A PROOF OF THE ALEXANDEROV’S UNIQUENESS THEOREM FOR
CONVEX SURFACES IN $\mathbb{R}^3$

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ABSTRACT. We give a new proof of a classical uniqueness theorem of Alexandrov [4]
using the weak uniqueness continuation theorem of Bers-Nirenberg [8]. We prove a
version of this theorem with the minimal regularity assumption: the spherical hessians
of the corresponding convex bodies as Radon measures are nonsingular.

We give a new proof of the following uniqueness theorem of Alexandrov, using the Weak
Unique Continuation Theorem of Bers-Nirenberg [8].

Theorem 1 (Theorem 9 in [4]). Suppose $M_1$ and $M_2$ are two closed strictly convex $C^2$
surfaces in $\mathbb{R}^3$, suppose $f(y_1, y_2) \in C^1$ is a function such that $\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2} > 0$. Denote $\kappa_1 \geq \kappa_2$
the principal curvatures of surfaces, and denote $\nu_{M_1}$ and $\nu_{M_2}$ the Gauss maps of $M_1$ and $M_2$ respectively. If
\begin{equation}
(1) \quad f(\kappa_1(\nu_{M_1}^{-1}(x), \kappa_1(\nu_{M_1}^{-1}(x))) = f(\kappa(\nu_{M_2}^{-1}(x), \kappa(\nu_{M_2}^{-1}(x))), \forall x \in S^2,
\end{equation}
then $M_1$ is equal to $M_2$ up to a translation.

This classical result was first proved for analytical surfaces by Alexandrov in [3], for
$C^4$ surfaces by Pogorelov in [20], and Hartman-Wintner [14] reduced regularity to $C^3$, see
also [21]. Pogorelov [22, 23] published certain uniqueness results for $C^2$ surfaces, these
general results would imply Theorem 1 in $C^2$ case. It was pointed out in [19] that the
proof of Pogorelov is erroneous, it contains an uncorrectable mistake (see page 301-302
in [19]). There is a counter-example of Martinez-Maure [15] (see also [19]) to the main
claims in [22, 23]. The results by Han-Nadirashvili-Yuan [13] imply two proofs of Theorem
1, one for $C^2$ surfaces and another for $C^{2,\alpha}$ surfaces. The problem is often reduced to a
uniqueness problem for linear elliptic equations in appropriate settings, either on $S^2$ or in
$\mathbb{R}^3$, we refer [4, 21]. Here we will concentrate on the corresponding equation on $S^2$, as in
[11]. The advantage in this setting is that it is globally defined.

If $M$ is a strictly convex surface with support function $u$, then the principal curvatures
at $\nu^{-1}(x)$ are the reciprocals of the principal radii $\lambda_1, \lambda_2$ of $M$, which are the eigenvalues
of spherical Hessian $W_u(x) = (u_{ij}(x) + u(x)\delta_{ij})$ where $u_{ij}$ are the covariant derivatives
with respect to any given local orthonormal frame on $S^2$. Set
\begin{equation}
(2) \quad \tilde{F}(W_u) = f(\frac{1}{\lambda_1(W_u)}, \frac{1}{\lambda_2(W_u)}) = f(\kappa_1, \kappa_2).
\end{equation}
In view of Lemma 1 in [5], if \( f \) satisfies the conditions in Theorem 1, then \( \tilde{F}^{ij} = \frac{\partial^2 F}{\partial w_{ij}} \in L^\infty \) is uniformly elliptic. In the case \( n = 2 \), it can be read off from the explicit formulas
\[
\lambda_1 = \frac{\sigma_1(W_u) - \sqrt{\sigma_1(W_u)^2 - 4\sigma_2(W_u)}}{2}, \quad \lambda_2 = \frac{\sigma_1(W_u) + \sqrt{\sigma_1(W_u)^2 - 4\sigma_2(W_u)}}{2}.
\]
As noted by Alexanderov [5], \( \tilde{F}^{ij} \) in general is not continuous if \( f(y_1, y_2) \) is not symmetric (even \( f \) is analytic).

We want to address when Theorem 1 remains true for convex bodies in \( \mathbb{R}^3 \) with weakened regularity assumption. In the Brunn-Minkowski theory, the uniqueness of Alexandrov-Fenchel-Jessen [1, 2, 10] states that, if two bounded convex bodies in \( \mathbb{R}^{n+1} \) have the same \( k \)th area measures on \( S^n \), then these two bodies are the same up to a rigidity motion in \( \mathbb{R}^{n+1} \). Though for a general convex body, the principal curvatures of its boundary may not be defined. But one can always define the support function \( u \), which is a function on \( \mathbb{S}^2 \). By the convexity, then \( W_u = (u_{ij} + u\delta_{ij}) \) is a Radon measure on \( \mathbb{S}^2 \). Also, by Alexandrov’s theorem for the differentiability of convex functions, \( W_u \) is defined for almost every point \( x \in \mathbb{S}^2 \). Denote \( \mathcal{N} \) to be the space of all positive definite \( 2 \times 2 \) matrices, and let \( G \) be a function defined on \( \mathcal{N} \). For a support function \( u \) of a bounded convex body \( \Omega_u, G(W_u) \) is defined for a.e. \( x \in \mathbb{S}^2 \). For fixed support functions \( u^l \) of \( \Omega_{u^l}, l = 1, 2 \), there is \( \Omega \subset \mathbb{S}^2 \) with \( |\mathbb{S}^2 \setminus \Omega| = 0 \) such that \( W_{u^1}, W_{u^2} \) are pointwise finite in \( \Omega \). Set \( P_{u^1, u^2} = \{ W \in \mathcal{N} \mid \exists x \in \Omega, W = W_{u^1}(x), \text{or } W = W_{u^2}(x) \} \), let \( P_{u^1, u^2} \) be the convex hull of \( P_{u^1, u^2} \) in \( \mathcal{N} \).

We establish the following slightly more general version of Theorem 1.

**Theorem 2.** Suppose \( \Omega_1 \) and \( \Omega_2 \) are two bounded convex bodies in \( \mathbb{R}^3 \). Let \( u^l, l = 1, 2 \) be the corresponding supporting functions respectively. Suppose the spherical Hessians \( W_{u^l} = (u^l_{ij} + \delta_{ij}u^l) \) (in the weak sense) are two non-singular Radon measures. Let \( G : \mathcal{N} \to \mathbb{R} \) be a \( C^{0,1} \) function such that
\[
\Lambda I \geq (G^{ij})(W) := (\frac{\partial G}{\partial W_{ij}})(W) \geq \lambda I > 0, \quad \forall W \in \mathcal{P}_{\gamma, \gamma},
\]
for some positive constants \( \Lambda, \lambda \). If
\[
G(W_{u^1}) = G(W_{u^2}),
\]
at almost every parallel normal \( x \in \mathbb{S}^2 \), then \( \Omega_1 \) is equal to \( \Omega_2 \) up to a translation.

Suppose \( u^1, u^2 \) are the support functions of two convex bodies \( \Omega_1, \Omega_2 \) respectively, and suppose \( \Omega^1, l = 1, 2 \) are defined and they satisfy equation (3) at some point \( x \in \mathbb{S}^2 \). Then, for \( u = u^1 - u^2 \), \( W_u(x) \) satisfies equation
\[
F^{ij}(x)(W_u(x)) = 0,
\]
with \( F^{ij}(x) = \int_0^1 \frac{\partial \tilde{F}}{\partial W_{ij}}(tW_{u^1}(x) + (1 - t)W_{u^2}(x))dt \). By the convexity, \( W_{u^l}, l = 1, 2 \) exist almost everywhere on \( \mathbb{S}^2 \). If they satisfy equation (3) almost everywhere, equation (4) is verified almost everywhere. Note that \( u \) may not be a solution (even in a weak sense) of partial differential equation (4). The classical elliptic theory (e.g., [16, 18, 8]) requires \( u \in W^{2,2} \) in order to make sense of \( u \) as a weak solution of (4). A main step in the proof
of Theorem 2 is to show that with the assumptions in the theorem, $u = u^1 - u^2$ is indeed in $W^{2,2}(S^2)$. The proof will appear in the last part of the paper.

Let’s now focus on $W^{2,2}$ solutions of differential equation (4), with general uniformly elliptic condition on tensor $F^{ij}$ on $S^2$:

$$\lambda |\xi|^2 \leq F^{ij}(x)\xi_i\xi_j \leq \Lambda |\xi|^2, \forall x \in S^2, \xi \in \mathbb{R}^2,$$

for some positive numbers $\lambda, \Lambda$. The aforementioned proofs of Theorem 1 ([20, 14, 21, 13]) all reduce to the statement that any solution of (5) is a linear function, under various regularity assumptions on $F^{ij}$ and $u$. Equation (4) is also related to minimal cone equation in $\mathbb{R}^3$ ([13]). The following result was proved in [13].

**Theorem 3** (Theorem 1.1 in [13]). Suppose $F^{ij}(x) \in L^\infty(S^2)$ satisfies (5), suppose $u \in W^{2,2}(S^2)$ is a solution of (4). Then, $u(x) = a_1x_1 + a_2x_2 + a_3x_3$ for some $a_i \in \mathbb{R}$.

There the original statement in [13] is for 1-homogeneous $W^{2,2}_{loc}(\mathbb{R}^3)$ solution $v$ of equation

$$\sum_{i,j=1}^3 a^{ij}(X)v_{ij}(X) = 0.$$

These two statements are equivalent. To see this, set $u(x) = \frac{v(X)}{|X|}$ with $x = \frac{X}{|X|}$. By the homogeneity assumption, the radial direction corresponds to null eigenvalue of $\nabla^2 v$, the other two eigenvalues coincide the eigenvaules of the spherical Hessian of $W = (u_{ij} + u\delta_{ij})$. $v(X) \in W^{2,2}_{loc}(\mathbb{R}^3)$ is a solution to (6) if and only if $u \in W^{2,2}(S^2)$ is a solution to (4) with $F^{ij}(x) = (e_1, A e_2)$, where $A = (a^{ij}(\frac{X}{|X|}))$ and $(e_1, e_2)$ is any orthonormal frame on $S^2$.

The proof in [13] uses gradient maps and support planes introduced by Alexandrov, as in [3, 20, 21]. We give a different proof of Theorem 3 using the maximum principle for smooth solutions and the unique continuation theorem of Bers-Nirenberg [8], working purely on solutions of equation (4) on $S^2$.

Note that $F$ in Theorem 2 (and Theorem 1) is not assumed to be symmetric. The weak assumption $F^{ij} \in L^\infty$ is needed to deal with this case. This assumption also fits well with the weak unique continuation theorem of Bers-Nirenberg. This beautiful result of Bers-Nirenberg will be used in a crucial way in our proof. If $u \in W^{2,2}(S^2)$, $u \in C^\alpha(S^2)$ for some $0 < \alpha < 1$ by the Sobolev embedding theorem. Equation (4) and $C^{1,\alpha}$ estimates for 2-d linear elliptic PDE (e.g., [16, 18, 8]) imply that $u$ is in $C^{1,\alpha}(S^2)$ for some $\alpha > 0$ depending only on $\|u\|_{C^0}$ and the ellipticity constants of $F^{ij}$. This fact will be assumed in the rest of the paper.

The following lemma is elementary.

**Lemma 4.** Suppose $F^{ij} \in L^\infty(S^2)$ satisfies (5), suppose at some point $x \in S^2$, $W_u(x) = (u_{ij}(x) + u(x)\delta_{ij})$ satisfies (4). Then,

$$|W_u|^2(x) \leq -\frac{2\Lambda}{\lambda} \det W_u(x).$$
Proof. At \( x \), by equation (4),

\[
\det W_u = -\frac{1}{F^{22}} \left( F^{11}W_{11}^2 + 2F^{12}W_{11}W_{12} + F^{22}W_{12}^2 \right) \leq -\frac{\lambda}{\Lambda} (W_{11}^2 + W_{12}^2),
\]
and similarly, \( \det W_u \leq -\frac{\lambda}{\Lambda} (W_{22}^2 + W_{21}^2) \). Thus,

\[
(W_{11}^2 + W_{12}^2 + W_{21}^2 + W_{22}^2) \leq -\frac{2\Lambda}{\lambda} \det W_u.
\]

\( \Box \)

For each \( u \in C^1(S^2) \), set \( X_u = \sum_i u_i e_i + ue_{n+1} \). For any unit vector \( E \) in \( \mathbb{R}^3 \), define

\[
\phi_E(x) = \langle E, X_u(x) \rangle, \quad \rho_u(x) = |X_u(x)|^2,
\]
where \( \langle , \rangle \) is the standard inner product in \( \mathbb{R}^3 \). The function \( \rho \) was introduced by Weyl in his study of Weyl’s problem [25]. It played important role in Nirenberg’s solution of the Weyl’s problem in [17]. Our basic observation is that there is a maximum principle for \( \rho_u \) and \( \phi_E \).

**Lemma 5.** Suppose \( U \subset S^2 \) is an open set, \( F^{ij} \in C^1(U) \) is a tensor in \( U \) and \( u \in C^3(U) \) satisfies equation (4), then there are two constants \( C_1, C_2 \) depending only on the \( C^1 \)-norm of \( F^{ij} \) such that

\[
F^{ij}(\rho_u)_{ij} \geq -C_1|\nabla \rho_u|, \quad F^{ij}(\phi_E)_{ij} \geq -C_2|\nabla \phi_E| \quad \text{in } U.
\]

**Proof.** Pick any orthonormal frame \( e_1, e_2 \), we have

\[
(X_u)_i = W_{ij}e_j, \quad (X_u)_{ij} = W_{ijk}e_k - W_{ij}x.
\]
By Codazzi property of \( W \) and (4),

\[
\frac{1}{2} F^{ij}(\rho_u)_{ij} = \langle X_u, F^{ij}W_{ijk}e_k \rangle + F^{ij}W_{ik}W_{kj} = -u_k F^{ij}W_{ij} + F^{ij}W_{ik}W_{kj}.
\]

On the other hand, \( \nabla \rho_u = 2W \cdot (\nabla u) \). At the non-degenerate points (i.e., \( \det W \neq 0 \)), \( \nabla u = \frac{1}{2} W^{-1} \cdot \nabla \rho_u \), where \( W^{-1} \) denotes the inverse matrix of \( W \). Now,

\[
2u_k F^{ij}_{,k}W_{ij} = W^{kl}(\rho_u)_{,l}F^{ij}_{,k}W_{ij} = (\rho_u)_{,l}F^{ij}_{,k} A^{kl}_{,j} W_{ij} \det W,
\]
where \( A^{kl} \) denote the co-factor of \( W_{kl} \).

The first inequality in (10) follows (8) and (12).

The proof for \( \phi_E \) follows the same argument and the following facts:

\[
F^{ij}(\phi_E)_{ij} = -\langle E, e_k \rangle F^{ij}_{,k}W_{ij}, \quad \nabla \phi_E = W \cdot \langle E, e_k \rangle.
\]

\( \Box \)

Lemma 5 yields immediately Theorem 1 in \( C^3 \) case, which corresponds to the Hartman-Wintner theorem ([14]).

**Corollary 6.** Suppose \( f \in C^2 \) and symmetric, \( M_1, M_2 \) are two closed convex \( C^3 \) surfaces satisfy conditions in Theorem 1, then the surfaces are the same up to a translation.
Proof. Since $f \in C^2$ is symmetric, $F^{ij}$ in (4) is in $C^1(S^2)$ and $u \in C^3(S^2)$. By Lemma 5 and the strong maximum principle, $X_u$ is a constant vector. □

To precede further, set
\[
\mathcal{M} = \{ p \in S^2 : \rho_u(p) = \max_{q \in S^2} \rho_u(q) \},
\]
for each unit vector $E \in \mathbb{R}^3$,
\[
\mathcal{M}_E = \{ p \in S^2 : \phi_E(p) = \max_{q \in S^2} \phi_E(q) \}.
\]

Lemma 7. $\mathcal{M}$ and $\mathcal{M}_E$ have no isolated points.

Proof. We prove the lemma for $\mathcal{M}$, the proof for $\mathcal{M}_E$ is the same. If point $p_0 \in \mathcal{M}$ is an isolated point, we may assume $p_0 = (0,0,1)$. Pick $\bar{U}$ a small open geodesic ball centered at $p_0$ such that $\bar{U}$ is properly contained in $S^2_+$, and pick a sequence of smooth 2–tensor $(F^{ij}_\epsilon) > 0$ which is convergent to $(F^{ij})$ in $L^\infty$-norm in $\bar{U}$. Consider
\[
(13) \begin{cases}
F^{ij}_\epsilon(u^i_j + u^\epsilon \delta_{ij}) = 0 \text{ in } \bar{U} \\
\quad\quad u^\epsilon = u \text{ on } \partial\bar{U}.
\end{cases}
\]
Since $x_3 > 0$ in $S^2_+$, one may write $u^\epsilon = x_3 v^\epsilon$ in $\bar{U}$. As $(x_3)_{ij} = -x_3 \delta_{ij}$, it easy to check $v^\epsilon$ satisfies
\[
F^{ij}_\epsilon v^\epsilon_{ij} + b_k v^\epsilon_k = 0, \quad \text{in } \bar{U}.
\]
Therefore, (13) is uniquely solvable.

Since $p_0 \in \mathcal{M}$ is an isolated point, there are open geodesic balls $U' \subset \bar{U}$ centered at $p_0$ and a small $\delta > 0$ such that
\[
(14) \quad \rho_u(p_0) - \rho_u(p) \geq \delta \quad \forall p \in \partial\bar{U}'.
\]

By the $C^{1,\alpha}$ estimates for linear elliptic equation in dimension two and the uniqueness of the Dirichlet problem ([16, 8, 18]), $\exists \epsilon_k$ such that
\[
\|u - u^{\epsilon_k}\|_{C^{1,\alpha}(U')} \to 0, \quad \left\| \rho_u - \rho_{u^{\epsilon_k}} \right\|_{C^0(\bar{U}')} \to 0.
\]
Together with (14), if $\epsilon_k$ small enough, there is a local maximal point of $\rho_{u^{\epsilon_k}}$ in $\bar{U}' \subset \bar{U}$. Since $u^{\epsilon_k}, F^{ij}_\epsilon \in C^\infty(\bar{U}')$ satisfy (13), it follows from Lemma 5 and the strong maximum principle that $\rho_{u^{\epsilon_k}}$ must be constant in $\bar{U}'$, $\forall \epsilon_k$ in small enough. This implies $\rho$ is constant in $\bar{U}'$. Contradiction. □

We now prove Theorem 3.

Proof of Theorem 3. For any $p_0 \in \mathcal{M}$, if $\rho_u(p_0) = 0$, then $u \equiv 0$. We may assume $\rho_u(p_0) > 0$. Set $E := \frac{x_u(p_0)}{|x_u(p_0)|}$. Choose another two unit constant vectors $\beta_1, \beta_2$ with $<\beta_i, \beta_j> = \delta_{ij}, \beta_i \perp E$ for $i, j = 1, 2$. Under this orthogonal coordinates in $\mathbb{R}^3$,
\[
(15) \quad x_u(p) = a(p)E + b_1(p)\beta_1 + b_2(p)\beta_2, \quad \forall p \in \mathcal{M}_E.
\]
On the other hand, \( \phi_E(p) = \rho_u^{1/2}(p_0), \forall p \in \mathcal{M}_E \). Thus,

\[
(16) \quad a(p) = \rho_u^{1/2}(p_0), \quad b_1(p) = b_2(p) = 0, \forall p \in \mathcal{M}_E.
\]

Consider the function \( \tilde{u}(x) = u(x) - \rho_u^{1/2}(p_0)E \cdot x \). (15) and (16) yield, \( \forall p \in \mathcal{M}_E \),

\[
(17) \quad \nabla_{e_i} \tilde{u}(p) = \nabla_{e_i} u(p) - \rho_u^{1/2}(p_0)(E, e_i) = \langle X_u(p), e_i \rangle - \rho_u^{1/2}(p_0)(E, e_i) = 0.
\]

Moreover, \( \tilde{u}(x) \) also satisfies equation (4). As pointed out in [8], if \( \tilde{u} \) satisfies an elliptic equation, \( \nabla \tilde{u} \) satisfies an elliptic system of equations. Lemma 7, (17) and the Unique Continuation Theorem of Bers-Nirenberg (P. 13 in [7]) imply \( \nabla \tilde{u} \equiv 0 \). Thus, \( \tilde{u}(x) \equiv \tilde{u}(p_0) = 0 \) and \( u(x) \) is a linear function on \( \mathbb{S}^2 \).

\[\square\]

**Theorem 1** is a direct consequence of Theorem 3. We now prove Theorem 2.

**Proof of Theorem 2.** The main step is to show \( u = u^1 - u^2 \in W^{2,2}(\mathbb{S}^2) \), using the assumption that \( W_{u^l}, l = 1, 2 \) are non-singular Radon measures. It follows from the convexity, the spherical hessians \( W_{u^l}, l = 1, 2 \) and \( W_u \) are defined almost everywhere on \( \mathbb{S}^2 \) (Alexandrov’s Theorem). So, we can define \( G(W_{u^l}) \), \( l = 1, 2 \) almost everywhere in \( \mathbb{S}^2 \). As \( W_{u^l}, l = 1, 2 \) are nonsingular Radon measures, \( W_{u^l} \in L^1(\mathbb{S}^2) \) (see [9]), we also have \( W_u \in L^1(\mathbb{S}^2) \). Since \( u^1, u^2 \) satisfy \( G(W_{u^1}) = G(W_{u^2}) \) for almost every parallel normal \( x \in \mathbb{S}^2 \), there is \( \Omega \subset \mathbb{S}^2 \) with \( |\mathbb{S}^2 \setminus \Omega| = 0 \), such that \( W_u \) satisfies following equation pointwise in \( \Omega \),

\[
G^{ij}(x)(u_{ij}(x) + u(x)\delta_{ij}) = 0, \quad x \in \Omega,
\]

where \( G^{ij} = \int_0^1 \frac{\partial^2 G}{\partial u_{ij}}(tW_u^1 + (1 - t)W_u^2)dt \). By Lemma 4, we can obtain that

\[
|W_{u^l}|^2 = W_{11}^2 + W_{12}^2 + W_{21}^2 + W_{22}^2 \leq -\frac{2\lambda}{\lambda} \det W_u, \quad x \in \Omega.
\]

On the other hand,

\[
\det W_u \leq \det W_{\tilde{u}},
\]

where \( \tilde{u} = u^1 + u^2 \). Thus, to prove \( u \in W^{2,2}(\mathbb{S}^2) \), it suffices to get an upper bound for \( \int_{\mathbb{S}^2} \det W_{\tilde{u}} \).

Recall that \( W_{u^l} \in L^1(\mathbb{S}^2) \), so \( u^l \in W^{2,1}(\mathbb{S}^2) \), \( l = 1, 2 \) and the same for \( \tilde{u} \). This allows us to choose two sequences of smooth convex bodies \( \Omega^l_\epsilon \) with supporting functions \( u^l_\epsilon \) such that \( ||\tilde{u}_\epsilon - \tilde{u}||_{W^{2,1}(\mathbb{S}^2)} \to 0 \) as \( \epsilon \to 0 \). By Fatou’s Lemma and continuity of the area measures,

\[
\int_{\mathbb{S}^2} \det W_{\tilde{u}} = \int_{\Omega} \det W_{\tilde{u}} \leq \liminf_{\epsilon \to 0} \int_{\mathbb{S}^2} \det W_{\tilde{u}_\epsilon} \leq V(\Omega^1) + V(\Omega^2) + 2V(\Omega^1, \Omega^2),
\]

where \( V(\Omega^1), V(\Omega^2) \) denote the volume of the convex bodies \( \Omega^1 \) and \( \Omega^2 \) respectively and \( V(\Omega^1, \Omega^2) \) is the mixed volume.

It follows that \( W_u \in L^2(\mathbb{S}^2) \) and thus, \( u \in W^{2,2}(\mathbb{S}^2) \). This implies that \( u \) is a \( W^{2,2} \) weak solution of the differential equation

\[
G^{ij}(x)(u_{ij}(x) + u(x)\delta_{ij}) = 0, \quad \forall x \in \mathbb{S}^2.
\]

Finally, the theorem follows directly from Theorem 3. \[\square\]
Remark 8. Alexanderov proved in [3] that, if \( u \) is a homogeneous degree 1 analytic function in \( \mathbb{R}^3 \) with \( \nabla^2 u \) definite nowhere, then \( u \) is a linear function. As a consequence, Alexanderov proved in [6] that if a analytic closed convex surface in \( \mathbb{R}^3 \) satisfying the condition \( (\kappa_1 - c)(\kappa_2 - c) \leq 0 \) at every point for some constant \( c \), then it is a sphere. Martinez-Maure gave a \( C^2 \) counter-example in [15] to this statement, see also [19]. The counter-examples in [15, 19] indicate that Theorem 3 is not true if \( F^{ij} \) is merely assumed to be degenerate elliptic. It is an interesting question that under what degeneracy condition on \( F^{ij} \) so that Theorem 3 is still true, even in smooth case. This question is related to similar questions in this nature posted by Alexandrov [4] and Pogorelov [21].

We shall wrap up this paper by mention a stability type result related with uniqueness. Indeed, by using the uniqueness property proved in Theorem 3, we can prove the following stability theorem via compactness argument.

Proposition 9. Suppose \( F^{ij}(x) \in L^\infty(\mathbb{S}^2) \) satisfies (5), and \( u(x) \in W^{2,2}(\mathbb{S}^2) \) is a solution of the following equation

\[
F^{ij}(x)(W_{u})_{ij} = f(x), \quad \forall x \in \mathbb{S}^2.
\]

Assume that \( f(x) \in L^\infty(\mathbb{S}^2) \) and there exists a point \( x_0 \in \mathbb{S}^2 \) such that \( \rho_u(x_0) = 0 \) (see (9) for the definition of \( \rho_u \)). Then,

\[
||u||_{L^\infty(\mathbb{S}^2)} \leq C_3 ||f||_{L^\infty(\mathbb{S}^2)}
\]

holds for some positive constant \( C_3 \) only depends on the ellipticity constants \( \lambda, \Lambda \).

Proof. As mentioned above, we will prove this proposition by a compactness argument. Suppose the desired estimate (19) does not hold, then there exists a sequence of functions \( \{f_n(x)\}_{n=1}^\infty \) on \( \mathbb{S}^2 \) with \( ||f||_{L^\infty(\mathbb{S}^2)} \leq C_4 \) and a sequence of points \( \{x_n\}_{n=1}^\infty \subset \mathbb{S}^2 \) such that \( \rho_{u_n}(x_n) = 0 \) and \( K_n := \frac{||u||_{L^\infty(\mathbb{S}^2)}}{||f||_{L^\infty(\mathbb{S}^2)}} \rightarrow +\infty \), where \( u_n(x) \) is the solution of equation (18) with right hand side replaced by \( f_n(x) \).

Let \( v_n(x) = \frac{u_n(x)}{K_n||f||_{L^\infty(\mathbb{S}^2)}} \), then \( ||v_n||_{L^\infty(\mathbb{S}^2)} = 1 \) and \( v_n(x) \) satisfies

\[
F^{ij}(x)(W_{v_n})_{ij} = \tilde{f}_n := \frac{f_n(x)}{K_n||f||_{L^\infty(\mathbb{S}^2)}}.
\]

By the interior \( C^{1,\alpha} \) estimates for linear elliptic equation in dimension two ([16, 8, 18]), we have

\[
||v_n||_{C^{1,\alpha}(\mathbb{S}^2)} \leq C_5 \left( ||v_n||_{L^\infty(\mathbb{S}^2)} + ||\tilde{f}_n||_{L^\infty(\mathbb{S}^2)} \right) \leq 2C_5
\]

for some positive constant \( C_5 = C_5(\lambda, \Lambda) \). In particular, this gives that \( ||\nabla v_n||_{L^\infty(\mathbb{S}^2)} \leq C_0 \). Now, apply the a priori \( W^{2,2} \) estimate for linear elliptic equation in dimension two ([16, 8, 18, 12]), we see that \( ||v_n||_{W^{2,2}(\mathbb{S}^2)} \leq C_7 \) for some constant \( C_7 = C_7(\lambda, \Lambda, C_0) \). It follows from this uniform estimate that, up to a subsequence, \( \{v_n(x)\}_{n=1}^\infty \) converges to some function \( v(x) \in W^{2,2}(\mathbb{S}^2) \) and \( v(x) \) satisfies

\[
F^{ij}(x)(W_v)_{ij} = 0, \quad \text{a.e.} \ x \in \mathbb{S}^2.
\]
Then, the previous uniqueness result Theorem 3 tells that \( v(x) \) must be a linear function, i.e., there exists a constant vector \( \vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3 \) such that \( v(x) = a_1x_1 + a_2x_2 + a_3x_3 \).

On the other hand, recall that, by the assumption at the beginning, there exists \( x_n \in S^2 \) such that \( \rho_v(x_n) = 0 \). Then, up to a subsequence, \( x_n \to x_\infty \in S^2 \) and \( \rho_v(x_\infty) = 0 \). This together with the linear property of \( v(x) \) imply that \( v(x) \equiv 0 \). However, this contradicts with the fact that \( ||v||_{L^\infty(S^2)} = 1 \) as \( ||v_n||_{L^\infty(S^2)} = 1 \).

\[ \square \]

As a direct corollary, we have the following stability property for convex surfaces.

**Theorem 10.** Suppose \( M_1, M_2 \) and \( f \) satisfy the same assumptions as in Theorem 3. Define \( \mu_1(x) := f(\kappa_1(\nu_{M_1}^{-1}(x)), \kappa_2(\nu_{M_1}^{-1}(x))) \) and \( \mu_2(x) := f(\kappa_1(\nu_{M_2}^{-1}(x)), \kappa_2(\nu_{M_2}^{-1}(x))) \) for all \( x \in S^2 \). If \( ||\mu_1 - \mu_2||_{L^\infty(S^2)} < \epsilon \), then, modulo a linear translation, \( M_1 \) is very close to \( M_2 \). More precisely, suppose \( u_1, u_2 \) are the supporting functions of \( M_1 \) and \( M_2 \) after modulo the linear translation, then there exists a constant \( C \) such that

\[
(21) \quad ||u_1 - u_2||_{L^\infty(S^2)} \leq C||\mu_1 - \mu_2||_{L^\infty(S^2)}.
\]

Finally, it is worth to remark that there are many stability type results for convex surfaces proved in the literature (see [24]). However, almost all the proofs need to use the assumption that \( f(\kappa_1, \kappa_2, \ldots, \kappa_n) \) satisfies divergence property. Here, we do not make such kind assumption in this dimension two case. There is one drawback in the above stability result: one could not get the sharp constant via the compactness argument. It would be an interesting question to derive a sharp estimate for (21).

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**References**

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