# A CONSTANT RANK THEOREM FOR QUASICONCAVE SOLUTIONS OF FULLY NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

BAOJUN BIAN, PENGFEI GUAN, XI-NAN MA, AND LU XU

Abstract. We prove a constant rank theorem for the second fundamental form of the convex level surfaces of solutions to equations $F(D^2u, Du, u, x) = 0$ under a structural condition introduced by Bianchini-Longinetti-Salani in [2] .

## 1. Introduction

A function $u$ is called *quasiconcave* if its level set $\{x|u(x) \geqslant c\}$ is convex for each constant $c$. The convexity of level-sets of solutions for partial differential equations was first studied by Gabriel [9] for harmonic function $u$ in convex ring domains of the form

$$\Omega = \Omega_0 \setminus \Omega_1, \quad \text{with boundary condition } u|_{\partial\Omega_0} = 0 \text{ and } u|_{\partial\Omega_1} = 1. \tag{1.1}$$

Lewis [15] extended the results in [9] to $p$-harmonic functions. Caffarelli-Spruck treated this problem for general inhomogeneous Laplace equation in [6] with the same boundary condition (1.1) in connection to a free boundary problem. Kawhol [12] proposed an approach of using quasi-concave envelop to study the level-set convexity of solutions to PDEs. Colesanti-Salani [7] carried out this approach for a class of elliptic equations. The technique was extended by Greco [10], Cuoghi-Salani [8] and Longinetti-Salani [16] for equation of type

$$F(D^2u, Du, u, x) = 0 \tag{1.2}$$

in convex ring under various structure conditions. General structure conditions on $F$ in equation (1.2) with Dirichlet condition (1.1) have been obtained in a recent paper [2] by Bianchini-Longinetti-Salani. All these type of results are of macroscopic nature. A different direction in the study of the convexity is the microscopic convexity principles. The *constant rank theorem* for the second fundamental forms of level sets of solutions to certain type of quasilinear equations was established by Korevaar [13], see also Xu [17] for recent generalization of results in [13].

Our interest is the microscopic counterpart of Theorem 1.1 in [2] by Bianchini-Longinetti-Salani. Let $\Omega$ be a domain in $\mathbb{R}^n$, $\mathcal{S}^n$ denotes the space of real symmetric $n \times n$ matrices

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and $\Lambda \subset \mathcal{S}^n$ is an open set, and $F = F(r,p,u,x)$ is a $C^{2,1}$ function in $\Lambda \times \mathbb{R}^n \times \mathbb{R} \times \Omega$. For each $(\theta, u) \in \mathbb{S}^{n-1} \times \mathbb{R}$ fixed, set

$$\Gamma_F = \Big\{(A,t,x) \in \Lambda \times (0,+\infty) \times \Omega : F(t^{-3}A, t^{-1}\theta, u, x) \geqslant 0\Big\}. \tag{1.3}$$

We will assume that $F$ satisfies the following conditions: there is $\gamma_0 > 0$ and $c_0 \in \mathbb{R}$,

$$F^{\alpha\beta} := \left(\frac{\partial F}{\partial r_{\alpha\beta}}(r,p,u,x)\right) > 0, \quad \forall\, (r,p,u,x) \in \Lambda \times \mathbb{R}^n \times (-\gamma_0 + c_0, \gamma_0 + c_0) \times \Omega, \tag{1.4}$$

and

$$\Gamma_F \text{ is locally convex for each } (\theta, u) \in \mathbb{S}^{n-1} \times (-\gamma_0 + c_0, \gamma_0 + c_0). \tag{1.5}$$

**Theorem 1.1.** *Suppose $u \in C^{3,1}(\Omega)$ is a solution of fully nonlinear equation (1.2) such that $(D^2u(x), Du(x), u(x)) \in \Lambda \times \mathbb{R}^n \times (-\gamma_0 + c_0, \gamma_0 + c_0)$ for each $x \in \Omega$. Suppose that, $F$ satisfies conditions (1.4) and (1.5), $Du \neq 0$ and the level sets $\{x \in \Omega | u(x) \geqslant c\}$ of $u$ is connected and locally convex for all $c \in (-\gamma_0 + c_0, \gamma_0 + c_0)$ for some $\gamma_0 > 0$. Then the second fundamental form of level surfaces $\{x \in \Omega | u(x) = c\}$ has the same constant rank for all $c \in (-\gamma_0 + c_0, \gamma_0 + c_0)$.*

*Remark* 1.2. The structural condition (1.5) is a localized version of a condition introduced by Bianchini-Longinetti-Salani (condition (1.2) in [2]). Under that condition and a weaker ellipticity condition, Bianchini-Longinetti-Salani proved (Theorem 1.1 in [2]) that any solution $u$ of equation (1.2) on convex ring $\Omega = \Omega_0 \setminus \Omega_1$ with the Dirichlet boundary condition (1.1) is quasiconcave, provided $|Du| \neq 0$. Theorem 1.1 implies the strict convexity of the level-sets in Theorem 1.1 in [2]. Also, Theorem 1.1 may yield macroscopic level-set convexity result if there is a homotopic path. As discussed in [2], condition (1.5) is satisfied by a class of elliptic operators, including Laplace operator, $p$-Laplace operators and Pucci's operator.

The proof of Theorem 1.1 uses the techniques developed in Bian-Guan [1] for the convexity of solutions of nonlinear partial differential equations. The convexity of level-sets is much more involved due to the distinguished gradient direction of the set $\{u = c\}$. This is also the main fact that the structural condition (1.5) is different from the structural condition considered in [1].

The organization of the paper is as follows. In section 2, we list some useful formulas for the second fundamental forms of level sets in terms of $u, Du, D^2u$. Main technique lemmas will be proved in section 3. The proof of Theorem 1.1 is given in section 4.

## 2. PRELIMINARIES

We recall some basic notation of differential geometry of hypersurfaces in $\mathbb{R}^n$. For a hypersurface $\Sigma$ given by a graph in a domain in $\mathbb{R}^{n-1}$,

$$x_n = v(x'), x' = (x_1, x_2, ..., x_{n-1}) \in \mathbb{R}^{n-1},$$

one may express the first fundamental form as

$$g_{ij} \;=\; \delta_{ij} + v_{x_i} v_{x_j},\ \forall\, i,j \leqslant n-1. \tag{2.1}$$

The upward normal direction $\vec{n}$ and the second fundamental form $II$ for a graph $x_n = v(x')$ are respectively given by

$$\vec{n} = \frac{1}{\sqrt{1+|\nabla_{x'} v|^2}}(-v_1, -v_2, ..., -v_{n-1}, 1), \tag{2.2}$$

$$h_{ij} = \frac{v_{x_i x_j}}{W}, \quad \forall\, i, j \leqslant n-1 \tag{2.3}$$

where $W = (1 + |\nabla_{x'} v|^2)^{\frac{1}{2}}$.

**Definition 2.1.** The graph of function $x_n = v(x')$ is *convex* with respect to *the upward normal*

$$\vec{n} = \frac{1}{\sqrt{1+|\nabla_{x'} v|^2}}(-v_1, -v_2, ..., -v_{n-1}, 1)$$

if the second second fundamental form $II := (h_{ij})$ defined in (2.3) is nonnegative definite.

The principal curvature $\kappa = (\kappa_1, ..., \kappa_{n-1})$ of the graph satisfies

$$\det(h_{ij} - \kappa g_{ij}) = 0.$$

Equivalently that $\kappa$ satisfies

$$\det(a_{ij} - \kappa \delta_{ij}) = 0,$$

where $a_{ij}$ is the symmetric Weingarten tensor defined as

$$\{a_{ij}\} = \{g^{ij}\}^{\frac{1}{2}}\{h_{ij}\}\{g^{ij}\}^{\frac{1}{2}}, \quad \forall\, i, j \leqslant n-1$$

here $\{g^{ij}\}$ is the inverse matrix to $\{g_{ij}\}$, and $\{g^{ij}\}^{\frac{1}{2}}$ is its positive square root. They are given explicitly by

$$\{g^{ij}\} = \{\delta_{ij} - \frac{v_{x_i} v_{x_j}}{W^2}\}, \quad \{g^{ij}\}^{\frac{1}{2}} = \{\delta_{ij} - \frac{v_{x_i} v_{x_j}}{W(1+W)}\}. \tag{2.4}$$

The Weingarten tensor of the hypersurface can be expressed as (e.g., see [5]),

$$a_{il} = \sum_{j,k=1}^{n-1} \frac{1}{W}\Big(v_{il} - \frac{v_i v_j v_{jl}}{W(1+W)} - \frac{v_l v_k v_{ki}}{W(1+W)} + \frac{v_i v_l v_j v_k v_{jk}}{W^2(1+W)^2}\Big), \quad \forall\, i, l \leqslant n-1. \tag{2.5}$$

Let $\Omega$ be a domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$, such that $|Du| \neq 0$ in $\Omega$. Denote the level surface of $u$ passing through the point $x_o \in \Omega$ as

$$\Sigma^{u(x_o)} := \{x \in \Omega | u(x) = u(x_o)\}.$$

We wish to express the Weingarten tensor of the level surface in terms of $u, Du, D^2u$.

At $x_0$, after proper rotation, we may assume $Du = (u_1, \cdots, u_n)$ with $u_n \neq 0$. By Implicity Function Theorem, the level set $\Sigma^{u(x_o)}$ can be locally represented as a graph

$$x_n = v(x'), x' = (x_1, x_2, ..., x_{n-1}) \in \mathbb{R}^{n-1}.$$

For $u(x_1, ..., x_{n-1}, x_n) \in C^2(\Omega)$, and the function $v(x')$ satisfies the following equation

$$u(x_1, x_2, ..., x_{n-1}, v(x_1, x_2, ..., x_{n-1})) = c. \tag{2.6}$$

Differentiate equation (2.6),

$$u_i + u_n v_i = 0, \quad v_i = -\frac{u_i}{u_n}, \quad W = (1 + |\nabla_{x'} v|^2)^{\frac{1}{2}} = \frac{|Du|}{|u_n|}.$$

It follows that the upward outer normal direction of the level sets is

$$\vec{n} = \frac{|u_n|}{|Du| u_n}(u_1, u_2, ..., u_{n-1}, u_n). \tag{2.7}$$

Differentiating (2.6) one more time,

$$u_{ij} + u_{in} v_j + u_{nj} v_i + u_{nn} v_i v_j + u_n v_{ij} = 0.$$

In turn,

$$v_{ij} = -\frac{1}{{u_n}^3}[u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn}]. \tag{2.8}$$

The second fundamental form $II$ of the level surface of function $u$ with respect to the upward normal direction (2.7) is

$$h_{ij} = -\frac{|u_n|(u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn})}{|Du| u_n^3}. \tag{2.9}$$

Note that expression (2.9) is valid locally near $x_0 \in \Omega$, independent of constant $c$ in (2.6).

**Definition 2.2.** For a function $u \in C^2(\Omega)$ with $|Du| \neq 0$ in $\Omega$, for each $y \in \Omega$, the level surface

$$\Sigma^{u(y)} = \{x \in \Omega | u(x) = u(y)\}$$

is called locally convex with respect to $Du$ near $x_0 \in \Sigma^{u(y)}$ if there is a local coordinate chart near $x_0$ (probably after some rotation) such that $u_n(x) > 0$ and the second fundamental form $h_{ij}$ defined in (2.9) is nonnegative definite near $x_0$ with respect to the upward normal direction $\vec{n}$ defined in (2.7) for $x \in \Sigma^{u(y)}$ close to $x_0$.

*Remark* 2.3. If $\{x \in \Omega | u(x) \geqslant c\}$ is locally convex, then the second fundamental form of $\Sigma^c$ is nonnegative definite with respect to $Du$ by Definition 2.2. For any $x_0 \in \Omega$, if $u_n(x_0) = |Du(x_0)|$ and the level set $\{x \in \Omega | u(x) = u(x_0)\}$ is locally convex near $x_0$, then (2.9) implies that the matrix $(u_{ij}(x_0))$ is nonpositive definite.

From (2.5) and (2.9),

$$a_{ij} = \sum_{k,l=1}^{n-1} (h_{ij} - \frac{u_i u_l h_{jl}}{W(1+W)u_n^2} - \frac{u_j u_l h_{il}}{W(1+W)u_n^2} + \frac{u_i u_j u_k u_l h_{kl}}{W^2(1+W)^2 u_n^4}). \tag{2.10}$$

With the above notation, at the point $x$ where $u_n(x) = |\nabla u(x)| > 0$, $u_i(x) = 0$, $a_{ij,k}$ is commutative. That is, they satisfy the Codazzi property $a_{ij,k} = a_{ik,j}$, $\forall i, j, k \leqslant n-1$.

## 3. Estimates

Since Theorem 1.1 is of local feature, we may assume level surface $\Sigma^c = \{x \in \Omega | u(x) = c\}$ is connected for each $c \in (c_0 - \gamma_0, c_0 + \gamma_0)$. Let $l(x)$ be the rank of the second fundamental form of $\Sigma^{u(x)}$ at $x$. Denote

$$l = \inf_{x \in \Omega} l(x). \tag{3.1}$$

Since the values of $l(x)$ are in $\mathbb{Z}$, there is $x_0 \in \Omega$ such that $l(x_0) = l$. We will concentrate in a neighborhood of some point $x_0 \in \Omega$ such that $l(x_0) = l$. We may assume $l \leqslant n-2$. We will assume $u \in C^{3,1}(\Omega)$, *$u_n > 0$ and the level surface $\Sigma^c$ is convex with respect to normal $Du$ for each $c$ in a small neighborhood of* $u(x_0)$ in the rest of the paper.

Let $\mathcal{O}$ be a small open neighborhood of $x_0$ such that for each $x \in \mathcal{O}$, there are $l$ "good" eigenvalues of $(a_{ij})$ which are bounded below by a positive constant, and the other $n-1-l$ "bad" eigenvalues of $(a_{ij})$ are very small. Denote $G$ the index set of these "good" eigenvalues and $B$ the index set of "bad" eigenvalues. For each $x \in \mathcal{O}$ fixed, we may express $(a_{ij})$ in a form of (2.10), by choosing $e_1, \cdots, e_{n-1}, e_n$ such that

$$|Du|(x) = u_n(x) > 0 \textit{ and the matrix } (u_{ij}), i,j = 1,..,n-1 \textit{ is diagonal at } x. \tag{3.2}$$

From (2.10), the matrix $(a_{ij}), i,j = 1,..,n-1$ is also diagonal at $x$, and without loss of generality we may assume $a_{11} \leqslant a_{22} \leqslant ... \leqslant a_{n-1,n-1}$. There is a positive constant $C_o > 0$ such that

$$\begin{aligned} &a_{n-1,n-1} \geqslant a_{n-2,n-2} \geqslant ... \geqslant a_{n-l,n-l} > C_o, \forall x \in \mathcal{O}, \\ &G = \{n-l, n-l+1, ..., n-1\}, \quad B = \{1, 2, ..., n-l-1\}. \end{aligned}$$

If there is no confusion, we also denote

$$B = \{a_{11}, ..., a_{n-l-1,n-l-1}\} \textit{ and } G = \{a_{n-l,n-l}, ..., a_{n-1,n-1}\}. \tag{3.3}$$

Note that for any $\delta > 0$, we may choose $\mathcal{O}$ small enough such that $a_{jj}(x) < \delta$ for all $j \in B$ and $x \in \mathcal{O}$. For two functions $f, h$ in $\mathcal{O}$, we write $h = O(f)$ if $|h(x)| \leqslant Cf(x)$ for $x \in \mathcal{O}$ with positive constant $C$ under control.

For each $c$ close to $u(x_0)$, let $a = (a_{ij})$ be the symmetric Weingarten tensor of $\Sigma^c$. Set

$$p(a) = \sigma_{l+1}(a_{ij}), \quad q(a) \; = \; \begin{cases} \frac{\sigma_{l+2}(a_{ij})}{\sigma_{l+1}(a_{ij})}, & if\ \sigma_{l+1}(a_{ij}) > 0 \\ 0, & otherwise. \end{cases} \tag{3.4}$$

Theorem 1.1 is equivalent to say $p(a) \equiv 0$ (defined in (3.4) ) in $\mathcal{O}$. For general fully nonlinear equation (1.2), as in the case for the convexity of solutions in [1], there are some technical difficulties to deal with $p(a)$ alone. A key idea introduced in [1] is to use some crucial concavity properties of function $q$ defined in (3.4). Set

$$\phi(a) = p(a) + q(a) \tag{3.5}$$

where $p$ and $q$ as in (3.4). Theorem 1.1 is equivalent to say $\phi(a) \equiv 0$.

To get around $p = 0$, for $\varepsilon > 0$ sufficiently small, consider

$$\phi_\varepsilon(a) = \phi(a_\varepsilon), \tag{3.6}$$

where $a_\varepsilon = a + \varepsilon I$. Denote $G_\varepsilon = \{a_{ii} + \varepsilon, i \in G\}$, $B_\varepsilon = \{a_{ii} + \varepsilon, i \in B\}$.

To simplify the notation, we will drop subindex $\varepsilon$ with the understanding that *all the estimates will be independent of* $\varepsilon$. In this setting, if $\mathcal{O}$ small enough, there is $C > 0$ independent of $\varepsilon$ such that

$$\phi(a(z)) \geqslant C\varepsilon, \quad \sigma_1(B(z)) \geqslant C\varepsilon, \quad \text{for all z} \in \mathcal{O}. \tag{3.7}$$

In what follows, we will use $i, j, \cdots$ as indices run from 1 to $n-1$ and $\alpha, \beta, \cdots$ as indices run from 1 to $n$. Denote

$$p_\alpha = \frac{\partial p}{\partial x_\alpha}, \quad p_{\alpha\beta} = \frac{\partial^2 p}{\partial x_\alpha \partial x_\beta}, \quad F^{\alpha\beta} = \frac{\partial F}{\partial u_{\alpha\beta}}, \quad 1 \leqslant \alpha, \beta \leqslant n,$$

and set

$$\mathcal{H}_\phi = \sum_{i,j \in B} |\nabla a_{ij}| + \phi. \tag{3.8}$$

**Lemma 3.1.** *For any fixed $x \in \mathcal{O}$, with the coordinate chart chosen as in (3.2) and (3.3),*

$$p_\alpha = \sigma_l(G) \sum_{j \in B} a_{jj,\alpha} + O(\mathcal{H}_\phi) \tag{3.9}$$

*and*

$$\begin{aligned} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} p_{\alpha\beta} \quad &\leqslant \quad -u_n^{-3} \sigma_l(G) \sum_{j \in B} [\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta jj} u_n^2 - 6 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta j} u_{nj} u_n \\ &+ \quad 6 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta} u_{nj}^2] + O(\mathcal{H}_\phi). \end{aligned} \tag{3.10}$$

**Proof of Lemma:** For each fixed point $x \in \mathcal{O}$, in a coordinate system as in (3.2),

$$-\frac{u_{jj}}{u_n} = a_{jj} = O(\mathcal{H}_\phi), \forall j \in B; \quad p_\alpha == \sigma_l(G) \sum_{j \in B} a_{jj,\alpha} + O(\mathcal{H}_\phi). \tag{3.11}$$

By (3.11),

$$p_{\alpha\beta} = \sigma_l(G)[\sum_{j \in B} a_{jj,\alpha\beta} - 2 \sum_{i \in G, j \in B} \frac{a_{ij,\alpha} a_{ij,\beta}}{a_{ii}}] + O(\mathcal{H}_\phi). \tag{3.12}$$

We now need to figure in the distinguished gradient direction $Du$ in the symmetric tensor $(a_{ij})$. Since $u_k = 0$ at $x$ for $k = 1, \cdots, n-1$, from (2.10),

$$u_n u_{ij\alpha} = -u_n^2 a_{ij,\alpha} + u_{nj} u_{i\alpha} + u_{ni} u_{j\alpha} + u_{n\alpha} u_{ij}, \quad \forall\, i, j \leqslant n-1, \tag{3.13}$$

and for each $j \in B$,

$$\begin{aligned} u_n^3 a_{jj,\alpha\beta} \quad &= \quad 2u_n u_{j\alpha} u_{nj\beta} + 2u_n u_{j\beta} u_{nj\alpha} + 2u_n u_{nj} u_{\alpha\beta j} - u_n^2 u_{\alpha\beta jj} - 2u_{n\alpha} u_{nj} u_{j\beta} \\ &\quad -2u_{n\beta} u_{nj} u_{j\alpha} - 2u_{nn} u_{j\alpha} u_{j\beta} + O(\mathcal{H}_\phi). \end{aligned}$$

Hence, for $j \in B$,

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} a_{jj,\alpha\beta} = \sum_{\alpha,\beta=1}^{n} \frac{F^{\alpha\beta}}{u_n^3}[-u_n^2 u_{\alpha\beta jj} - 4u_{n\alpha}u_{nj}u_{j\beta} + 4u_n u_{j\alpha}u_{nj\beta} \\ +2u_n u_{nj}u_{\alpha\beta j} - 2u_{nn}u_{j\alpha}u_{j\beta}] + O(\mathcal{H}_\phi). \tag{3.14}$$

Using the fact that $\sum_{\alpha=1}^{n} F^{\alpha n}u_{n\alpha} = (\sum_{\alpha,\beta=1}^{n} - \sum_{\beta=1}^{n-1}\sum_{\alpha=1}^{n})F^{\alpha\beta}u_{\alpha\beta}$, $\forall j \in B$,

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{n\alpha}u_{j\beta} = u_{nj}(\sum_{\alpha,\beta=1}^{n} - \sum_{\beta=1}^{n-1}\sum_{\alpha=1}^{n})F^{\alpha\beta}u_{\alpha\beta} + O(\mathcal{H}_\phi),$$

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{j\alpha}u_{nj\beta} = u_{nj}(\sum_{\alpha,\beta=1}^{n} - \sum_{\alpha=1}^{n-1}\sum_{\beta=1}^{n})F^{\alpha\beta}u_{\alpha\beta j} + O(\mathcal{H}_\phi),$$

and

$$\begin{aligned} & -2u_{nn}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{j\alpha}u_{j\beta} = -2u_{nn}F^{nn}u_{nj}^2 + O(\mathcal{H}_\phi) \\ = \; & -2u_{nj}^2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta} + 4u_{nj}^2\sum_{\alpha=1}^{n-1} F^{\alpha n}u_{n\alpha} + 2u_{nj}^2\sum_{\alpha,\beta=1}^{n-1} F^{\alpha\beta}u_{\alpha\beta} + O(\mathcal{H}_\phi). \end{aligned}$$

Put above to (3.14),

$$\begin{aligned} & \sum_{j\in B}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_n^3 a_{jj,\alpha\beta} \\ = \; & -u_n^2\sum_{j\in B}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta jj} + 6u_n\sum_{j\in B}u_{nj}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta j} \\ & -6\sum_{j\in B}u_{nj}^2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta} - 4u_n\sum_{j\in B}u_{nj}\sum_{\alpha=1}^{n-1}\sum_{\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta j} \\ & +8\sum_{j\in B}u_{nj}^2\sum_{\alpha=1}^{n-1} F^{\alpha n}u_{n\alpha} + 6\sum_{j\in B}u_{nj}^2\sum_{\alpha,\beta=1}^{n-1} F^{\alpha\beta}u_{\alpha\beta} + O(\mathcal{H}_\phi). \end{aligned} \tag{3.15}$$

By (3.13), for $j \in B$,

$$
\begin{aligned}
& u_n \sum_{\alpha=1}^{n-1} \sum_{\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta j} = u_n \sum_{\alpha=1}^{n} \left( \sum_{i\in B} F^{\alpha i} u_{ij\alpha} + \sum_{i \in G} F^{\alpha i} u_{ij\alpha} \right) \\
= & \sum_{\alpha=1}^{n} \sum_{i\in G} F^{\alpha i} (-u_n^2 a_{ij,\alpha} + u_{i\alpha} u_{jn} + u_{j\alpha} u_{in}) \\
& + \sum_{\alpha=1}^{n} \sum_{i\in B} F^{\alpha i} (u_{i\alpha} u_{jn} + u_{j\alpha} u_{in}) + O(\mathcal{H}_\phi) \\
= & -u_n^2 \sum_{\alpha=1}^{n} \sum_{i\in G} F^{\alpha i} a_{ij,\alpha} + u_{nj} \sum_{i\in G} F^{ii} u_{ii} + 2u_{nj} (\sum_{i=1}^{n-1} F^{ni} u_{ni}) + O(\mathcal{H}_\phi). \qquad (3.16)
\end{aligned}
$$

(3.15) and (3.16) yield that, $\forall j \in B$,

$$
\begin{aligned}
\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_n^3 a_{jj,\alpha\beta} = & -u_n^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta jj} + 6u_n u_{nj} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta j} - 6u_{nj}^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta} \\
& + 4u_n^2 u_{nj} \sum_{\alpha=1}^{n} \sum_{i\in G} F^{\alpha i} a_{ij,\alpha} + 2u_{nj}^2 \sum_{i\in G} F^{ii} u_{ii} + O(\mathcal{H}_\phi). \qquad (3.17)
\end{aligned}
$$

From (3.17), $\forall j \in B$,

$$
\begin{aligned}
& \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} [a_{jj,\alpha\beta} - 2 \sum_{i\in G} \frac{a_{ij,\alpha} a_{ij,\beta}}{a_{ii}}] \qquad (3.18) \\
= & -u_n^{-3} \Big[ \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_n^2 u_{\alpha\beta jj} - 6u_n \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{jn} u_{\alpha\beta j} \\
& + 6u_{jn}^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta} \Big] - 2 \sum_{i\in G} \frac{1}{a_{ii}} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} \\
& + 4u_n^{-1} u_{nj} \sum_{\alpha=1}^{n} \sum_{i\in G} F^{\alpha i} a_{ij,\alpha} + 2u_n^{-3} u_{nj}^2 \sum_{i\in G} F^{ii} u_{ii} + O(\mathcal{H}_\phi).
\end{aligned}
$$

**Claim:** $\forall i, j$, $\quad \dfrac{-1}{a_{ii}} \displaystyle\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} + \frac{2u_{nj}}{u_n} \sum_{\alpha=1}^{n} F^{\alpha i} a_{ij,\alpha} + \frac{u_{nj}^2 F^{ii} u_{ii}}{u_n^3} \leqslant 0.$

Assuming the **Claim**, by (3.12)

$$
\begin{aligned}
\frac{\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} p_{\alpha\beta}}{\sigma_l(G)} \leqslant & -u_n^{-3} \sum_{j\in B} \sum_{\alpha,\beta=1}^{n} [F^{\alpha\beta} u_n^2 u_{jj\alpha\beta} - 6u_n F^{\alpha\beta} u_{jn} u_{j\alpha\beta} \\
& + 6u_{jn}^2 F^{\alpha\beta} u_{\alpha\beta}] + O(\mathcal{H}_\phi). \qquad (3.19)
\end{aligned}
$$

We need to check the **Claim**. It is equivalent to the following inequality,

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} - 2u_n^{-1} u_{nj} a_{ii} \sum_{\alpha=1}^{n} F^{\alpha i} a_{ij,\alpha} + u_n^{-2} u_{nj}^2 F^{ii} a_{ii}^2 \geqslant 0. \tag{3.20}$$

We may assume $i=1$ and $j$ is fixed. Set $X_0 = u_n^{-1} a_{11} u_{jn}$ and $X_\alpha = a_{1j,\alpha}$ for $1 \leqslant \alpha \leqslant n$, (3.20) follows from the fact that $(n+1)\times(n+1)$ matrix

$$\begin{bmatrix} F^{11} & -F^{11} & -F^{12} & \cdots & -F^{1n} \\ -F^{11} & F^{11} & F^{12} & \cdots & F^{1n} \\ -F^{21} & F^{21} & F^{22} & \cdots & F^{2n} \\ \cdots & & & & \\ -F^{n1} & F^{n1} & F^{n2} & \cdots & F^{nn} \end{bmatrix}$$

is semi-positive definitive. □

**Lemma 3.2.** *$q \in C^{1,1}(\mathcal{O})$ and for any fixed $x \in \mathcal{O}$, with the coordinate chosen as in (3.2) and (3.3),*

$$q_\alpha = \frac{\partial q}{\partial x_\alpha} = \sum_{j\in B} \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} a_{jj,\alpha} + O(\mathcal{H}_\phi), \tag{3.21}$$

*and*

$$\begin{aligned} & \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} q_{\alpha\beta} \\ = \; & -u_n^{-3} \sum_{j\in B} \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \sum_{\alpha,\beta=1}^{n} \Big[ F^{\alpha\beta} u_{\alpha\beta jj} u_n^2 - 6F^{\alpha\beta} u_{\alpha\beta j} u_{jn} u_n + 6F^{\alpha\beta} u_{\alpha\beta} u_{jn}^2 \Big] \\ & - \frac{1}{\sigma_1^3(B)} \sum_{\alpha,\beta=1}^{n} \sum_{i\in B} F^{\alpha\beta} [\sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j\in B} a_{jj,\alpha}][\sigma_1(B) a_{ii,\beta} - a_{ii} \sum_{j\in B} a_{jj,\beta}] \\ & - \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^{n} \sum_{i\neq j\in B} F^{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} + O(\mathcal{H}_\phi). \end{aligned} \tag{3.22}$$

**Proof:** The fact $q \in C^{1,1}(\mathcal{O})$ follows Corollary 2.2 in [1]. Though it was stated for nonnegative matrix function $W = (u_{ij})$ with $u \in C^{3,1}$, the proof works for any nonnegative matrix function $W \in C^{1,1}$.

Identity (3.21) follows directly from Lemma 2.4 in [1]. Again, by Lemma 2.4 in [1],

$$\begin{aligned} q_{\alpha\beta} = & \sum_{j\in B} \frac{\sigma_1^2(B|j)-\sigma_2(B|j)}{\sigma_1^2(B)}\Big[a_{jj,\alpha\beta} - 2\sum_{i\in G}\frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}}\Big] \\ & -\frac{1}{\sigma_1^3(B)}\sum_{i\in B}\Big[\sigma_1(B)a_{ii,\alpha} - a_{ii}\sum_{j\in B}a_{jj,\alpha}\Big]\Big[\sigma_1(B)a_{ii,\beta} - a_{ii}\sum_{j\in B}a_{jj,\beta}\Big] \\ & -\frac{1}{\sigma_1(B)}\sum_{i\neq j\in B} a_{ij,\alpha}a_{ij,\beta} + O(\mathcal{H}_\phi). \end{aligned} \tag{3.23}$$

The lemma now follows from (3.18) and the **Claim** in the proof of Lemma 3.1. □

## 4. A STRONG MAXIMUM PRINCIPLE

We start this section with a discussion on the structure condition imposed in Theorem 1.1. For any function $F(r, Du, u, x)$, write $F^{\alpha\beta} = \frac{\partial F}{\partial r_{\alpha\beta}}, F^{u_l} = \frac{\partial F}{\partial u_l}, \cdots$ as derivatives of $F$ with respect to corresponding arguments. For $\Gamma_F$ defined in (1.3), denote

$$\mathcal{T}\Gamma_F = \{V = ((X_{\alpha\beta}), Y, (Z_i)) \in \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n :< V, \nabla_{(A,t,x)}F(t^{-3}A, t^{-1}\theta, u, x) >= 0\}.$$

**Lemma 4.1.** *If $F$ satisfies condition* (1.5)*, then*

$$\begin{aligned} Q(V,V) \ = \ & F^{\alpha\beta,rs}X_{\alpha\beta}X_{rs} + 2F^{\alpha\beta,u_l}\theta_l X_{\alpha\beta}Y + 2F^{\alpha\beta,x_k}X_{\alpha\beta}Z_k \\ & + F^{u_l,u_s}\theta_l\theta_s Y^2 + 2F^{u_l,x_k}\theta_l Y Z_k + F^{x_i,x_j}Z_iZ_j + 2tF^{u_l}\theta_l Y^2 \\ & + 6tF^{\alpha\beta}X_{\alpha\beta}Y - 6t^{-1}F^{\alpha\beta}A_{\alpha\beta}Y^2 \\ \leqslant \ & 0, \end{aligned} \tag{4.1}$$

*for every*

$$(X_{\alpha\beta}, Y, (Z_i)) = ((t^{-3}\widetilde{X}_{\alpha\beta} - 3t^{-4}A_{\alpha\beta}\widetilde{Y}), -t^{-2}\widetilde{Y}, (Z_i)), \textit{with} \quad \widetilde{V} = ((\widetilde{X}_{\alpha\beta}), \widetilde{Y}, (Z_i)) \in \mathcal{T}\Gamma_F,$$

*where $F^{\alpha\beta,rs}, F^{\alpha\beta,u_l}$, etc. are evaluated at $(t^{-3}A, t^{-1}\theta, u, x)$, and the Einstein summation convention is used.*

**Proof:** Denote $\widetilde{F}(A, t, x) = F(t^{-3}A, t^{-1}\theta, u, x)$, condition (1.5) implies that $\widetilde{F}(A, t, x)$ is locally convex with respect to the normal $\nabla\widetilde{F}$. That is, for each tangential vector $\widetilde{V} = ((\widetilde{X}_{ij}), \widetilde{Y}, (\widetilde{Z}_i))$:

$$\begin{aligned} & \widetilde{F}^{\alpha\beta,rs}\widetilde{X}_{\alpha\beta}\widetilde{X}_{rs} + 2\widetilde{F}^{\alpha\beta,t}\widetilde{X}_{\alpha\beta}\widetilde{Y} + 2\widetilde{F}^{\alpha\beta,x_k}\widetilde{X}_{\alpha\beta}\widetilde{Z}_k \\ & + \widetilde{F}^{t,t}\widetilde{Y}^2 + 2\widetilde{F}^{t,x_k}\widetilde{Y}\widetilde{Z}_k + \widetilde{F}^{x_i,x_j}\widetilde{Z}_i\widetilde{Z}_j \\ & \leqslant 0. \end{aligned} \tag{4.2}$$

At $(A,t,x)$,

$$\begin{aligned}
&\widetilde{F}^{\alpha\beta}=t^{-3}F^{\alpha\beta},\quad \widetilde{F}^{\alpha\beta,rs}=t^{-6}F^{\alpha\beta,rs},\quad \widetilde{F}^{\alpha\beta,x_k}=t^{-3}F^{\alpha\beta,x_k},\quad \widetilde{F}^{x_i,x_j}=F^{x_i,x_j},\\
&\widetilde{F}^{\alpha\beta,t}=-3t^{-4}F^{\alpha\beta}-3t^{-7}F^{\alpha\beta,rs}A_{rs}-t^{-5}F^{\alpha\beta,u_l}\theta_l,\\
&\widetilde{F}^{t}=-3t^{-4}F^{\alpha\beta}A_{\alpha\beta}-t^{-2}F^{u_l}\theta_l,\quad \widetilde{F}^{t,x_k}=-3t^{-4}F^{\alpha\beta,x_k}A_{\alpha\beta}-t^{-2}F^{u_l,x_k}\theta_l,\\
&\widetilde{F}^{t,t}=\frac{12F^{\alpha\beta}A_{\alpha\beta}}{t^5}+\frac{9F^{\alpha\beta,rs}A_{\alpha\beta}A_{rs}}{t^8}+\frac{6F^{\alpha\beta,u_l}A_{\alpha\beta}\theta_l}{t^6}+\frac{2F^{u_l}\theta_l}{t^3}+\frac{F^{u_l,u_s}\theta_s\theta_l}{t^4}.
\end{aligned}$$

(4.2) is equivalent to

$$\begin{aligned}
&t^{-6}F^{\alpha\beta,rs}\widetilde{X}_{\alpha\beta}\widetilde{X}_{rs}-2\left[3t^{-4}F^{\alpha\beta}+3t^{-7}F^{\alpha\beta,rs}A_{rs}+t^{-5}F^{\alpha\beta,u_l}\theta_l\right]\widetilde{X}_{\alpha\beta}\widetilde{Y}\\
&+2t^{-3}F^{\alpha\beta,x_k}\widetilde{X}_{\alpha\beta}\widetilde{Z}_k-2\left[3t^{-4}F^{\alpha\beta,x_k}A_{\alpha\beta}+t^{-2}F^{u_l,x_k}\theta_l\right]\widetilde{Y}\widetilde{Z}_k+F^{x_i,x_j}\widetilde{Z}_i\widetilde{Z}_j\\
&+\Big[12t^{-5}F^{\alpha\beta}A_{\alpha\beta}+9t^{-8}F^{\alpha\beta,rs}A_{\alpha\beta}A_{rs}\\
&+6t^{-6}F^{\alpha\beta,u_l}A_{\alpha\beta}\theta_l+2t^{-3}F^{u_l}\theta_l+t^{-4}F^{u_l,u_s}\theta_l\theta_s\Big]\widetilde{Y}^2
\end{aligned}$$

$$\leqslant 0. \tag{4.3}$$

The left side of (4.3) can be written as

$$\begin{aligned}
&t^{-8}F^{\alpha\beta,rs}\left(\widetilde{X}_{\alpha\beta}\widetilde{X}_{rs}t^2-6t\widetilde{X}_{\alpha\beta}A_{rs}\widetilde{Y}+9A_{\alpha\beta}A_{rs}\widetilde{Y}^2\right)\\
&-2t^{-6}F^{\alpha\beta,u_l}\theta_l\left[t\widetilde{X}_{\alpha\beta}-3A_{\alpha\beta}\widetilde{Y}\right]\widetilde{Y}+2t^{-4}F^{\alpha\beta,x_k}\left[t\widetilde{X}_{\alpha\beta}-3A_{\alpha\beta}\widetilde{Y}\right]\widetilde{Z}_k\\
&+t^{-4}F^{u_l,u_s}\theta_l\theta_s\widetilde{Y}^2-2t^{-2}F^{u_l,x_k}\theta_l\widetilde{Y}\widetilde{Z}_k\\
&+F^{x_i,x_j}\widetilde{Z}_i\widetilde{Z}_j+2t^{-3}F^{u_l}\theta_l\widetilde{Y}^2-6t^{-5}F^{\alpha\beta}\left[t\widetilde{X}_{\alpha\beta}-3A_{\alpha\beta}\widetilde{Y}\right]\widetilde{Y}-6t^{-5}F^{\alpha\beta}A_{\alpha\beta}\widetilde{Y}^2\\
=\;&F^{\alpha\beta,rs}X_{\alpha\beta}X_{rs}+2F^{\alpha\beta,u_l}\theta_lX_{\alpha\beta}Y+2F^{\alpha\beta,x_k}X_{\alpha\beta}Z_k\\
&+F^{u_l,u_s}\theta_l\theta_sY^2+2F^{u_l,x_k}\theta_lYZ_k+F^{x_i,x_j}Z_iZ_j+2tF^{u_l}\theta_lY^2\\
&+6tF^{\alpha\beta}X_{\alpha\beta}Y-6t^{-1}F^{\alpha\beta}A_{\alpha\beta}Y^2
\end{aligned}\tag{4.4}$$

where $X_{\alpha\beta}=t^{-4}\left[t\widetilde{X}_{\alpha\beta}-3A_{\alpha\beta}\widetilde{Y}\right]$, $Y=-t^{-2}\widetilde{Y}$, and $Z_i=\widetilde{Z}_i$. (4.1) follows from (4.3) and (4.4). ☐

Theorem 1.1 is a direct consequence of the following proposition and the strong maximum principle.

**Proposition 4.2.** Suppose $F,u$ satisfying assumptions in Theorem 1.1. If $l=l(x_0)$ ($l$ defined in (3.1)) for some point $x_0\in\Omega$, then there exist a neighborhood $\mathcal{O}$ of $x_0$ and a positive constant $C$ independent of $\phi$ (defined in (3.5)), such that

$$\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}\phi_{\alpha\beta}(x)\leqslant C(\phi(x)+|\nabla\phi(x)|),\ \forall\ x\in\mathcal{O}. \tag{4.5}$$

**Proof of Proposition 4.2.** Let $u \in C^{3,1}(\Omega)$ be a solution of equation (1.2) and $(u_{ij}) \in \mathcal{S}^n$. Suppose $l(x_0) = l$ for some $x_0 \in \Omega$. We work on a small open neighborhood $\mathcal{O}$ of $x_0$. We may assume $l \leqslant n-2$. Lemma 3.2 implies $\phi \in C^{1,1}(\mathcal{O})$, $\phi(x) \geqslant 0, \quad \phi(x_0) = 0$. For $\epsilon > 0$ sufficient small, let $\phi_\epsilon$ defined as in (3.5) and (3.6). For each fixed $x$, choose a local coordinate chart $e_1, \cdots, e_{n-1}, e_n$ so that (3.2) and (3.3) are satisfied. We want to establish differential inequality (4.5) for $\phi_\varepsilon$ defined in (3.6) with constant $C$ independent of $\varepsilon$. In what follows, we will omit the subindex $\varepsilon$ with the understanding that all the estimates are independent of $\varepsilon$.

By Lemma 3.1 and Lemma 3.2

$$
\begin{aligned}
&\sum_{\alpha,\beta=1}^n F^{\alpha\beta}\phi_{\alpha\beta} = \sum_{\alpha,\beta=1}^n F^{\alpha\beta}(p_{\alpha\beta}+q_{\alpha\beta}) \\
\leqslant\ & -u_n^{-3}\sum_{j\in B}\left[\sigma_l(G) + \frac{\sigma_1^2(B|j)-\sigma_2(B|j)}{\sigma_1^2(B)}\right]\Big[\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_n^2 u_{jj\alpha\beta} \\
& -6u_n\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{jn}u_{j\alpha\beta} + 6u_{jn}^2\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{\alpha\beta}\Big] \\
& -\frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta=1}^n\sum_{i\in B}F^{\alpha\beta}[\sigma_1(B)a_{ii,\alpha} - a_{ii}\sum_{j\in B}a_{jj,\alpha}][\sigma_1(B)a_{ii,\beta} - a_{ii}\sum_{j\in B}a_{jj,\beta}] \\
& -\frac{1}{\sigma_1(B)}\sum_{\alpha,\beta=1}^n\sum_{i\neq j, i,j\in B}F^{\alpha\beta}a_{ij,\alpha}a_{ij,\beta} + O(\mathcal{H}_\phi).
\end{aligned} \tag{4.6}
$$

For each $j \in B$, differentiating equation (1.2) in $e_j$ direction at $x$,

$$
\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{\alpha\beta j} + F^{u_n}u_{jn} + F^{u_j}u_{jj} + F^{x_j} = 0, \tag{4.7}
$$

$$
\begin{aligned}
&\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{\alpha\beta jj} \\
=\ & -\sum_{\alpha,\beta,r,s=1}^n F^{\alpha\beta,rs}u_{\alpha\beta j}u_{rsj} - 2\sum_{\alpha,\beta,l=1}^n F^{\alpha\beta,u_l}u_{\alpha\beta j}u_{lj} - 2\sum_{\alpha,\beta=1}^n F^{\alpha\beta,u}u_{j\alpha\beta}u_j \\
& -2\sum_{\alpha,\beta=1}^n F^{\alpha\beta,x_j}u_{\alpha\beta j} - \sum_{l,s=1}^n F^{u_l,u_s}u_{lj}u_{sj} - 2\sum_{l=1}^n F^{u_l,u}u_{lj}u_j \\
& -2\sum_{l=1}^n F^{u_l,x_j}u_{lj} - F^{u,u}u_j^2 - 2F^{u,x_j}u_j - F^{x_j,x_j} - \sum_{l=1}^n F^{u_l}u_{ljj} - F^u u_{jj}.
\end{aligned} \tag{4.8}
$$

It follows from (3.13) that, at $x$

$$\begin{aligned}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta jj} &= -\sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}u_{\alpha\beta j}u_{rsj} - 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,u_n}u_{j\alpha\beta}u_{nj} \\ &\quad -2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,x_j}u_{\alpha\beta j} - F^{u_n,u_n}u_{jn}^2 - 2F^{u_n,x_j}u_{jn} \\ &\quad -F^{x_j,x_j} - 2\frac{F^{u_n}}{u_n}u_{jn}^2 + O(\mathcal{H}_\phi). \end{aligned} \tag{4.9}$$

Since $u_{\alpha\beta jj} = u_{jj\alpha\beta}$, (4.6) and (4.9) yield

$$\begin{aligned} & F^{\alpha\beta}\phi_{\alpha\beta} \\ = & \sum_{j\in B} u_n^{-3}\left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right]\Bigg\{\Big[\sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}u_{\alpha\beta j}u_{rsj} \\ & +2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,u_n}u_{j\alpha\beta}u_{jn} + 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,x_j}u_{j\alpha\beta} \\ & +F^{u_n,u_n}u_{jn}^2 + 2F^{u_n,x_j}u_{jn} + F^{x_j,x_j} + 2\frac{F^{u_n}}{u_n}u_{jn}^2\Big]u_n^2 \\ & + 6\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{j\alpha\beta}u_{jn}u_n - 6\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta}u_{jn}^2\Bigg\} \\ & -\frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\in B} F^{\alpha\beta}[\sigma_1(B)a_{ii,\alpha} - a_{ii}\sum_{j\in B} a_{jj,\alpha}][\sigma_1(B)a_{ii,\beta} - a_{ii}\sum_{j\in B} a_{jj,\beta}] \\ & -\frac{1}{\sigma_1(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\neq j, i,j\in B} F^{\alpha\beta}a_{ij,\alpha}a_{ij,\beta} + O(\mathcal{H}_\phi). \end{aligned} \tag{4.10}$$

For each $j \in B$, set

$$\begin{aligned} S_j = & \Big[\sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}u_{j\alpha\beta}u_{rsj} + 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,u_n}u_{j\alpha\beta}u_{jn} + 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,x_j}u_{j\alpha\beta} \\ & + F^{u_n,u_n}u_{jn}^2 + 2F^{u_n,x_j}u_{jn} + F^{x_j,x_j} + 2\frac{F^{u_n}}{u_n}u_{jn}^2\Big]u_n^2 \\ & + 6\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{j\alpha\beta}u_{jn}u_n - 6\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta}u_{jn}^2 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} X_{nn} &= u_{nnj}u_n + \frac{F^{u_j}}{F^{nn}}u_{jj}u_n; \\ X_{\alpha\beta} &= u_{\alpha\beta j}u_n, \forall(\alpha,\beta)\neq(n,n); \\ Y &= u_{jn}u_n, \quad \text{and} \quad Z_i = \delta_{ij}u_n. \end{aligned} \tag{4.12}$$

In the coordinate system (3.2),

$$(D^2u(x), Du(x), u(x), x) = (D^2u, (0, ..., 0, |Du(x)|), u, x) = (t^{-3}A, t^{-1}\theta, u, x).$$

Accordingly, the components of $\widetilde{V}$ defined in Lemma 4.1 are

$$\begin{aligned} \widetilde{X}_{nn} &= \frac{u_{nnj}}{{u_n}^2} - \frac{3u_{nn}u_{jn}}{u_n^3} + \frac{F^{u_j}u_{jj}}{F^{nn}u_n^2}; \\ \widetilde{X}_{\alpha\beta} &= \frac{u_{\alpha\beta j}}{{u_n}^2} - \frac{3u_{\alpha\beta}u_{jn}}{u_n^3}, \forall(\alpha,\beta)\neq(n,n); \\ \widetilde{Y} &= -\frac{u_{jn}}{u_n}, \quad \widetilde{Z}_i = \delta_{ij}u_n. \end{aligned}$$

At $(t^{-3}A, t^{-1}\theta, u, x)$,

$$\nabla_{(A,t,x)}F = ((F^{\alpha\beta}u_n^3), -3\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta}u_n - F^{u_n}u_n^2, (F^{x_i})).$$

By (4.7),

$$\begin{aligned} & \frac{<\widetilde{V}, \nabla_{(A,t,x)}F>}{u_n} \\ = \;& u_n^2 \sum_{\alpha\beta=1}^{n} F^{\alpha\beta}(\frac{u_{\alpha\beta j}}{{u_n}^2} - \frac{3u_{\alpha\beta}u_{jn}}{u_n^3}) + F^{u_j}u_{jj} + \frac{u_{jn}}{u_n}(3\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta} + F^{u_n}u_n) + F^{x_j} \\ = \;& 0 \end{aligned}$$

That is $\widetilde{V} \in \mathcal{T}\Gamma_F$. It follows from Lemma 4.1 and the fact $u_{jj} = O(\phi)$ for $j \in B$,

$$S_j \leqslant C(\phi). \tag{4.13}$$

Condition (1.4) implies

$$(F^{\alpha\beta}) \geqslant \delta_0 I, \quad \text{for some } \delta_0 > 0, \text{ and } \forall x \in \mathcal{O}. \tag{4.14}$$

Set

$$V_{i\alpha} = \sigma_1(B)a_{ii,\alpha} - a_{ii}\sum_{j\in B} a_{jj,\alpha}.$$

Combine (4.14), (4.13) and (4.10),

$$F^{\alpha\beta}\phi_{\alpha\beta} \leqslant C(\phi + \sum_{i,j\in B}|\nabla a_{ij}|) - \delta_0[\frac{\sum_{i\neq j\in B,\alpha=1}^{n} a_{ij\alpha}^2}{\sigma_1(B)} + \frac{\sum_{i\in B,\alpha=1}^{n} V_{i\alpha}^2}{\sigma_1^3(B)}]. \tag{4.15}$$

By Lemma 3.3 in [1], for each $M \geqslant 1$, for any $M \geqslant |\gamma_i| \geqslant \frac{1}{M}$, there is a constant $C$ depending only on $n$ and $M$ such that, $\forall \alpha$,

$$\sum_{i,j\in B} |a_{ij\alpha}| \leqslant C(1+\frac{1}{\delta_0^2})(\sigma_1(B) + |\sum_{i\in B}\gamma_i a_{ii\alpha}|) + \frac{\delta_0}{2}[\frac{\sum_{i\neq j\in B}|a_{ij\alpha}|^2}{\sigma_1(B)} + \frac{\sum_{i\in B} V_{i\alpha}^2}{\sigma_1^3(B)}]. \tag{4.16}$$

Set

$$\gamma_j = \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}, \forall j \in B,$$

the Newton-MacLaurine inequality implies

$$\sigma_l(G) + 1 \geqslant \gamma_j \geqslant \sigma_l(G), \quad \forall j \in B.$$

We conclude from Lemma 3.1, Lemma 3.2 and (4.16) that $\sum_{i,j\in B}|\nabla a_{ij}|$ is controlled by the rest terms on the right hand side in (4.15) together with $\phi + |\nabla\phi|$. The proof is complete. $\square$

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## References

1. B. Bian and P. Guan, *A microscopic convexity principle for nonlinear partial differential equations*, Inventiones Math., **177**, (2009), 307-335.
2. C. Bianchini, M. Longinetti and P. Salani, *Quasiconcave solutions to elliptic problems in convex rings*, Indiana Univ. Math. J. **58** (2009), 1565-1590.
3. L. Caffarelli and A. Friedman, *Convexity of solutions of some semilinear elliptic equations*, Duke Math. J. **52** (1985), 431-455.
4. L. Caffarelli, P. Guan and X. Ma, *A constant rank theorem for solutions of fully nonlinear elliptic equations*, Comm. Pure Appl. Math. **60**(12) (2007), 1769-1791.
5. L. Caffarelli, L. Nirenberg and J. Spruck, *Nonlinear second order elliptic equations IV: Starshaped compact Weingarten hypersurfaces*, Current topics in partial differential equations, Y.Ohya, K.Kasahara and N.Shimakura (eds), Kinokunize, Tokyo, 1985, 1-26.
6. L. Caffarelli and J. Spruck, *Convexity properties of solutions to some classical variational problems*, Comm. Part. Diff. Eq. **7** (1982), 1337-1379.
7. A. Colesanti and P. Salani, *Quasi-concave envelope of a function and convexity of level sets of solutions to elliptic equations*, Math. Nachr. **258** (2003), 3-15.
8. P. Cuoghi and P. Salani, *Convexity of level sets for solutions to nonlinear elliptic problems in convex rings*, Electronic J. Diff. Eq. **2006**(124) (2006), 1-12. URL: http://ejde.math.txstate.edu
9. R. Gabriel, *A result concerning convex level surfaces of 3-dimensional harmonic functions*, J. London Math. Soc. **32** (1957), 286-294.
10. A. Greco, *Quasi-concavity for semilinear elliptic equations with non-monotone and anisotrpic nonlinearities*, Boundary Value Problems (2006),article ID80347, 1-15.
11. P. Guan and X. Ma, *The Christoffel-Minkowski Problem I: Convexity of Solutions of a Hessian Equations*, Inventiones Math., 151, (2003), 553-577.
12. B. Kawhol, *Rearrangements and convexity of level sets in PDE*, Springer Lecture Notes in Math. **1150** (1985).
13. N. Korevaar, *Convexity of level sets for solutions to elliptic ring problems*, Comm. Part. Diff. Eq. **15**(4) (1990), 541-556.

14. N. Korevaar and J. Lewis, *Convex solutions of certain elliptic equations have constant rank hessians*, Arch. Rational Mech. Anal. **91** (1987), 19–32.
15. J. Lewis, *Capacitary functions in convex rings*, Arch. Rat. Mech. Anal. **66** (1977), 201-224.
16. M. Longinetti and P. Salani, *On the Hessian matrix and Minkowski addition of quasiconvex functions*. J. Math. Pures Appl. **88**, (2007), 276–292.
17. L. Xu, *A Microscopic convexity theorem of level sets for solutions to elliptic equations*, submitted (2008).

Department of Mathematics, Tongji University, Shanghai, 200092, China.
*E-mail address*: `bianbj@mail.tongji.edu.cn`

Department of Mathematics, McGill University, Montreal, Quebec. H3A 2K6, Canada.
*E-mail address*: `guan@math.mcgill.ca`

Department of Mathematics, University of Science and Technology of China, Hefei, 230026, Anhui Province, China.
*E-mail address*: `xinan@ustc.edu.cn`

Wuhan Institute of Physics and Mathematics, The Chinese Academy of Science, Wuhan,430071, HuBei Province, China
*E-mail address*: `xulu@wipm.ac.cn`