \left(\prod_{j=1}^{n-1} b_j \right)^{\frac{1}{n-1}} \right]. \end{aligned}$$

If $i = n$, therefore $\left| \frac{\partial}{\partial x_n} f^{\frac{1}{n-1}} \right| \leq C$ and

$$\begin{aligned} M_n \left[\prod_{j=1}^{n-1} b_j \right]^{\frac{1}{n-1}} &\geq \frac{1}{b_1} \left(\prod_{j=1}^{n-1} b_1 \right)^{\frac{1}{n-1}} = 1 \\ \Rightarrow \left(\frac{\partial}{\partial y_n} \left(\tilde{f}^{\frac{1}{n-1}}(y) \right) \right) &\leq C. \end{aligned}$$

If $i \leq n-1$, therefore $f^{\frac{1}{n-1}} \in C^{1,1}(\overline{\Omega})$, $f \geq 0 \Rightarrow |D_\tau f^{\frac{1}{n-1}}| \leq C\, f^{\frac{1}{2(n-1)}}$ for τ tangential. $\Rightarrow$

$$\begin{aligned} \left| \frac{\partial}{\partial y_i} \tilde{f}^{\frac{1}{n-1}}(y) \right| &\geq C\, f^{\frac{1}{n-1}} \Big/ \left[M_i \left(\prod b_j \right)^{\frac{1}{n-1}} \right] \\ &= C\, \tilde{f}^{\frac{1}{2(n-1)}} \Big/ \left(M_i \left[\prod_{j}^{n-1} b_j \right]^{\frac{1}{2(n-1)}} \right). \end{aligned}$$

Since $M_i^{2(n-1)} (\prod_j^{n-1} b_j) \geq M_1^{2(n-1)} \prod_{j=1}^{n-1} b_j \geq M_1^{2(n-1} b_1^{n-1} = 1$.

$$\Rightarrow \left| \frac{\partial}{\partial y_i} \tilde{f}^{\frac{1}{2(n-1)}}(y) \right| \leq C, \ \forall\, i \leq n-1.$$

Therefore, if $\sup_w \tilde{f} \to \infty \Rightarrow \inf_w \tilde{f} \to \infty$. If $\sup_w \tilde{f}$ is very large, $\inf \tilde{f}$ very large a standard barrier argument gives $\inf_w v < 0$ contradiction. This proves the claim, so Lemma 9. As for Lemma 10, we first prove:

Sublemma: Letting γ be inner normal to $\partial\Omega$, τ be a tangent vector, $\forall\, x \in \partial\Omega$, if $|x| \leq \frac{1}{2}\sqrt{b_{n-1}}$, then $|u_{\gamma\tau}(x)| \leq C\sqrt{b_{n-1}}$ (Note, $n = 2$, Lemma 10 follows).

Proof.

Let $M = \frac{1}{b_{n-1}}$, $x \to y = T(x)$ with

$$\begin{cases} y' = \sqrt{M} x' \\ y_n = M x_n. \end{cases}$$

Set $v(y) = M^2 u(x)$, we have

$$\det(D_y^2 v) = \tilde{f} \overset{\det}{\equiv} M^{n-1} f(T^{-1}(y)), \quad \tilde{\Omega} = T(\Omega)$$

$\partial\tilde{\Omega}$ can be expressed:

$$y_n = \frac{1}{2}\,(y')^2 + O(|y'|^3).$$

On $\partial\tilde{\Omega}$:

$$v(y) = \tilde{g}(y') = \frac{1}{2}\sum_i^{n-1} \tilde{b}_i y_i^2 + R(y') + O(|y'|^4),$$

where $\tilde{b}_i = \frac{b_i}{b_{n-1}} \leq 1$.

As before, we can show $\partial\tilde{\Omega}$, $\tilde{\varphi} \in C^{3,1}$ (with norms independent of M). Similarly, for

$$\begin{aligned} w &= \{y \in \tilde{\Omega} | y_n < 1\} \\ \sup_w \tilde{f} &\leq C. \end{aligned}$$

This gives $|v_{\tilde{\gamma}}(y)| \leq C$ for $y \in \partial\Omega$ $(y_n \leq \frac{1}{2})$

$$\begin{aligned} &\Rightarrow \quad |Dv| \leq C \text{ in } \partial\tilde{\Omega} \cap \left\{ y_n \leq \frac{1}{2} \right\} \\ &\Rightarrow \quad |Dv| \leq C \text{ in } \tilde{\Omega} \cap \left\{ y_n \leq \frac{1}{2} \right\} \end{aligned}$$

therefore, $\partial\tilde{\Omega}$ uniformly convex in $y_n \leq \frac{1}{2}$, using the same argument as in the proof for the mixed derivative bounds in homogeneous case, we get $|v_{\tilde{\gamma}\tilde{\tau}}| \leq C$ on $\partial\tilde{\Omega} \cap \left\{ y_n \leq \frac{1}{4} \right\} \Rightarrow$ $|u_{\gamma\tau}| \leq C$. □

Now, we prove Lemma 10. For $i = 1, \cdots, n-1$, let us denote by $\tau_i = \tau_i(x)$ the tangential direction of $\partial\Omega$ at $x \in \partial\Omega$, which lies in the two dimensional plane parallel to the x_i and x_n axes and passes through the point x. Our induction hypothesis is that for some $k = 1, \cdots, n-2$ and $i = k+1, \cdots, n-1$, there exists a constant $\theta_i > 0$, depending on $\Omega, |\varphi|_{3,1}$ and $|\tilde{f}|_{1,1}$, such that for $x \in \partial\Omega$, with $|x| \leq \theta_i \sqrt{b_i}$, we have the estimates

$$|u_{\gamma\tau_j}(x)| \leq C\sqrt{b_i}, \forall j = 1, \cdots, i, \tag{53}$$

where C is a constant depending on Ω, $|\varphi|_{3,1}$ and $|\widetilde{f}|_{1,1}$. When $k = n-2$, (53) is exactly Sublemma with $\theta_{n-1} = \frac{1}{2}$. We shall prove that there exists a constant θ_k, also depending on Ω, $|\varphi|_{3,1}$, and $|\widetilde{f}|_{1,1}$, such that for $x \in \partial\Omega$, with $|x| \le \theta_k\sqrt{b_k}$, we have

$$|u_{\gamma\tau_j}(x)| \le C\sqrt{b_k}, \forall j = 1, \cdots, k, \tag{54}$$

where C is a constant depending on Ω, $|\varphi|_{3,1}$ and $|\widetilde{f}|_{1,1}$. The Lemma then follows from Sublemma and (54) by induction.

To prove (54), we introduce the dilation, $x \to y = T(x)$, defined by

$$y_i = M_i x_i, \quad i = 1, \cdots, n,$$

where$M_i = \sqrt{M}$, for $i = 1, \cdots, k$; and $M_i = \sqrt{b_i}M$, for $i = k+1, \cdots, n-1$; with $M_n = M$ where $M = \frac{1}{b_k}$. We may suppose that $b_k \le \frac{1}{2}b_{k+1}$, otherwise (54) follows immediately from (53). Let $v(y) = M^2 u(x)$. Then v satisfies

$$\det(D_y^2 v) = g(y) =: f(\frac{y_i}{M_i})M^{2n} / \prod_{i=1}^{n} M_i^2$$

in $\widetilde{\Omega} = T(\Omega)$. Near the origin $\partial\widetilde{\Omega}$ is represented by

$$y_n = \widetilde{\rho}(y') = \frac{1}{2}\sum_{i=1}^{n-1} d_i y_i^2 + O(|y'|^3), \tag{55}$$

where $d_i = 1$, for $i \le k$, and $d_i = \frac{b_k}{b_i}$, for $i \ge k+1$. After the transformation we have,

$$v(y) = \widetilde{\varphi}(y') = \frac{1}{2}\sum_{i=1}^{n-1} \widetilde{b}_i y_i^2 + R(y') + O(|y'|^4), \quad y = (y', \widetilde{\rho}(y')) \in \partial\widetilde{\Omega}, \tag{56}$$

where $\widetilde{b}_i = 1, \quad i \ge k$, and $\widetilde{b}_i = \frac{b_i}{b_k}, \quad i < k$. As above we see that near the origin, both $\widetilde{\varphi}$ and $\partial\widetilde{\Omega}$ are $C^{3,1}$ smooth and their $C^{3,1}$-norms are independent of M.

Let

$$\omega = \{y \in \widetilde{\Omega} \mid y_n < \beta^2, \ |y_i| < \beta, \ i = k+1, \cdots, n-1\},$$

where β will be chosen small such that the third and high order terms in (55) and (56) do no harm to the following estimation. As before we may assume ω is small and, by (55), ω is bounded independently of M. By the convexity of v we have,

$$v \le C \quad \text{in } \omega.$$

Similar to the proof of Lemma 9 we have

$$\sup\{g(y); \ y \in \omega\} \le C.$$

To prove (54) it is crucial to establish a bound for the normal derivative of v near the origin. The main difficulty is that we cannot control the convexity of $\partial\widetilde{\Omega}$ near the origin. We construct a lower barrier v^* by setting

$$v^*(y) = \frac{\sigma}{2}|y'|^2 + \frac{1}{2}Ky_n^2 - K^2y_n,$$

where $\sigma > 0$ small and $K > 1$ large will be chosen so that

$$\det(D^2v^*) = \sigma^{n-1}K \geq \sup_{\omega} g(y).$$

We claim $v^* \leq v$ on $\partial\omega$ (with appropriate choice of β, σ and K). For later application we will prove the stronger inequality

$$v^* \leq \frac{1}{2}v \quad \text{on } \partial\omega. \tag{57}$$

To prove (57) we first consider the piece $\partial_1\omega := \partial\omega \cap \partial\widetilde{\Omega}$. For $y \in \partial_1\omega$ we have, by (56),

$$v(y) = \frac{1}{2}\sum_{i-1}^{n-1}\widetilde{b}_i y_i^2 + O(|y'|^3) \geq \frac{1}{4}|\widetilde{y}|^2 - C|\hat{y}|^2 \tag{58}$$

provided β is small, where $\hat{y} = (y_1, \cdots, y_k)$, and $\widetilde{y} = (y_{k+1}, \cdots y_{n-1})$. By (55) we have

$$v^*(y) \leq \frac{\sigma}{2}|y'|^2 - \frac{1}{2}K^2y_n \leq \frac{\sigma}{2}|\widetilde{y}|^2 - \frac{1}{4}K^2|\hat{y}|^2. \tag{59}$$

Hence (57) holds on $\partial_1\omega$. On $\partial_2\omega := \partial\omega \cap \{y_n = \beta^2\}$ we have $v \geq 0$. For $\sigma > 0$ small and $K > 1$ large, we have $v^* \leq -\frac{1}{2}K^2\beta^2$, so that (57) also holds on $\partial_2\omega$.

Finally we consider the piece $\partial_3\omega := \partial\omega \cap \{|y_i| = \beta, \quad \text{for some } i = k+1, \cdots, n-1\}$. We only consider the piece $\partial_3'\omega := \partial\omega \cap \{y_{n-1} = \beta\}$ since other pieces of $\partial_3\omega$ can be handled in the same way. First we prove that

$$v(y) \geq \frac{1}{4}\beta^2 \quad \text{on } \partial_3'\omega \cap \{y_n < \varepsilon_0\beta^2\} \tag{60}$$

provided ε_0 is small enough. If $\partial_3'\omega \subset \{y_n \geq \varepsilon_0\beta^2\}$, we have nothing to prove, so we may suppose $\partial_3'\omega \cap \{y_n < \varepsilon_0\beta^2\} \neq \emptyset$. To prove (60) we first fix a point $p = (\hat{p}, \widetilde{p}, p_n) \in \partial_3'\omega$, where $\hat{p} = (p_1, \cdots, p_k) \neq 0$, $\widetilde{p} = (p_{k+1}, \cdots, p_{n-1})$, $p_n < \varepsilon_0\beta^2$. For $\delta \geq 0$, sufficiently small, we then fix a further point $p^* = (0, \widetilde{p}, p_n + \delta)$ so that the straight line through p and p^* meets $\partial_3'\omega$ in a point $\overline{p}$ satisfying

$$\frac{1}{2}|p - p^*| \leq |\overline{p} - p^*| \leq |\overline{p} - p|. \tag{61}$$

In view of the convexity of $\widetilde{\Omega}$ and the representation (55), we may accomplish (61) by taking

$$\delta = |\nabla_{\hat{y}}\widetilde{\rho}(0, \widetilde{p})|\,|\hat{p}| \leq O(|\beta|^3)$$

and β sufficiently small. Now let $p^0 = (0, \widetilde{p}, \widetilde{\rho}(0, \widetilde{p}))$ be the projection of p^* on $\partial\widetilde{\Omega}$. We claim

$$|v_{\widetilde{\gamma}}(p^0)| \le C, \tag{62}$$

where $\widetilde{\gamma}$ is the unit inner normal at p^0. Indeed, for any $y = (0, \widetilde{y}, \widetilde{\rho}(0, \widetilde{y})) \in \partial\widetilde{\Omega} \cap B_{\theta_{k+1}}(0)$, with θ_{k+1} as given in (53), we have, (for $x \in T^{-1}(y)$),

$$|x_i| = |y_i/M_i| \le \theta_{k+1} b_k / b_i^{1/2}, \quad i = k+1, \cdots, n-1.$$

Hence by (53),

$$|u_\gamma(x)| \le u_\gamma(0) + C \sum_{i=k+1}^{n-1} \sup\{|x_i| \cdot |u_{\gamma\tau_i}(x)|;\ |x_i| \le \theta_{k+1} b_k / b_i^{1/2}\} \le C b_k,$$

and since $u_\gamma(0) = 0$, we obtain

$$|\nabla u(x)| \le C b_k at x = T^{-1}(y).$$

By the definition of T,

$$|v_{y_n}(y)| = M |u_{x_n}(x)| \le C M b_k = C.$$

Noticing that $v = \widetilde{\varphi} \in C^3$ on $\partial\widetilde{\Omega}$, we obtain (61). From (61) and the convexity of v, we have

$$v(p^*) \ge v(p^0) - C|p_n^* - p_n^0|, \tag{63}$$

while, from (56),

$$v(p^0) = \frac{1}{2} \sum_{i=k+1}^{n-1} |p_i^0|^2 + O(|p^0|^3),$$

and

$$v(\overline{p}) = \frac{1}{2} \sum_{i=1}^{n-1} \widetilde{b}_i |\overline{p}_i|^2 + O(|\overline{p}|^3) = v(p^0) + \frac{1}{2} \sum_{i=1}^{k} \widetilde{b}_i |\overline{p}_i|^2 + O(|\overline{p}|^3)$$

Noticing that $p_{n-1}^0 = \beta$, we have $v(p^0) \ge \frac{1}{2}\beta^2 - O(\beta^3) \ge \frac{3}{8}\beta^2$ if β is small enough. Since $d_i = 1$ in (55) when $i \le k$, we see that

$$\sum_{i=1}^{k} \widetilde{b}_i |\overline{p}_i|^2 \le C(|\overline{p}_n - p_n^0| + \beta^3) \le C(\overline{p}_n + \beta^3).$$

Hence we may first fix β small and then choose ε_0 small so that $\overline{p}_n \le 2\varepsilon_0\beta^2$ and

$$v(\overline{p}) \le \frac{9}{8} v(p^*),$$

whence by (63), $v(p^*) \geq \frac{1}{3}\beta^2$. Hence by (61) and the convexity of v, we have

$$v(p) \geq 3v(p^*) - 2v(\overline{p}) \geq \frac{3}{4}v(p^*) \geq \frac{1}{4}\beta^2$$

and (60) is proved.

From (60) we have

$$v(y) \geq \frac{1}{8}\beta^2 - Cy_n \quad \text{on } \partial_3'\omega,$$

where C depends on β and ε_0. On the other hand,

$$v^*(y) \leq C\sigma - K^2 y_n.$$

Hence $v^* < \frac{1}{2}v$ on $\partial_3'\omega$ if σ is small and K large enough. This completes the proof of inequality (57).

The next step in our proof is to adapt the barrier v^* to obtain a normal derivative bound near the origin. For any point $y_0 \in \partial_1\omega$, let $\xi = (\xi_1, \cdots, \xi_n) = A(y - y_0)$ be an orthogonal basis at y_0 so that ξ_n coincides with the inner normal, where A is some orthogonal matrix. We now set

$$v * (y) = \frac{\sigma}{2}|\xi'|^2 + \frac{1}{2}K\xi_n^2 - K^2\xi_n + \ell(\xi'),$$

where $\sigma > 0$ small and $K > 1$ large, $\xi' = (\xi_1, \cdots, \xi_{n-1})$, ℓ is a linear function such that $|\ell(\xi') - v(y)| = o(|\xi'|)$ as $\xi' \to 0$. Since both $\widetilde{\varphi}$ and $\partial\widetilde{\Omega}$ are $C^{3,1}$ smooth near the origin, arguing as above and by virtue of (57)) we see that $v^* \leq \frac{1}{2}v$ on $\partial\omega$ if $|y_0|$ and σ (> 0) are sufficiently small and $K > 1$ is sufficiently large. By the comparison principle it follows that $v^*(y) \leq v(y)$ on ω. We therefore obtain, by the convexity of v,

$$|v_{\widetilde{\gamma}}(y)| \leq C, \quad \forall y \in \partial\widetilde{\Omega}.$$

if y is near the origin. By choosing a new β we can then ensure the above estimate holds for all $y \in \partial_1\omega = \partial\omega \cap \partial\widetilde{\Omega}$, subsequently, from the convexity of v,

$$|Dv| \leq C \quad \text{in } \omega.$$

We can now complete the proof of the Lemma by standard arguments. Let

$$Lw = v^{ij}w_{ij},$$

where $\{v^{ij}\}$ denotes the inverse of the Hessian D^2v, and $T = (T_1, \cdots, T_k)$, where

$$T_i = \partial_i + (y_i\partial_n - y_n\partial_i), \quad i = 1, \cdots, k.$$

Applying the operator T to both sides of the equation

$$F(D^2v) := \log\det(D^2v) = \log g,$$

we obtain

$$L(Tv) = T(\log g),$$

where $L = v^{ij}\partial_i\partial_j$. Setting

$$w(y) = \pm T(v - \widetilde{\varphi})(y) + B|y|^2,$$

we have, since $d_i = 1$ for $i \leq k$ in (55),

$$|w(y)| \leq C_B(|y|^2 + y_n) on \partial\omega. \tag{64}$$

Similar to the proof of Lemma9 we have

$$|\nabla g^{1/2(n-1)}(y)| \leq C,$$

so that

$$|\nabla g| \leq C g^{(2n-3)/2(n-1)}(y).$$

Hence

$$\begin{aligned} Lw \quad &\geq B\sum v^{ii} - C(g^{-1/2(n-1)} + \sum v^{ii}) && (65)\\ &\geq \tfrac{1}{2}B\sum v^{ii} - Cg^{-1/2(n-1)} && (66)\\ &\geq \tfrac{1}{2}Bg^{-1/n} - Cg^{-1/2(n-1)} \geq 0. && (67)\\ & && (68) \end{aligned}$$

provided B is large enough. Now set

$$\widetilde{w}(y) = A(v^* - v - y_n) + w(y),$$

where $A > 1$ is a sufficient large constant to be chosen later. By (57) and the concavity of F, we have

$$L(v^* - v) \geq F(D^2v^*) - F(D^2v) \geq 0.$$

Consequently

$$L\widetilde{w} \geq 0,$$

which implies the function $\widetilde{w}$ attains the maximum on the boundary of ω.

We claim $\widetilde{w}(y) \leq 0$ on $\partial\omega$ for sufficiently large A. This is because by (60) and (64),

$$\widetilde{w}(y) \leq C_B(|y'|^2 + y_n) - \frac{1}{2}A(v(y) + y_n).$$

Using (55) and (56), we then choose A large enough so that $\widetilde{w}(y) \leq 0$ on $\partial\omega$.

Noticing that $\widetilde{w}(0) = 0$, we have therefore $\frac{\partial}{\partial y_n}\widetilde{w}(0) \leq 0$. Namely, $|v_{in}(0)| \leq C$, $i = 1, \cdots, k$. Similarly we have $|v_{\gamma\widetilde{\tau_i}}(y)| \leq C$, $i = 1, \cdots, k$, for $y \in \partial\omega \cap \partial\widetilde{\Omega}$ near the origin, where $\widetilde{\tau}_i$ is a tangential direction of $\partial\widetilde{\Omega}$ at y which lies in the plane parallel to the y_i and

y_n axes and passes through the point y. Pulling back to the x-coordinates we obtain the desired estimate. $\square$

We note that for degenerate Monge-Ampére Equations, $C^{1,1}$ regularity is the best that can be expected. This is readily seen by letting $B = B_1(0)$ in $\mathbf{R}^2$,

$$u(x,y) = [\max\{(x^2 - \frac{1}{2})^+,\ (y^2 - \frac{1}{2})^+\}]^2.$$

u is C^w at $\partial B_1(0)$, $\det(D^2u) = 0$ in $B_1(0)$.

The following example indicates that even if $u \in C^{2,\alpha}$, $f \geq 0$, $f \in C^\infty$ (or C^w), u may not be necessary in C^3. $u(x) = |x|^{2+\frac{2}{n}}$, we have $\det(D^2u) = C_n|x|^2$, $C_n > 0$, but we have only $u \in C^{2,\frac{2}{n}}$. In this case f has the best possible degeneracy (i.e. $f_{ij}(0) = I$). One may wish to know when we have higher regularity of u (better than $C^{1,1}$). In the two dimensional case, this is possible provided that: (i) f is of finite type near $\{f = 0\}$; (ii) Hessian matrix D^2u has at least one positive eigenvalue. The proof of the result depends on some tools in micro-local analysis, we won't deal with here. There is no such analogue result for $n \geq 3$ to date.

6 Mass Transport Problem – A second boundary value problem for MA

Suppose Ω, Ω^* are two domains in $\mathbf{R}^n$, f and g are two probability measures on Ω and Ω^* respectively. We want to find a map $T : \Omega \to \Omega^*$. T preserves the measures: $\int_E f = \int_{T(E)} g$, $\forall\ E \subset \Omega$, E Borel, such that it minimizes the cost function

$$\int_\Omega |T(x) - x|^2, \qquad \text{(among the measure-preserving class)}.$$

A Theorem of Rockafeller tells us, if such minimization T exists, T is a convex potential, i.e., $T = Du$, for some convex function in Ω.

If u is C^2, we readily see that

$$\det(D^2u) = f(x)/g(Du), \qquad x \in \Omega.$$

The above problem was first solved by Y. Brinier, he obtained a (unique) weak solution (in general, his weak solution is not the weak solution in Alexandrov sense for MA). In the case $n = 2$, Ω, Ω^* are strongly convex, Delano proved the classical solvability of the problem (the existence of regular solutions). Caffarelli obtained $C^{1,\alpha}$ interior regularity (if Ω^* is convex (1992)), $C^{1,\alpha}$ global regularity (if both Ω, Ω^* are convex, 1992), and $C^{2,\alpha}$ global regularity (if both Ω, Ω^* are strictly convex, 1996). The problem was studied by Pogolerov in $n = 2$.

Here, we will prove global $C^{2,\alpha}$ regularity of the solution to the problem under the assumption that both Ω, Ω^* are strongly convex. The proof is due to J. Urban (1997). In an interesting article, J. Wolfson considers $n = 2$, $f \equiv g \equiv 1$. He obtained C^∞ regularity of the solution under the assumption $\min_{\partial\Omega_1} k + \min_{\partial\Omega_2} k \geq C_0 > 0$. His approach is in the direction of sympletic geometry. It's important to understand what is the geometric obstruction for the regularity of the problem.

Now, we try to put the problem in PDE setting: We want to find $T = Du$, such that, $Du : \Omega \to \Omega^*$ diffeomorphism, and $\int_E f = \int_{\nabla u(E)} g$, $\forall\ E \subset \Omega$. Let h be the defining function of Ω^*, normalizing it as: $\Omega^* = \{p \in \mathbf{R}^n | h(p) > 0\}$, $|Dh|_{\partial\Omega} = 1$. We may assume $-(h_{p_ip_j}) \geq CI > 0\ \forall\ p \in \Omega^*$ (therefore Ω^* is strongly convex). Therefore, the problem is equivalent to the following PDE (second boundary value problem):

$$\begin{cases} \det(D^2u) = \frac{f(x)}{g(Du)} & \text{in } \Omega \\ h(Du) = 0 & \text{on } \partial\Omega. \end{cases} \tag{69}$$

We may consider a little more general model:

$$\begin{cases} \det(D^2u) = (f(x)/g(Du))e^{\varepsilon u}, & \Omega \\ h(Du) = 0, & \partial\Omega. \end{cases} \tag{70}$$

Theorem 11 *If $0 < \frac{1}{C} \leq f \leq C < \infty$, in Ω, $0 < \frac{1}{C} \leq g \leq C < \infty$ in Ω^*, and $f, g \in C^2$ ($\int_\Omega f = \int_{\Omega^*} g$). If both Ω and Ω^* are strongly convex. $\partial\Omega$, $\partial\Omega^* \in C^{2,1}$, then there is (a unique, up to a constant) convex solution u of equation (69), $u \in C^{3,\alpha}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$ $\forall$ $0 < \alpha < 1$.*

Theorem 11 can be deduced from the following:

Proposition 2 *There is constant c, C (Ω, Ω^*, f, g, as in Theorem 11)* $\forall\ 0 \leq \varepsilon < 1$. *Equation (70) has a unique solution u_ε, such that* $\max_\Omega |D^2 u_\varepsilon| \leq C$, *and* $|h_p(Du_\varepsilon) \cdot \nu| \geq c_0 > 0$ *on $\partial\Omega$ where ν the normal at $\partial\Omega$. (c, C are independent of ε).*

Proof of Theorem 11. $|Du_\varepsilon|$ is bounded and we know $\int_\Omega f = \int_\Omega^* g = \int_\Omega e^{\varepsilon u_\varepsilon} f$. Therefore, u_ε must change sign somewhere in Ω, i.e., $\exists\ x_0 \in \Omega$. $u_\varepsilon(x_0) = 0 \Rightarrow |u_\varepsilon|_{L^\infty(\overline{\Omega})} \leq C$. ($|Du_\varepsilon|$ bounded). By Proposition, the problem is elliptic (uniformly) and oblique. So $\|u_\varepsilon\|_{C^{2,\alpha}(\overline{\Omega})} \leq C$ by Liberman-Trudinger Theorem, C independent of ε. Let $\varepsilon \to 0$, we get u. □

The rest will be devoted to the proof of the proposition.

Proof–Step 1. The obliqueness (ν inner normal): Let $\chi = \sum_k h_{p_k}(Du)\nu_k$. We want to show $\chi \geq C > 0$. Let $H(x) = h(Du(x))$. $\forall\ \tau$ tangential to $\partial\Omega$, $H \equiv 0$ and Ω,

$$\begin{aligned} D_\tau H &= h_{p_k} D_{k\tau} u = 0 \text{ on } \partial\Omega \text{ and} \\ D_\nu H &= h_{p_k} D_{k\nu} u \geq 0 \text{ on } \partial\Omega \\ D_i H &= h_{p_k} D_{ik} u = (D_\nu H)\nu_i \end{aligned} \tag{71}$$

we have

$$\chi = h_{p_k}\nu_k = (D_\nu H)u^{\nu\nu} \geq 0, \tag{72}$$

where

$$u^{\nu\nu} = u^{ij}\nu_i\nu_j.$$

From 71

$$h_{p_i}h_{p_j}D_{ij}u = \chi D_\nu H, \tag{73}$$

so

$$\chi = \sqrt{u^{ij}\nu_i\nu_j u_{k\ell} h_{p_k} h_{p_\ell}} \qquad \text{(which is symmetric)}. \tag{74}$$

Let $x_0 \in \partial\Omega$, such that $\min_{\partial\Omega} \chi = \chi(x_0)$.

We may assume $\nu(x_0) = \vec{e}_n$, $\vec{e}_1, \ldots, \vec{e}_{n-1}$ tangential at x_0, we may also assume ν extended into $\overline{\Omega}$ such that $-[D_k\nu_\ell] \geq CI$ on $\partial\Omega$.

Consider $V = \chi + Ah(Du)$, A is a constant to be chosen. (Note that x_0 is still a minimum point of V on $\partial\Omega$). We have

$$D_j V(x_0) = 0, \qquad j = 1, \ldots, n-1.$$

Assertion: $D_n V(x_0) \geq -C$.

Claim: $LV \stackrel{\text{def}}{\equiv} u^{ij}D_{ij}V - G_{p_i}D_iV \le C\sum u^{ii}$ in Ω (where $G(x,z,p) = \log(e^{\varepsilon z}f(x)/g(p)))$.

If the claim is true, we construct a barrier function $w = V - e^{-B\rho} + 1$, where ρ is a defining function of Ω.

We have (therefore, $-(\rho_{ij}) \ge c^*I > 0$).

$$Lw \le C\Sigma u^{ii} - C^*B\Sigma u^{ii} - B^2u^{ij}\rho_i\rho_j - BG_{p_i}\rho_i$$

in Ω. Pick B large, $Lw \le 0$ in Ω (since at the point $\Sigma u^{ii} >> 1$, the second term dominates others, if $\Sigma u^{ii} \le M$, $\Rightarrow$ ($u^{ii} \ge CI$, the 2nd and 3rd terms together will dominate 1st and 4th terms: $G_{p_i}\rho_i \le \frac{1}{\alpha\beta}|G_p|^2 + \alpha\beta p_i^2 \Rightarrow \min_\Omega w = \min_{\alpha\Omega} w = w(x_0) \Rightarrow w_n(x_0) \ge 0$ $\Rightarrow D_nV(x_0) \ge -Bp_n(x_0) = -B$. □

Now to the **claim**: we compute

$$\begin{aligned} D_iV &= h_{p_kp_\ell}D_{\ell_i}u\nu_k + h_{p_k}D_i\nu_k + Ah_{p_k}d_{ki}u, \\ D_{ij}V &= h_{p_kp_\ell p_m}u_{\ell i}u_{mj}u\nu_k + h_{p_kp_\ell}u_{ij\ell}\nu_k + h_{p_kkp_\ell}u_{\ell_i}D_j\nu_k \\ &\quad + h_{p_kp_\ell}u_{\ell j} + D_i\nu_k + h_{p_k}D_{ij}\nu_k + Ah_{p_kp_\ell}u_{ki}u_{ij} + Ah_{p_k}u_{ijk}. \end{aligned}$$

Making use of the fact $u^{ij}u_{ijk} = G_{x_k} + g_zu_k + G_{p_i}u_{ik}$. Acting u^{ij} on $D_{ij}V$, g_{p_i} on D_iV, we get

$$\begin{aligned} &u^{ij}D_{ij}V - g_{p_i}D_iV \\ &\quad \le h_{p_kp_\ell p_m}u_{\ell m} + Ah_{p_kp_\ell}u_{k\ell} + C(HA) + C\Sigma u^{ii} \end{aligned}$$

≤ 0 if A large (therefore $\Sigma u^{ii} \ge (\det(u_{ij}))^{-\frac{1}{n}} \ge C > 0$.)

$$\le \tilde{C}\Sigma u^{ii}.$$

□

Back to the obliqueness:
$D_jV(x_0) = 0$, $j \le n-1$.

$$\Rightarrow h_{p_np_\ell}u_{\ell j} + h_{p_k}D_\alpha v_k + Ah_{p_k}u_{kj} = 0 \text{ at } x_0 \tag{75}$$

$D_nV(x_0) \ge -C$.

$$\Rightarrow h_{p_np_\ell}u_{\ell n} + h_{p_k}D_nv_k + Ah_{p_k}u_{kn} \ge -C. \tag{76}$$

Multiply h_{p_j} to 75, h_{p_n} to 76, sum up:

$$\begin{aligned} Au_{k\ell}h_{p_k}h_{p_\ell} &\ge -Ch_{p_n} - (D_kv_\ell)h_{p_k}h_{p_\ell} - h_{p_k}h_{p_np_\ell}u_{k\ell} \\ &\ge -Ch_{p_n} - (D_kv_\ell)h_{p_k}h_{p_\ell} \end{aligned}$$

(since $h_{p_k}u_{kj} = 0$, $j \le n-1$, $h_{p_k}u_{kn} \ge 0$ at x_0).

Note at x_0, $\chi = h_{p_n}$. We may assume $Ch_{p_n} \le \frac{1}{2} - (D_k v_\ell)h_{p_k}h_{p_\ell}$, this yields:

$$u_{k\ell}h_{p_k}h_{p\ell} \ge C > 0 \Leftrightarrow u_{nn} \ge C > 0.$$

Similarly, we also have $u^{\nu\nu}(x_0) \ge C > 0$, since by Legendre transform: $u^*(p) = xDu(x) - u(x)$, $P = Du(x)$, $\det(D^2u^*) = 1/f(Du^*, \Sigma p_k \cdot D_k u^* - u^*, p)$, $Du^*(\Omega^*) = \Omega$ (with v and Dh exchanged, and note that $\frac{\partial u^*}{\partial p_i} = x_i$, $\frac{\partial^2 u^*}{\partial p_i \partial p_j} = u^{ij}(x)$). Therefore, $\chi(x_0) \ge C > 0$.

Step 2: $|D^2u| \le C$.

Let $\beta = h_p(Du)$ the oblique direction (as a vector field). Let $M = \max_\Omega |D^2u|$. We already have some information: $D_{\tau\beta}u = 0$ on $\partial\Omega$, $D_{\nu\beta}u \ge 0$ on $\partial\Omega$, and $M \le C(H \max_{\partial\Omega}|D^2u|)$.

Lemma 11 *$\forall\ \delta > 0$, $u_{\beta\beta} \le C(\delta) + \delta M$, on $\partial\Omega$.*

Proof. Let $H = h(Du)$, $D_iH = h_{p_k}D_{pk}u$, $D_{ij}H = h_{p_kp_\ell}u_{ik}u_{j\ell} + h_{p_k}u_{ijk}$.

$$\begin{aligned} u^{ij}D_{ij}H &= h_{p_kp_\ell}u_{k\ell} + h_{p_k}(G_{x_k} + G_z u_k + G_{p_i}u_{ik}) \\ &\ge -C(H|D^2u|) \ge -(C(\delta) + \delta M)\Sigma u^{ii} \end{aligned}$$

(therefore $\Sigma u^{ii} \ge C|D^2u|^{\frac{1}{n-1}}$, this is a useful fact for us).

We may apply similar barrier argument as before to get $D_{\nu\beta}u \le A(C(\delta) + \delta M)$ on $\partial\Omega$ $\Rightarrow D_{\beta\beta}u \le A(C(\delta) + \delta M)$, since $\nu = x\beta + \tau$, and $x \ge C_0 > 0$, $D_{\tau\beta}u$ is bounded. □

Let $\max\limits_{\substack{x,\tau \\ |\tau|=1 \text{tangent} \\ x\in\partial\Omega}} D_{\tau\tau}u(x) = D_{\tau_0\tau_0}u(x_0)$. We may assume $\tau = e_1$, and $e_1, \ldots, e_{n-1}$ tangential at $x_0 = 0$, e_n inner normal. We may assume $u_{11}(0) \ge 1$ (otherwise, noting to prove). We calculate: (for $\tau = \tau(e_1)$, the projection of $\vec{e}_1$ to tangent directions at $\partial\Omega$).

$$\begin{aligned} D_{11}u &= D_{\tau\tau}u + 2\,\frac{\nu_1}{\beta\cdot\nu}\,D_{\tau\beta}u + \frac{\nu_1^2}{(\beta\cdot\nu)^2}\,D_{\beta\beta}u \\ &\le |\tau|^2u_{11}(0) + (C(\delta) + \delta M)\nu_1^2 \end{aligned}$$

therefore $\tau(e_1) = e_1 - \nu_1\nu - \frac{\nu_1}{\beta\cdot\nu}\,\beta^T$, where $\beta^T = \beta - (\beta\cdot\nu)\nu$

$$\begin{aligned} &\Rightarrow \quad |\tau(e_1)|^2 \le 1 + C\nu_1^2 - 2\nu_1\frac{\beta_1^T}{\beta\cdot\nu} \\ &\Rightarrow \quad D_{11}u \le [1 + C\nu_1^2 + 2\nu_1 B^T/(\beta\cdot\nu)]u_{11}(0) + (C(\delta) + \delta M)\nu_1^2. \end{aligned}$$

Therefore,

$$\begin{aligned} W &\overset{\text{def}}{\equiv} \frac{u_1 1}{u_{11}(0)} + \frac{2\nu_1\beta_1^T}{\beta\cdot\nu} \\ &\le 1 + \left[C + \frac{C(\delta) + \delta M}{u_{11}(0)}\right]\nu_1^2 \le 1 + \left[\tilde{C}(\delta) + \frac{\delta M}{u_{11}(0)}\right]\nu_1^2 \end{aligned}$$

on $\partial\Omega$.

$$\begin{aligned} &\Rightarrow \quad M \le C[1 + \tilde{C}(\delta) + \delta M + u_{11}(0)] \\ &\Rightarrow \quad M \le \tilde{C}(\delta) u_{11}(0). \end{aligned}$$

Again, we compute as before

$$u^{ij} D_{ij} W - G_{p_i} D_i w \ge -(C(\delta) + \delta M)(\Sigma u^{ii})$$

in Ω. A standard barrier argument yields

$$w_n(0) \ge C(\delta) + \delta M.$$

So $D_\beta w(0) \ge C(\delta) + \delta M$

$$\Rightarrow (D_\beta u_{11})(0) \le (c(\delta) + \delta M)(D_{11} u(0).$$

Finally, differentiate $h(Du) = 0$ twice in $\vec{e}_1$ direction, evaluate at 0.

$$D_\beta u_{11} + h_{p_k p_\ell} u_{1k} u_{1\ell} + k_1 D_{\nu\beta} u = 0$$

(k_1 normal curvature if $\partial\Omega$ in $\vec{e}_1$ direction at 0).

$$\begin{aligned} &\Rightarrow \quad -h_{p_k p_\ell} u_{1k} u_{1\ell} \le [C(\delta) + \delta M] u_{11} \text{ at } 0. \\ &\Rightarrow \quad u_{11}(0) \le C(\delta) + \delta M \\ &\Rightarrow \quad M \le C(1 + C(\delta) + \delta M), \end{aligned}$$

pick δ small enough, $M \le C$. □

7 Weak Solution Theory

Pogolerov's example: $x = (x', x_n)$, $u(x', x_n) = |x'|^\alpha f(x_n)$, using polar coordinates in x', at $x' = (x, 0, \ldots, 0)$

$$D^2 u = \begin{bmatrix} \alpha(\alpha-1)|x'|^{\alpha-2} f & 0 & \alpha|x'|^{\alpha-1} f' \\ 0 & \alpha|x|^{\alpha-2} f & 0 \\ \cdots & \cdots & 0 \\ \alpha|x'|^{\alpha-1} f' & 0 & |x'|^\alpha f'' \end{bmatrix}$$

$$\det(D^2 u) = \alpha^n(\alpha-1)|x'|^{n\alpha-2(n-1)}[(\alpha-1)ff'' - (f')^2]f^{n-2}.$$

Let $n\alpha - 2(n-1) = 0$. $\alpha = 2 - \frac{2}{n}$ $(n > 2)$.

$$f \text{ satisfies: } (\alpha-1)f^{n-1}f'' - (f')^2 f^{n-2} = \frac{1}{\alpha^n(\alpha-1)}.$$

First ode: $(1 - \frac{2}{n})f'' = \frac{1+(f')^2 f^{n-2}}{f^{n-1}}$, $f(0) = 1$, $f'(0) = 0$

$\Rightarrow$ $f'' > 0$ if f is defined $\Rightarrow$ f convex .

$\Rightarrow$ $u(x', x_n)$ well defined in a strip $-a < x_n < a$.

Restrict $u|_{B_\rho}$, $\rho < a$, let $g|_{\partial B_\rho} = u|_{\partial B_\rho}$ smooth g to C^4 g_ε, such that $g_\varepsilon - \varepsilon \le g \le g_\varepsilon + \varepsilon$ find a classical solution $u_\varepsilon \Rightarrow u_\varepsilon + \varepsilon \ge u \ge u_\varepsilon - \varepsilon$.

$$u \notin C^{1+\beta} \text{ if } \beta > 1 - \frac{2}{n} \Rightarrow u \notin C^{1,1}.$$

Caffarelli's example: $u(x', x_n) = |x'| + |x'|^\alpha f(x_n)$, $\alpha = \frac{n}{2}$, $n > 3$, $f(t) = 1 + t^2$.

Definition of a weak solution.

(i) Alexandrov sense, u convex $\det D^2 u = d\mu$.

Consider the mapping Du, where $Du(x_0)$ is defined as multivalued mapping, $Du(x_0) = \{$ all slopes of all "tangent" supporting planes to u at $x_0\}$.

$$\det(D^2 u) = d\mu,$$

if $|Du(E)| = d\mu(E)$, $\forall\ E \subset \Omega$.

(ii) Viscosity: $\forall\ v \in C^2$, $\det(D^2 v) \ge (\le) du \Rightarrow u - v$ can't have local min. (max.).

It can be shown that (i) and (ii) are equivalent.

In most of the cases we will treat $d\mu = f dx$, $0 < \lambda \le f \le \Lambda < \infty$. But, we will also consider $d\mu$, with following **Doubling Property:** Let S be a section of the group of U, i.e., $S = \{x | u(x) - \ell(x) < 0 \text{ for some } \ell \text{ linear function}\}$ then $\mu(S) \le C\mu(\frac{1}{2}\ S)$. (making origin as the center of mass of Ω).

Lemma 12 *Let u be a (weak, convex) solution of* $\det(u_{ij}) = d\mu$ *in* Ω, $u = 0$ *on* $\partial\Omega$ *and* $B_1 \subset \Omega^* \subset B_R$ *with center of mass at 0. Suppose that for some* $0 < \delta,\ \lambda < 1$, $\mu(\lambda\Omega) \geq \delta\mu(\Omega) > 0$. *Then*

$$|u(x)| \leq Cd^\alpha(x, \partial\Omega), \ |\inf_\Omega u| \sim C\mu^{\frac{1}{n}}(\Omega) \quad (C, \alpha \textit{ dependent on } \delta, \lambda, R).$$

Proof. Alexandrov estimates $|u(x)|^n \leq Cd(x, \partial\Omega)\mu(\Omega)$, on $\lambda\Omega$, $(1-\lambda)|\nabla u(x)| \leq |u(x)| \leq |\inf_\Omega u|$. $\Rightarrow \mu(\lambda\Omega) = \mathrm{vol}(Image \nabla u(\lambda\Omega)) \leq (1-\lambda)^{-n}|\inf_\Omega u|^n$.

Basic Theorem: Let u be a locally Liptshitz convex (weak) solution of $\det(u_{ij}) = d\mu$. Assume that u satisfies doubling property. Then if ℓ_{x_0} is a supporting linear function to u at x_0, we have

(i) $\{u = \ell_{x_0}\} = \{x_0\}$
orœ

(ii) $\{u = \ell_{x_0}\}$ has no external points in the Ω.

Proof. We may assume $\ell_{x_0} = 0$ (by considering $n - \ell_{x_0}$). $\{u = 0\}$ is the set we have to deal with. If $\{u = 0\}$ has an extremal point in $\Omega \Rightarrow \exists$ half space. Say $\{x_n \geq 0\}$, if $\{x_n \geq 0\} \cap \{u = 0\} = K_0$, $K_0 \subset\subset \Omega$ and $\emptyset \neq \{x_n = 0\} \cap \{u = 0\} \neq K_0$, $K_\varepsilon = \{u \leq \varepsilon x_n\} \cap \{x_n \geq 0\}$, $K_\varepsilon \to K_0$, $f(x) = x_n$. Let $x_0 \in K_0$, such that $f_0(x_0) = \max_{K_0} f(x)$, K_ε convex. T_ε affine transform. $TK_\varepsilon = K_\varepsilon^*$. $B_1 \subset K_\varepsilon^* \subset B_n$. Let $u_\varepsilon = u - \varepsilon x_n$. $u_\varepsilon^*(x) = (\det T_\varepsilon)^{2/n} u_\varepsilon(T_\varepsilon^{-1}(x))$. We focus on $x_0^\varepsilon = T_\varepsilon(x_0)$. We have

$$\frac{u_\varepsilon^*(x_0^\varepsilon)}{\inf_{K_\varepsilon^*} u_\varepsilon^*} = \frac{u_\varepsilon(x_0)}{\inf_{K_\varepsilon} u_\varepsilon} = \frac{-\varepsilon x_n^\varepsilon}{-\varepsilon \max_{K_\varepsilon} x_n} \geq \frac{1}{2}$$

($K_\varepsilon \to K_0$, if ε small $x_n^\varepsilon = \max_{K_\varepsilon} x_n$). Also, by lemma, $\inf_{K_\varepsilon} u_\varepsilon^* \leq -C\mu^{\frac{1}{n}}(K_\varepsilon^*) \leq -C\mu^{\frac{1}{n}}(B_1) \Rightarrow u_\varepsilon^*(x_0^\varepsilon) \leq -Cu^{\frac{1}{n}}(Bu)/2$. Let $\pi_1 = \{x_n = 0\}$, $\pi_2 = \{x_n = x_n^0\}$. $\pi_3^\varepsilon = \{x_n = \sup_{K_\varepsilon} y = x_n^{0,\varepsilon}\}$. The ratio

$$\frac{d(\pi_2, \pi_3^\varepsilon)}{d(\pi_1, \pi_3^\varepsilon)} = \frac{x_n^{0,\varepsilon} - x_n^0}{x_n^{0,\varepsilon}}$$

is invariant under T_ε. Since π_1^*, π_3^* are parallel, opposite supporting planes

$$\begin{aligned}
&\Rightarrow \quad 2 \leq d(\pi_1^*, \pi_3^*) \leq 2n \\
&\Rightarrow \quad d(\pi_2^*, \pi_3^*) \leq 2n \, \frac{x^{0,\varepsilon} - x_n^0}{x_n^{0,\varepsilon}} \overset{\varepsilon \to 0}{\to} 0 \\
&\Rightarrow \quad d(x_0^*, \partial K_\varepsilon^*) \leq d(\pi_2^*, \pi_3^*) \to 0
\end{aligned}$$

By lemma 12 $|u_\varepsilon^*(x_0^*)| \leq Cd^\alpha(x_0^*, \partial K_\varepsilon^*) \to 0$. Contradiction.

Remark: If u_k as in previous theorem. If $u_k \to u$ unif. in compact subsets $\Rightarrow$ then the same conclusion holds for u.

Lemma 13 *Let u be a solution of* $\det(u_{ij})$ *du in Ω. Suppose: (a) $u = 1$ on $\partial\Omega$, $B_1 C \subset \Omega \subset B_n$, (b) $\inf_\Omega u = u(x_0) = 0$, (c) du satisfies doubling property ($u(\Omega) \sim 1$ by Lemma 1). Let h_α be the cone generated by x_0 and level surface $u = \alpha$. i.e., $h_\alpha(x - x_0)$ is homogeneous of degree 1 and $h_\alpha(x-x_0) = \alpha$, $\forall\, x \in \{u = \alpha\} \Rightarrow \exists\, \delta < 1$ such that $h_{1/2}(x-x_0) < \delta h_1(x-x_0)$.*

Proof. Suppose $\exists\ u_k$ such that $\sup_{x \neq x_0} \frac{h_{1/2}(x-x_0)}{h_1(x-x_0)} \geq 1 - y_k$. From Lemma 1 $\{u_k = \frac{1}{2}\}$ and $\{u_k = 1\}$ stay unif. away from each other $\Rightarrow \exists\ 0 < C_1 < C_2$ such that $C_1|x - x_0| \leq h_{1/2}(x - x_0) \leq h_1(x - x_0) \leq C|x - x_0|$ unif. for k. We have picked a subeqn such that $\{x_0\} = \{u_k = 0\}$ remains fixed, $\Omega_k = \{u_k \leq 1\}$ conv. to $\Omega = \{u \leq 1\}$, $u_k \to u$ unif. in compact subset of Ω. $\Rightarrow$ graph of u has a segment starting at x_0. x_0 extreme pt. Contradiction.

Theorem 12 *Let u be a solution of* $\det(D^2u) = d\mu$, *$d\mu$ satisfies "doubling property", $u|_{\partial\Omega} = 0$, $B_1 \subset \Omega \subset B_n$. Let $x_0 \in \Omega$, $u(x_0) = \inf_\Omega u(x)$. Then u is $C^{1,\alpha}$ at x_0, α and $C^{1,\alpha}$ norm depending only on $d\mu$.*

Proof. We normalize the level surface $u = 2^{-k}$ by affine transform T_k. Since doubling property is invariant under affine transform. By Lemma 2, $h_{2^{-k}} \leq \gamma^k h_1$ after iteration k times, with $0 < \gamma < 1$. Since $u \in C^{0,1}$, $h_1(x) \leq C|x|$. Let $\alpha > 0$, such that $2^{-\alpha} = \gamma$, then

$$h_{2^{-k}}(x) \leq C(2^{-k})^\alpha |x| \leq 2^{-k}, \text{ if } |x| \leq \frac{1}{C}(2^{-k})^{(1-\alpha)}.$$

Since $u \leq h_{2^{-k}}(x)$ if $H_{2^{-k}}(x) \leq 2^{-k}$, we get $u(x) \leq 2^{-k}$ if $|x| \leq \frac{1}{c}\ 2^{-k(1-\alpha)}$. This allows us to estimate the slope s of supporting planes at any point $\frac{1}{4C}\ 2^{-k(1-\alpha)} \leq y \leq \frac{1}{2C}\ 2^{-k(1-\alpha)}$ by

$$\begin{aligned} |s| &\leq \left| \frac{u(\frac{1}{C}\ 2^{-k(1-\alpha)} - u(y)}{\frac{1}{C}\ 2^{-k(1-\alpha)} - y} \right| \leq \frac{2^{-k}}{\frac{1}{2C}\ 2^{-k(1-\alpha)}} \leq 2C2^{-k\alpha} \\ &\leq \tilde{C}|y|^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

□

Lemma 14 *Suppose u is a weak solution of det(D^2u) $d\mu$ in Ω, $u|_{\partial\Omega} = 0$, $B_1 \subset \Omega \subset B_n$, $d\mu$ satisfies doubling property. Then, $\forall\ \varepsilon > 0$, $\exists\ \delta_0 > 0$, such that if $0 < \delta \leq \delta_0$, $d(x_0, \delta\Omega) \geq \varepsilon$, ℓ is a supporting plane of u at x_0, then $\{\ell + \delta \geq u\} \subset\subset \Omega$.*

Proof. If not, $\exists\, \Omega_k$, u_k, $x_{0,k}$, du_k, such that $B_1 \subset \Omega_k \subset B_n$, $\det(D^2u_k) = du_k$, $\text{dist}(x_{0,k}, \partial\Omega_k) \geq \varepsilon$, but $\{\ell_{x_{0,k}} + \frac{1}{k} \geq u_k\} \cap \partial\Omega_k \neq \emptyset$. Passing to a subsequence, we get $\{\ell_{x_0} \geq u\}\partial\Omega \neq \emptyset$. Therefore $\{\ell_{x_0} = u\} = \{\ell_{x_0} \geq u\}$, $x_0 \in \{\ell_0 uu\}$. $\{\ell_0 = u\}$ is nontrivial, since $u \not\equiv 0$. Therefore, $\{\ell_0 = u_0\}$ has at least one extreme point in Ω. Contradiction.

Corollary 1 *If det(D^2u) $= du$ in Ω, $u|_{\partial\Omega} = 0$, du satisfies doubling property, then $\forall\ \Omega' \subset\subset \Omega$, $\exists\ \alpha > 0$, $u \in C^\infty(\bar{\Omega}')$, ($\|u\|_{C^{1,\alpha}(\bar{\Omega}')} \leq C$, where C depending only on du and dist($\Omega', \partial\Omega$).*

Proof. Be Lemma 14, $S_\delta = \{u \le \ell + \delta\} \subset\subset \Omega$. We can apply Theorem 1, if we can control the eccentricity of S_δ. We have

Lemma 15 *If $S_\delta = \{u \le \ell + \delta\} \subset\subset \Omega$, if $E \subset S_\delta \subset nE$ (E an ellipsoid), if $\lambda_1, \ldots, \lambda_n$ the principal axes of E then $\lambda_i \ge \delta^{n/2}$.*

Proof. Let $T : E \to B_1$. Let $T(S_\delta) = \tilde{\Omega}$, we have $B_1 \subset \tilde{\Omega} \subset B_n$. Let

$$\tilde{u} = \frac{1}{(\det(T))^{\frac{2}{n}}} (u - \ell - \delta),$$

we have $\det(D^2\tilde{u}) = d\tilde{u}$. $\tilde{u}|_{\partial\Omega} = 0$. Since $d\tilde{u}$ is the same as du (just change the variables) by Lemma 9. $\inf_{\tilde{\Omega}_0} \tilde{u} \sim -1$. $\inf_{S_0} u = -\delta \Rightarrow (\det T)^{2/n} \sim \delta \Rightarrow \mathrm{vol}(S_\delta) \sim \det T \sim \delta^{u/2}$ $E \subset S_\delta \subset hE \Rightarrow \lambda_i \ge \delta^{n/2}$. □

When x_0 is strictly convex.

Lemma 16 *Let u be a convex weak solution of $\det(D^2u) = d\mu$, Ω, $u|_{\partial\Omega} = f$, (continuous). $d\mu$ satisfies DBP. Let $\Gamma(f)$ be the convex envelope of f in Ω, i.e., $\Gamma(f) = \sup_{\ell \le f \text{ on } \partial\Omega} \ell$. If $u(x_0) < \Gamma(f)(x_0)$, then u is strictly convex, i.e., if L_{x_0} is a support plane of u at x_0, diam$\{u \le \ell_x + \rho\} \le \sigma(\rho)$ where $\sigma(\rho) \to 0$, $\rho \to 0$, σ depends only on the modulus of continuity of f, $\Gamma(f)(x_0) - u(x_0)$ and $d\mu$.*

Proof. If not, $\exists\ u_k$, $u_k \to u_0$ is compact subsets of $\Omega \Rightarrow \{u_0 = \ell_0 = 0\}$ is not a point. $\{u_0 - \ell_0 = 0\}$ has not extreme point $\Rightarrow u_0(x_0) = \Gamma(f_0)(x_0)$. ($x \notin \partial\Omega_0$, therefore f inf. cont. $u - \Gamma(f) \to 0$ as $x \to \partial\Omega$.

Corollary 2 $0 < \lambda \le \det(D^2u) \le A < \infty$. *If $u|_{\partial\Omega}$ is $C^{1+\beta}$, $\Rightarrow$ u is strictly convex.* $u \in C^{1,\alpha}$.

Proof. We only need to show $\forall\ x_0 \in \Omega$, $u(x_0) < \Gamma(f)(x_0)$. If not, after subtracting ℓ, we may assume that (a) $u \ge 0$; (b) $u = 0$ on a segment $\overline{x_1 x_2}$ with $x_1 = t_0 e_n$, $x_2 = t_0 e_n$ on $\partial\Omega$; (c) $\Gamma(\pm t_0 e_n + x^*) \le C|x^*|^{1+\alpha}$ $a < u \le \Gamma \Rightarrow u(\pm t_0 e_n + x^*) \le C|x^*|^{1+\alpha}$ for $\pm t_0 e_n + x \in \partial\Omega$. From convexity $\Rightarrow$ for x, with $\pm t_0 e_n + x \in \Omega \Rightarrow u(\pm t_0 e_n + x) \le C|x|^{1+\alpha}$.

We construct upper barrier for u that becomes zero at origin that will be a contradiction.

$$B(t, x) = \frac{\lambda}{2n}\left[a^{n-1}t^2 + \frac{1}{a}|x|^2\right] \Rightarrow \det(D^2B) < \det(D^2u).$$

Let's compare u and B on $|t| < t/2$, $|x| < \varepsilon$ for $|x| = \varepsilon$, $B \ge \frac{C_1}{a}\varepsilon^2$, $u \le c_2\varepsilon^{1+\alpha} \Rightarrow B > u$ if $a = \frac{\varepsilon^{1-\alpha}}{M}$, M large (independent of ε) on the top $u \ge C\varepsilon^{1+\alpha}$, $B \ge a^{n-1}t_0 = \frac{\varepsilon^{(n-1)(1-\alpha)}}{M^{n-1}}(\frac{t_0}{2})^2 \Rightarrow u < B_\varepsilon$. □

Corollary 3 *If $\det(D^2u) = d\mu$, $d\mu$ satisfies doubling property and $u|_{\partial\Omega} = 0$. Then u is strictly convex.*

Proof. Since if $\{u(x) = \Gamma(0)(x)\} \cap \Omega \neq \emptyset$ it is nontrivial $\Rightarrow \{u = \ell_x\}$ has an extreme point in Ω. Contradiction.

Corollary 4 *Let M be a convex set in $\mathbf{R}^n$. Assume that the Gauss curvature of ∂M (in Alexandrov sense) satisfies doubling property, then, ∂M is strictly convex, and $\partial M \in C^{1,\alpha}$.*

Proof. It's enough to show locally that any supporting plane to ∂M touches ∂M only at point (with estimate).

Claim: If ℓ is a supporting plane to ∂M, ν is the inner unit normal, $\exists\ \delta$ (depending only on du), such that $\ell + \delta\nu$ separates a portion of M that projecting in the ν direction.

Proof of the claim. If not, $\exists\ M_k \to M$, such that $\partial M \cap \ell$ is nontrivial. If $x_0 \in \partial M \cap \ell$, we can always locally project $M \Rightarrow\ x_0$ can't be an extremal point to $\partial M \cap \ell$ by main Theorem $\Rightarrow\ \partial M \cap \ell$ is unbounded. Contradiction. □

The following corollary shows Pogolerov's example in some sense is the extreme case: We now turn on to $C^{2,\alpha}$ regularity. First, we state Pogolerov's Theorem:

Theorem 13 *Let $0 < r < R < \infty$, $B_r \subset \Omega \subset B_R$, Ω convex (no smooth assumption on $\partial\Omega$ and Ω is not necessary strongly convex!). Then*

$$\begin{cases} \det(D^2u) = 1 & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}$$

has a unique convex solution $u \in C^\infty(\Omega)$. $\forall\ \varepsilon > 0$ let $\Omega_\varepsilon = \{u < -\varepsilon\}$ we have, $\forall\ k \geq 2$, $0 < \alpha < 1$ $\|u\|_{C^{k,\alpha}\Omega)\varepsilon)} \leq C$, where C depending only on $\varepsilon, n, k, \alpha, r, R$.

Proof. By Evans-Krylov Theorem, we only need to show $\|u\|_{C^{k,\alpha}\Omega)\varepsilon)} \leq C$. (Note: $u(x) \geq -Cd^{\frac{2}{n}}(x, \partial\Omega)$ by a standard barrier argument). We have $|\nabla u|_{\Omega_{\varepsilon/2}} \leq C$. After proper scaling, we may assume $\max_\Omega |u| \leq 1$ ($\max_\Omega |u| \sim 1$). Set $h = |u - \frac{\varepsilon}{2}|(D^2u)e^{\frac{1}{2}|Du|^2}$, where D unit vector field. Let $h_{D_1}(x_0) = \max_{\substack{|D|=1 \\ x \in \Omega_{\varepsilon/2}}} h \Rightarrow x_0 \in \Omega_{\varepsilon/2}$ ($h = 0$ on $\partial\Omega_{\varepsilon/2}$). We may assume $x_0 = 0$, $D_1 = \frac{\partial}{\partial x_1}$ $u_{11}(0)$ is the max of $D^2u(0) \Rightarrow u_{1j}(0) = 0\ \forall\ j \geq 2$. After rotating $x_2, \ldots, x_n$. We may assume $(u_{ij}(0))$ is diagonal at 0. Let $L = \Sigma \frac{1}{u_{ii}(0)} D_{ii} \Rightarrow L(h) \geq 0$ at 0, and $\nabla h(0) = 0$

$$\Rightarrow \frac{1}{u_{11}} L(u_{11}) - \Sigma_i \frac{1}{u_{11}^2} \left[\frac{1}{u_{ii}}(u_{11i})^2\right] + u_1 L(u_1) + \Sigma_i \frac{u_{1i}^2}{u_{ii}} + \frac{1}{u} Lu - \Sigma_i \frac{u_i^2}{u_{ii}} \leq 0$$

therefore $L(u) = n$, $L(u_1) = 0$, $L(u_{11}) = \Sigma_{k,\ell} u_{1k\ell}^2 / u_{kk} u_{\ell\ell}$.

$$\Rightarrow 0 \geq \Sigma_{k,\ell} \frac{1}{u_{11}} \frac{(u_{1k\ell})^2}{u_{kk} u_{\ell\ell}} - \Sigma_i \frac{1}{u_{11}^2} \frac{u_{11i}^2}{u_{ii}} + \Sigma_i \frac{u_{1i}^2}{u_{ii}} + \frac{n}{u} - \Sigma \frac{u_i^2}{u_{ii}}$$

$$\Rightarrow \Sigma_{\substack{k\neq 1\\ \ell}} \frac{1}{u-11} \frac{u_{1k\ell}^2}{u_{kk}u_{\ell\ell}} + u_{11} + \frac{n}{u} - \frac{u_i^2}{u_{ii}} \geq 0.$$

Therefore $h_i = 0 \Rightarrow 0 = \frac{u_{ii1}}{u_{11}} + u_1 u_{1i} + \frac{u_i}{u} = \frac{u_{11i}}{u_{11}} + \frac{u_i}{u}$, $i \geq 2$

$$\Rightarrow u_i = -u\frac{u_{11i}}{u_{11}} \Rightarrow \frac{u_i^2}{u_{ii}} = \frac{1}{u_{ii}} \left(\frac{uu_{11i}}{u_{11}}\right)^2.$$

For $i \neq 1$, $\sigma_{i\neq 1}\frac{u_i^2}{u_{ii}}$ can be absorbed by $\sigma_{\substack{k\neq 1\\ \ell=1}} \frac{1}{u_{11}} \frac{u_{1k\ell}^2}{u_{kk}u_{\ell\ell}}$

$$\Rightarrow u_{11} \leq \frac{n}{u} + \frac{u_1^2}{u_{11}} = \frac{n}{|u|} + \frac{u_1^2}{u_{11}} \leq \frac{2n}{\varepsilon} + \frac{u_1^2}{u_{11}}$$

$$\Rightarrow u_{11}(0) \leq C \Rightarrow h(0) \leq C \Rightarrow |D^2 u| \leq C.$$

□

Lemma 17 *If $B_1 \subset \Omega \subset B_n$, $1-\varepsilon \leq f \leq 1+\varepsilon$, $\det(D^2u) = f$ $u|_{\partial\Omega} = 0$. Suppose w be the solution of $\det(D^2w) = 1$ $w|_{\partial\Omega} = 0$. Then:*

$$(1+\varepsilon)^{\frac{1}{n}} w \leq u \leq (1-\varepsilon)^{\frac{1}{n}} w.$$

Proof. Comparison Lemma.

Lemma 18 *If $\det(D^2u) = f$, $0 \leq f \leq C_0 < \infty$ in $\Omega \subset B_n(0)$ $u|_{\partial\Omega} = 0$. If $\min_\Omega u = u(0) = -1$, then $\exists$ r depending only on n and C_0 such that $B_r(0) \subset \Omega$.*

Proof. If re_n is the closest point of $\partial\Omega$ to 0. Consider barrier $\tilde{h} = \tilde{C}(4n)^2 h\left(\frac{x'}{4n}, \frac{r-x_n}{4n}\right)$, where $h = -x_n^{\frac{1}{n}}\left(1 - \frac{|x'|^2}{2}\right)$. □

Lemma 19 *Suppose $1-\varepsilon \leq f \leq 1+\varepsilon$ (ε small) $\det(D^2u) = f$ in Ω, $B_1 \subset \Omega \subset B_n$. Let $u(x_0)$ be the minimum point of u. Then $\exists$ $\mu_0 > 0$ such that if $0 < \mu < \mu_0$. If $S_\mu = \{x | u(x) \leq \mu(x_0) + \mu\}$, $\partial S_\mu \subset N_\delta(\partial E)$ where $E = \{\Sigma(\frac{x_i - x_i^0}{\alpha_i})^2 = 1\}$, $0 < C_1 \leq \alpha_i \leq C_2 < \infty$, $\Pi\frac{\alpha_i}{2} = 1$, $\gamma = \mu^{1/2}$, $\delta = C(\gamma^{3/2} + \varepsilon^{1/2})$. ($N_\delta$ mean δ-neighbourhood). Furthermore, if $\partial\Omega$ is a σ-neighbourhood of B, then $|\alpha_i - 2| < C\sigma^{1/3}$. (The constants C_1, C_2, C in the lemma are universal).*

Proof. Let w be a solution as in Pogolerov's Theorem. Since $w \in C^{2,1}$ in $B_{\frac{3r}{4}}(x_0)$ (r is the constant in Lemma 18), ε small, μ small. $S_\mu(w) = \{w = \min w + \mu\} \subset B_r(x_0)$. Taylor development for w yields that the lemma is true for w (at $w(\tilde{x}) = \min w$) with $\delta = C\gamma^{3/2}$. Now, by Lemma 17 $|u - w| \leq c\varepsilon \Rightarrow$

$$S_{\mu - c\varepsilon}(w) \subset S_\mu(u) \subset S_{\mu + c\varepsilon}(w).$$

The first conclusion of the lemma follows.

If $\partial\Omega$ is in σ-neighbourhood of B_1, we have (by comparison lemma)

$$1 - c\sigma + \Sigma\frac{x_i^2}{2} \le w \le 1 + c\sigma + \Sigma\frac{x_i^2}{2} .$$

Since $w \in C^{3,1}$, interpolation yields:

$$|D_{ij}w - \frac{1}{2}\delta_{ij}|_{B_{1/2}} \le C\sigma^{1/3}.$$

□

Theorem 14 *Let Ω be a bounded convex domain in $\mathbf{R}^n$. Suppose $det(D^2u) = f$ in Ω, f positive in Ω, $f \in C^\alpha(\Omega)$, $u|_{\partial\Omega} = 0$. Then $\forall\ \Omega' \subset C\Omega$, $u \in C^{2,\alpha}(\bar{\Omega}')$, and*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega}')} \le C,$$

where C depending only on f, $dist(\Omega', \partial\Omega)$.

Proof. $\forall\ x_0 \in \Omega'$, we know that u is strictly convex at x_0 and $\exists\ \delta_0 > 0$ (δ_0 depending on f and $\mathrm{dist}(\Omega', \partial\Omega)$). $\mathrm{dist}(\{u \le \ell_{x_0} + \delta_0\}, \partial\Omega) \ge \frac{\mathrm{dist}(\Omega', \partial\Omega)}{2}$. Since f is C^α, by Lemma 16, we may pick δ_0 such that $|f(x) - f(x_0)| \le (\frac{\mu_0}{2})^{3/2}$, $\forall\ x \in S_{\delta_0} = \{u \le \ell_{x_0} + \delta_0\}$, where μ_0 is the constant in Lemma 19. We may also assume $f(x_0) = 1$. Now we normalize S_{δ_0} using John's Lemma. After the normalization, we may assume (still denote u for its normalization, since we want avoid too many notations).

$$\begin{cases} \det(D^2u) = f, & B_i \subset \Omega \subset B_n, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{77}$$

$|f - 1| \le (\frac{M_0}{2})^{3/2}$, and $f \in C^\alpha(\overline{\Omega})$ (C^α norm of f may change, depending on δ_0, but δ_0 is fixed from now on).

So, we need to show u is $C^{2,\alpha}$ at $u(x_0) = \min_\Omega u$ (with norm controlled). We may assume $x_0 = 0$ (Lemma 18).

We will show by induction that $\forall\ k \in \mathbf{Z}^+$, $\forall\ \frac{\mu_0}{2} \le \mu \equiv \mu_0$, $\exists\ E_k = \{Z^*C_kz = 1\}$, $(C_k) > 0$, such that $\det(C_k) = \frac{1}{2^n}$, $\|C_k - C_{k+1}\| \le c\mu^{k/12}$, $\partial S_{\mu^k} \subset N_{\delta_k}(M^{\frac{k}{2}}E_k)$, where $\delta_k = C\mu^{\frac{3k}{4}}$, $k = 1$ follows directly from Lemma 19. Suppose we have above for k. We consider $u^* = \frac{1}{\mu^k}U(\mu^{k/2}T_k^{-1}(x))$ where $T_k : E_k \to B_{\sqrt{2}}$. Let $\Omega^* = T_kS_{\mu^k}$. By kth hypothesis Ω^* is in $C\mu^{k\alpha/4}$ neighborhood of $B_{\sqrt{2}}$, by Lemma 19, $\partial S_\mu(u^*)$ is in δ-neighborhood of γE, where $\tilde{E} = \{\Sigma(\frac{x_i}{\tilde{\alpha}_i})^2 = 1\}$, $\Pi\frac{\tilde{\alpha}_i}{2} = 1$, $\gamma = \mu^{1/2}$, $\delta = C(\gamma^{3/2} + \varepsilon^{1/2}) \sim c\mu^{3/4}$, and $|\tilde{\alpha}_i - 2| \le C\mu^{k/12}$ scale back, let $E_{k+1} = T_k^{-1}\tilde{E} = \tilde{T}E_k$ where $\tilde{T} : \tilde{E} \to B_{\sqrt{2}}$, we get the desired property.

Claim: Let $P_k(x) = X^*C_kX$, then

(i) $|P_k(x) - u(x)|_{B_{\mu^{k/2}}} \le C\mu^{k(1+\alpha')}$, $(\alpha' = \min(\frac{\alpha}{2}, \frac{1}{4}))$

(ii) $|C_k - C_{k+1}| \le C\mu^{k/12}$

Proof of the Claim: (ii) has already been proved. As for (i) since $S_{\mu^k} \sim \mu^{\frac{k}{2}} E_k$ (E_k is unif. elliptic) . Since f is $C^\alpha \Rightarrow |f(x) - 1| \le C(\mu^{k/2})^\alpha = C\mu^{\frac{k\alpha}{2}}$, $\forall\ x \in S_{\mu^k}$. On $\partial(\mu^{\frac{k}{2}} E_k)$, $|u(x) - P_k(x)| \le \delta_k \sim \mu^{\frac{3k}{4}}$. Comparison Lemma (applied to $u_k = \frac{1}{\mu^k} u(\mu^{k/2} T_k^{-1}(x))$). We get $\|u(x) - P_k(x)| \le C \max(\mu^{\frac{k}{4}+k}) \le C\mu^{k(1+\alpha')}$. □

By the claim, $P_k \to P$, P is a quadratic polynomial. Now, $\forall\ |x| \sim \mu^{\ell/2}$, we have

$$
\begin{aligned}
|u(x) - P(x)| &\le |u(x) - P_\ell(x)| + \sum_{k=\ell}^{\infty} |P_k(x) - P_{k+1}(x)| \\
&\le C\mu^{\frac{\ell}{2}(2+\alpha')} + |x|^2 \sum_{k=\ell}^{\infty} |C_k - C_{k+1}| \\
&\le \tilde{C}\mu^{\frac{\ell}{2}(2+\alpha')} \qquad (|x| \sim u^{\ell/2}). \\
\Rightarrow \quad & u \in C^{2,\alpha'} \Rightarrow u \in c^{2,\alpha'}(\overline{\Omega}') \Rightarrow u \in C^{2,\alpha}(\overline{\Omega}')
\end{aligned}
$$

Since $\alpha' > 0$, and $f \in C^\alpha$. □

Corollary 5 *Let M be a convex set in $\mathbf{R}^n$. Assume that the Gauss curvature of ∂M is positive (bounded below and above) and is C^α for some $\alpha > 0$, then $\partial M \in C^{2,\alpha}$.*

Proof. We already have proved ∂M is strictly convex (with estimate). □

Corollary 6 *If f is a positive C^α function on S^n, $\int_{S^n} \frac{x_i}{f} = 0$, $i = 1, \dots, n+1$. Then, there is a $C^{2,\alpha}$ convex surface M in $\mathbf{R}^{n+1}$, such that $f(x) = k(\vec{n}_m^{-1}(x))$, where k is the Gauss curvature of M.*

Proof. By Minkowski-Alexandrov, M exists (weak). Weak solution theory applies. □

Theorem 15 (Jurgen, Calabi, Pogolerov) . *Let u be a global (viscosity) solution of $det(D^2u) = 1$. Normalize it so that $u \ge 0$, $u(0) = 0$. Then, $u(x) = \sum_{i=1}^n a_i x_i^2$, $a_i > 0$, $\prod_{i=1}^n (a_i/2) = 1$.*

Proof. Step 1. $u(x) > 0\ \forall\ |x| > 0$. If not, $\{u(x) = 0\}$ contains an infinite line. $\{u(x) < 1\}$ must contain an open cylinder (around the line). So it contain ellipsoids E, with $V(E) = M \to \infty$. $E = \{\Sigma \frac{x_i^2}{a_i^2} = 1\}$. Let $P = 1 + M^{1/n}[\frac{x_i^2}{2a_i} - 1]$, $P|_{\partial E} = 1$, $\det(DP) = 1$ but $P(0) < 0$. □

Step 2. $\Gamma_\lambda = \{w < \lambda\} \sim B_{\lambda^{1/2}}$. Therefore u is strictly convex $C^{2,\alpha}$ at 0. ($\{u = 1\}$ is bounded, convex). If we normalize it, we may assume $D^2u(0) = \frac{1}{2}\mathrm{Id}$. Let $T(P_\lambda) = \Gamma_\lambda^* \sim B_1$ (i.e., $B_1 \subset \Gamma_\lambda^* \subset B_n$). T can be taken diagonal. $u^*(x) = \frac{1}{(\det T)^{2/n}} u(Tx)$, $\min u^* \sim 1$, and $\mathrm{osc}(u) = \lambda \Rightarrow (\det T)^{2/n} \sim \lambda \Rightarrow Id \sim D^2u^*(0) = \frac{T^t(D^2u(0))T}{(\det T)^{2/n}} = \frac{2^{-n}T^tT}{(\det T)^{2/n}}$, if we write $T = \begin{bmatrix} u_1 & 0 \\ 0 & u_n \end{bmatrix} \Rightarrow \begin{bmatrix} u_1^2/\lambda & 0 \\ 0 & u_n^2/\lambda \end{bmatrix} \sim Id \Rightarrow u_i \sim \lambda^{1/2}$.

Step 3. $\forall\ 0 < \alpha < 1$, $[D_{ii}u]_\alpha \equiv 0$ (Even: $\|u^*\|_{C^{2,\alpha}_{(B_{1/3n})}} \leq C(0,n)\ \|D^2u^*\|_{C^\alpha(B_{(1/2n)})} \leq C(0,n)$. Therefore T is basically a dilation,

$$\|D^2u\|_{C^\alpha(B_{\frac{1}{4}\lambda^{\frac{1}{2}}})} \leq \frac{C(n)}{\lambda^{\alpha/2}}$$

$\lambda \to \infty \Rightarrow \|D^2u\|_{C^\alpha} \equiv 0 \Rightarrow D^2u \equiv Id/2$.

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