RANKIN-SELBERG L-FUNCTIONS AND CYCLES ON UNITARY SHIMURA VARIETIES

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Contents

1. Introduction

The purpose of these notes is to give a summary of the results and proofs of [BHY], under some simplifying hypotheses, and with some of the more technical calculations omitted. The main result is in the same spirit as the Gross-Zagier theorem, and the methods and ideas draw heavily from work of Borcherds [Bo1, Bo2], Gross [Gr], Gross-Zagier [GZ], Kudla [Ku2], Kudla-Rapoport [KR1] and [KR2], and earlier work of the authors of [BHY] and their collaborators [Br], [BF], [BY], [Ho1],

[Ho2], and [KRY]. Our goal is an equality, up to simple factors,

$$[\widehat{\Theta}(F) : Y_\Lambda] \doteq \frac{d}{ds} L(F, \vartheta_\Lambda, s)\big|_{s=0},$$

where the quantities involved are as follows. On the right hand side

- F is a cuspidal modular form of weight $n \geq 2$,
- Λ is a positive definite Hermitian lattice of rank $n-1$,
- ϑ_Λ is the weight $n-1$ theta series associated to Λ,
- $L(F, \vartheta_\Lambda, s)$ is the Rankin-Selberg convolution of F and ϑ_Λ, normalized so that the center of its functional equation is at $s = 0$.

On the left hand side

- M is the integral model of a Shimura variety of type $\mathrm{GU}(n-1,1)$,
- Y_Λ is a cycle of CM points on M, depending on Λ,
- $\widehat{\Theta}(F)$ is a metrized line bundle on M, depending on F,
- $[\widehat{\Theta}(F) : Y_\Lambda]$ is the degree of the pullback of $\widehat{\Theta}(F)$ to Y_Λ.

For the precise statement see Theorem 3.5.1.

Throughout these notes, the following data is fixed. Let $\boldsymbol{k} = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of odd discriminant $-D$, and assume that $\boldsymbol{k}$ has class number one. In particular, this implies that D is prime. Fix an embedding $i_\mathbb{C} : \boldsymbol{k} \to \mathbb{C}$, and let $\delta = \sqrt{-D}$ be the choice of square root such that $i_\mathbb{C}(\delta)$ lies in the upper half plane. Let χ be the quadratic Dirichlet character associated with the extension $\boldsymbol{k}/\mathbb{Q}$.

A *Hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice* is a finitely generated projective $\mathcal{O}_{\boldsymbol{k}}$-module L equipped with a nondegenerate pairing $\langle \cdot, \cdot \rangle : L \times L \to \mathcal{O}_{\boldsymbol{k}}$ that is $\mathcal{O}_{\boldsymbol{k}}$-linear in the first variable, and satisfies $\langle x, y \rangle = \overline{\langle y, x \rangle}$. We say that L is *self-dual* if the map $y \mapsto \langle \cdot, y \rangle$ induces an isomorphism $L \to \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(L, \mathcal{O}_{\boldsymbol{k}})$. A Hermitian lattice L has *signature* (p,q) if $L \otimes_{\mathcal{O}_{\boldsymbol{k}}} \boldsymbol{k}$ can be written as the direct sum of a p-dimensional subspace on which the Hermitian form is positive definite, and a q-dimensional subspace on which the Hermitian form is negative definite.

2. Unitary Shimura varieties and their special cycles

We define the Shimura varieties that we'll be working on, their integral models, and special cycles in dimension one and codimension one.

2.1. **Unitary Shimura varieties.** Let $M_{(p,q)}(\mathbb{C})$ be the moduli space of triples (A, κ, ψ) over $\mathbb{C}$ in which

- A is an abelian variety of dimension n,
- $\kappa : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A)$ is an action of $\mathcal{O}_{\boldsymbol{k}}$,
- $\psi : A \to A^\vee$ is a principal polarization.

We require that ψ be *$\mathcal{O}_{\boldsymbol{k}}$-linear*, in the sense that

$$\kappa(\overline{\alpha})^\vee \circ \psi = \psi \circ \kappa(\alpha)$$

for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$, and that κ satisfies the *naive signature (p,q)-condition*: each $\alpha \in \mathcal{O}_{\boldsymbol{k}}$ acts on the $\mathbb{C}$-vector space $\mathrm{Lie}(A)$ with characteristic polynomial

$$\det(T - \alpha) = (T - i_\mathbb{C}(\alpha))^p \cdot (T - i_\mathbb{C}(\overline{\alpha}))^q. \tag{2.1.1}$$

Using the isomorphism

$$\mathcal{O}_{\boldsymbol{k}} \otimes_\mathbb{Z} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$$

defined by $\alpha \otimes x \mapsto (i_{\mathbb{C}}(\alpha)x, i_{\mathbb{C}}(\overline{\alpha})x)$, we obtain two idempotents $e = (1,0)$ and $e' = (0,1)$ in $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathbb{C}$. These idempotents induce a splitting of the Lie algebra

$$\mathrm{Lie}(A) = e\mathrm{Lie}(A) \oplus e'\mathrm{Lie}(A),$$

and the naive signature condition is equivalent to the summands on the right having dimensions p and q, respectively.

Remark 2.1.1. Strictly speaking, the above moduli problem is not representable. Really $M_{(p,q)}(\mathbb{C})$ is the complex orbifold associated to a Deligne-Mumford stack over $\mathbb{C}$.

Proposition 2.1.2. *There is an isomorphism of complex orbifolds*

$$\bigsqcup_{L} \mathrm{Aut}(L)\backslash\mathcal{D}_L \cong M_{(p,q)}(\mathbb{C}), \tag{2.1.2}$$

where the disjoint union is over the finitely many isomorphism classes of self-dual Hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattices L of signature (p,q), $\mathcal{D}_L$ is the space of q-dimensional negative definite $\mathbb{C}$-subspaces of $L_{\mathbb{C}} = L \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathbb{C}$, and $\mathrm{Aut}(L)$ *is the automorphism group of the Hermitian lattice L.*

Proof. Although $L_{\mathbb{C}}$ is already a complex vector space, it carries a family of different complex structures parametrized by $\mathcal{D}_L$. Indeed, for each $z \in \mathcal{D}_L$ we decompose

$$L_{\mathbb{C}} = z^{\perp} \oplus z,$$

and let I_z be the unique $\mathbb{R}$-linear endomorphism of $L_{\mathbb{C}}$ satisfying

$$I_z(v) = \begin{cases} i \cdot v & \text{if } v \in z^{\perp} \\ -i \cdot v & \text{if } v \in z. \end{cases}$$

Obviously $I_z^2 = -1$. Define a complex torus

$$A_z(\mathbb{C}) = L_{\mathbb{C}}/L$$

where the complex structure is determined by I_z. There is an obvious action κ_z of $\mathcal{O}_{\boldsymbol{k}}$ on A_z, and the $\mathbb{Z}$-valued symplectic form

$$\psi_z(x,y) = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}}\langle \delta^{-1}x, y\rangle$$

on L defines an $\mathcal{O}_{\boldsymbol{k}}$-linear principal polarization of A_z. The desired uniformization (2.1.2) is $z \mapsto (A_z, \kappa_z, \psi_z)$. ∎

Remark 2.1.3. Using the uniformization (2.1.2) it's not hard to see that $M_{(p,q)}(\mathbb{C})$ has dimension pq. In particular, the orbifolds $M_{(m,0)}(\mathbb{C})$ and $M_{(0,m)}(\mathbb{C})$ are zero dimensional.

Remark 2.1.4. The stack $M_{(1,0)}(\mathbb{C})$ is just the moduli space of elliptic curves A_0 over $\mathbb{C}$ with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$, where the action $\mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A_0)$ is normalized so that $\mathcal{O}_{\boldsymbol{k}}$ acts on $\mathrm{Lie}(A_0)$ through the fixed embedding $i_{\boldsymbol{k}} : \boldsymbol{k} \to \mathbb{C}$. Because we assume that $\boldsymbol{k}$ has class number one, there is a unique such A_0. Thus $M_{(1,0)}(\mathbb{C})$ consists of a single point with automorphism group $\mathcal{O}_{\boldsymbol{k}}^{\times}$.

The following result, a consequence of the class number one hypothesis on $\boldsymbol{k}$ and the strong approximation theorem, implies that in most cases there is exactly one L contributing to (2.1.2).

Proposition 2.1.5. *Suppose $pq > 0$. Up to isomorphism, there is a unique self-dual Hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature (p,q).*

Proof. First we recall the classification of Hermitian spaces over local fields. Let v be a place of $\mathbb{Q}$. For each Hermitian space V of dimension $p+q$ over $\boldsymbol{k}_v$ we define the *invariant*

$$\mathrm{inv}(V) = \chi_v(\det(V)),$$

where $\chi : \mathbb{A}^\times \to \{\pm 1\}$ is the idele class character associated with $\boldsymbol{k}$. There are three cases to consider.

(1) If $v < \infty$ is split in $\boldsymbol{k}$, then there is a unique Hermitian space over $\boldsymbol{k}_v$ of dimension $p+q$, and its invariant is 1. This Hermitian space adimits a self-dual lattice, and all such lattices lie in the same $U(V)$-orbit.
(2) If $v < \infty$ is inert in $\boldsymbol{k}$, then there are two Hermitian spaces over $\boldsymbol{k}_v$ of dimension $p+q$. One has invariant 1, the other has invariant -1. The one of invariant 1 admits a self-dual lattice, and all such lattices lie in the same $U(V)$-orbit. The space of invariant -1 does not admit any self-dual lattices.
(3) If $v < \infty$ is ramified in $\boldsymbol{k}$, then there are again two Hermitian spaces over $\boldsymbol{k}_v$ of dimension $p+q$. One has invariant 1, the other has invariant -1. Each one admits a self-dual lattice, and all such lattices lie in the same $U(V)$-orbit[1].
(4) If $v = \infty$ then there is a unique quadratic space of signature (p,q), and its invariant is $(-1)^q$.

Over the global field $\boldsymbol{k}$, a quadratic space V is uniquely determined by its collection of localizations, which must satisfy the reciprocity law

$$\prod_v \mathrm{inv}(V_v) = 1.$$

Conversely, given any collection of Hermitian spaces $\{V_v\}$ of the same dimension $p+q$ such that $\mathrm{inv}(V_v) = 1$ for all but finitely many v, and $\prod_v \mathrm{inv}(V_v) = 1$, there is a unique Hermitian space over $\boldsymbol{k}$ giving rise to this local collection.

Now we prove the claim. There is a unique Hermitian space V over $\boldsymbol{k}$ of signature (p,q) such that $\mathrm{inv}(V_v) = 1$ for all finite $v \neq D$, and such that $\mathrm{inv}(V_D) = (-1)^q$. From what was said above, this is the unique Hermitian space of signature (p,q) that admits a self-dual lattice. Let $L \subset V$ be a self-dual lattice. Abbreviate $G = U(L)$ and $G_0 = SU(L)$. As any two self-dual lattices in V lie in the same $G(\mathbb{A}_f)$-orbit, the set of all such lattices is in bijection with the set $G(\mathbb{Q})\backslash G(\widehat{\mathbb{Q}})/G(\widehat{\mathbb{Z}})$. To see that this set has only one element, one first uses the class number one hypothesis to show that the natural map

$$G_0(\mathbb{Q})\backslash G_0(\widehat{\mathbb{Q}})/G_0(\widehat{\mathbb{Z}}) \to G(\mathbb{Q})\backslash G(\widehat{\mathbb{Q}})/G(\widehat{\mathbb{Z}})$$

is a bijection. As G_0 is simply connected and non-compact at ∞, the strong approximation theorem implies

$$G_0(\mathbb{Q})\backslash G_0(\widehat{\mathbb{Q}})/G_0(\widehat{\mathbb{Z}}) = \{1\}.$$

□

The moduli problem defining $M_{(p,q)}(\mathbb{C})$ makes sense if we replace $\mathbb{C}$ by an arbitrary $\mathcal{O}_{\boldsymbol{k}}$-scheme, S. Just replace the map $i_{\mathbb{C}}$ in (2.1.1) by the structure map $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$. The result is a stack over $\mathcal{O}_{\boldsymbol{k}}$ which is smooth of relative dimension

[1] This is due to Jocobowitz [Jac]. We are using here the fact that D is odd; when 2 ramifies in $\boldsymbol{k}$ there are more orbits of self-dual lattices

pq after inverting D, but if one does not invert D, the stack is poorly behaved. For example, Pappas [Pa] has shown that it is neither flat nor equidimensional, and has proposed a less naive form of the signature condition which, at least conjecturally, corrects these defects. In the signature $(m, 0)$ and $(m, 1)$ cases much more can be said, and we now discuss these special cases.

2.2. **Integral models for signature (m,0).** Let $M_{(m,0)}$ be the moduli stack parametrizing triples (B, κ, ψ) in which

- $B \to S$ is an abelian scheme of dimension m over an $\mathcal{O}_{\boldsymbol{k}}$-scheme S,
- $\kappa : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(B)$ is an action of $\mathcal{O}_{\boldsymbol{k}}$,
- $\psi : B \to B^\vee$ is an $\mathcal{O}_{\boldsymbol{k}}$-linear principal polarization.

We require further that the triple satisfies the *signature* $(m, 0)$*-condition* that each $\alpha \in \mathcal{O}_{\boldsymbol{k}}$ acts on $\mathrm{Lie}(B)$ through the structure map $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$.

Points of $M_{(m,0)}(\mathbb{C})$ are easy to describe using Proposition 2.1.2. Starting from a self-dual Hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice Λ of signature $(m, 0)$, define a complex torus

$$B_\Lambda(\mathbb{C}) = \Lambda_{\mathbb{C}}/\Lambda$$

with the obvious $\mathcal{O}_{\boldsymbol{k}}$-action. The symplectic form

$$\psi(x, y) = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}}\langle \delta^{-1}x, y\rangle$$

on Λ defines a principal polarization on B_Λ, and the construction $\Lambda \mapsto B_\Lambda$ establishes a bijection from the set of all such Λ to $M_{(m,0)}(\mathbb{C})$. In particular, it is clear from this description that every B_Λ is isomorphic, as an abelian variety with $\mathcal{O}_{\boldsymbol{k}}$-action, to the product of m copies of $\mathbb{C}/\mathcal{O}_{\boldsymbol{k}}$ (just pick an isomorphism of $\mathcal{O}_{\boldsymbol{k}}$-modules $\Lambda \cong \mathcal{O}_{\boldsymbol{k}}^m$), although this isomorphism need not identify the polarization on B_Λ with the product polarization. In particular, all points of $M_{(m,0)}(\mathbb{C})$ are defined over a number field, and have potentially good reduction.

Theorem 2.2.1 (Canonical lifting theorem)**.** *If $\tilde{R} \to R$ is a surjection of $\mathcal{O}_{\boldsymbol{k}}$-algebras with nilpotent kernel, then*

1. *each $B \in M_{(m,0)}(R)$ admits a unique deformation to $\tilde{B} \in M_{(m,0)}(\tilde{R})$,*
2. *for any $B_1 \in M_{(m_1,0)}(R)$ and $B_2 \in M_{(m_2,0)}(R)$ the reduction map*

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\tilde{B}_1, \tilde{B}_2) \to \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(B_1, B_2)$$

is an isomorphism.

Corollary 2.2.2. *The stack $M_{(m,0)}$ is smooth and proper over $\mathcal{O}_{\boldsymbol{k}}$ of relative dimension* 0*. If R is a complete local ring with residue field $\mathbb{F}$, the reduction map*

$$M_{(m,0)}(R) \to M_{(m,0)}(\mathbb{F})$$

is a bijection.

In order to prove the canonical lifting theorem, we need some basic notions from deformation theory. A thorough reference is [Lan], to which we refer for more details. First, for any abelian scheme $\pi : A \to S$ we define the *relative deRham cohomology*

$$H^1_{dR}(A) = \mathbb{R}^1\pi_*\Omega^\bullet_{A/S}$$

as the hypercohomology of the de Rham complex

$$0 \to \mathcal{O}_A \to \Omega^1_{A/S} \to \Omega^2_{A/S} \to \cdots .$$

The *relative deRham homology* is its dual

$$H_1^{dR}(A) = \underline{\mathrm{Hom}}_{\mathcal{O}_S}(H^1_{dR}(A), \mathcal{O}_S).$$

It is a locally free $\mathcal{O}_S$-module of rank $2\dim(A)$, and there is a canonical short exact sequence

$$0 \to \mathrm{Fil}^1(A) \to H_1^{dR}(A) \to \mathrm{Lie}(A) \to 0$$

in which $\mathrm{Fil}^1(A)$ is the *Hodge filtration*. Each of $\mathrm{Fil}^1(A)$ and $\mathrm{Lie}(A)$ is locally free of rank $\dim(A)$. Any polarization of A induces an $\mathcal{O}_S$-valued alternating form on $H_1^{dR}(A)$, under which $\mathrm{Fil}^1(A)$ is totally isotropic.

Now suppose $S = \mathrm{Spec}(R)$, and we start with an abelian scheme over R. Let $\tilde{R} \to R$ be a surjection whose kernel I satisfies $I^2 = 0$. We want to understand all ways of deforming A to an abelian scheme $\tilde{A}$ over $\tilde{R}$. Here are the fundamental facts.

(1) There is at least one deformation of A to $\tilde{R}$.

(2) For any two deformations $\tilde{A}_1$ and $\tilde{A}_2$, there is a canonical isomorphism of $\tilde{R}$-modules $H_1^{dR}(\tilde{A}_1) \cong H_1^{dR}(\tilde{A}_2)$. Define an $\tilde{R}$-module

$$\tilde{H}_1^{dR}(A) = H_1^{dR}(\tilde{A})$$

for one (any) deformation $\tilde{A}$.

(3) For any deformation $\tilde{A}$ the Hodge filtration $\mathrm{Fil}^1(\tilde{A})$ determines a local direct summand

$$\mathrm{Fil}^1(\tilde{A}) \subset \tilde{H}_1^{dR}(A).$$

This establishes a bijection from the set of all deformations of A to the set of "lifts of the Hodge filtration": local direct summands of $\tilde{H}_1^{dR}(A)$ whose image under $\tilde{H}_1^{dR}(A) \to H_1^{dR}(A)$ is $\mathrm{Fil}^1(A)$.

(4) Suppose B is another abelian scheme over R, and $f : A \to B$ is a morphism. The induced map $f : H_1^{dR}(A) \to H_1^{dR}(B)$ has a distinguished lift

$$\tilde{f} : \tilde{H}_1^{dR}(A) \to \tilde{H}_1^{dR}(B). \tag{2.2.1}$$

(5) Suppose $\tilde{A}$ and $\tilde{B}$ are deformations of A and B to $\tilde{R}$, and that we are given a morphism $f : A \to B$. How do we determine whether or not f lifts to a morphism $\tilde{A} \to \tilde{B}$? The deformations $\tilde{A}$ and $\tilde{B}$ correspond to lifts of the Hodge filtrations

$$\mathrm{Fil}^1(\tilde{A}) \subset \tilde{H}_1^{dR}(A) \quad \text{and} \quad \mathrm{Fil}^1(\tilde{B}) \subset \tilde{H}_1^{dR}(B),$$

and f lifts (necessarily uniquely) to a morphism $\tilde{A} \to \tilde{B}$ if and only if the induced (2.2.1) respects these lifts:

$$\tilde{f}(\mathrm{Fil}^1(\tilde{A})) \subset \mathrm{Fil}^1(\tilde{B}).$$

Equivalently, f lifts if and only if the *obstruction to deforming* f, defined as the composition

$$\begin{tikzcd} \mathrm{Fil}^1(\tilde{A}) \arrow[r] \arrow[rrr, bend right, "\mathrm{obst}^*(f)"'] & H_1^{dR}(\tilde{A}) \arrow[r, "\tilde{f}"] & H_1^{dR}(\tilde{B}) \arrow[r] & \mathrm{Lie}(\tilde{B}), \end{tikzcd}$$

vanishes.

(6) Any polarization ψ of A induces an alternating pairing $\tilde{\psi}$ on $\tilde{H}_1^{dR}(A)$ lifting the pairing on $H_1^{dR}(A)$. If $\tilde{A}$ is a deformation of A, then the polarization ψ lifts (necessarily uniquely) to a polarization of $\tilde{A}$ if and only if $\mathrm{Fil}^1(\tilde{A})$ is totally isotropic under $\tilde{\psi}$.

Let $S = \mathrm{Spec}(R)$ and $\tilde{S} = \mathrm{Spec}(\tilde{R})$, so that we have a closed immersion $S \hookrightarrow \tilde{S}$, and suppose we are given abelian schemes A and B over S, together with deformations $\tilde{A}$ and $\tilde{B}$ to $\tilde{S}$. By what we have said, any morphism $f : A \to B$ induces a map

$$f : H_1^{dR}(A) \to H_1^{dR}(B),$$

which has a canonical extension to

$$\tilde{f} : H_1^{dR}(\tilde{A}) \to H_1^{dR}(\tilde{B}).$$

Expressed differently, $f : A \to B$ induces a section over S of the locally free $\mathcal{O}_{\tilde{S}}$-module

$$\boldsymbol{H} = \underline{\mathrm{Hom}}_{\mathcal{O}_{\tilde{S}}}(H_1^{dR}(\tilde{A}), H_1^{dR}(\tilde{B})),$$

and this section has a canonical extension $\tilde{f}$ to all of $\tilde{S}$. The sheaf $\boldsymbol{H}$ is endowed with its *Gauss-Manin connection*

$$\nabla : \boldsymbol{H} \to \boldsymbol{H} \otimes \Omega^1_{\tilde{S}/\mathbb{Z}},$$

and, intuitively speaking, one should think of $\tilde{f}$ as being obtained from f by parallel transport[2]. The section $\tilde{f}$ induces a section $\mathrm{obst}^*(f)$ of the sheaf

$$\underline{\mathrm{Hom}}_{\mathcal{O}_{\tilde{S}}}(\mathrm{Fil}^1(\tilde{A}), \mathrm{Lie}(\tilde{B})),$$

and the zero locus of this section is the largest closed subscheme between S and $\tilde{S}$ to which f can be extended.

Proof of the Canonical Lifting Theorem. Start with a triple $(B, \kappa, \psi) \in M_{(m,0)}(R)$. It suffices to show that our triple lifts uniquely through any surjection $\tilde{R} \to R$ whose kernel I satisfies $I^2 = 0$. Here is the idea. First one shows that, locally on $\mathrm{Spec}(R)$, $H_1^{dR}(B)$ is free of rank m over $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} R$ (use Nakayama's lemma to reduce to the case where R is a field).

Denoting by $i_R : \mathcal{O}_{\boldsymbol{k}} \to R$ the structure map, the ring $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} R$ has two distinguished ideals

$$J = \ker\big(\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} R \xrightarrow{\alpha \otimes x \mapsto i_R(\alpha) x} R\big)$$
$$J' = \ker\big(\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} R \xrightarrow{\alpha \otimes x \mapsto i_R(\overline{\alpha}) x} R\big).$$

As R-modules, each is free of rank 1. In fact, if we pick any $\pi \in \mathcal{O}_{\boldsymbol{k}}$ such that $\mathcal{O}_{\boldsymbol{k}} = \mathbb{Z}[\pi]$ then J and J' are generated by

$$j = \pi \otimes 1 - 1 \otimes i_R(\pi)$$
$$j' = \overline{\pi} \otimes 1 - 1 \otimes i_R(\pi)$$

respectively. The signature condition on $\mathrm{Lie}(B)$ says precisely that

$$\alpha x = i_R(\alpha) x$$

[2] If S is a $\mathbb{Q}$-scheme this is more than intuition: $\tilde{f}$ is the unique parallel section extending f.

for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$ and all $x \in \mathrm{Lie}(B)$. In particular $J\mathrm{Lie}(B) = 0$, and so $JH_1^{dR}(B) \subset \mathrm{Fil}^1(B)$. But both $JH_1^{dR}(B)$ and $\mathrm{Fil}^1(B)$ are local R-module direct summands of rank m, and so we must have

$$\mathrm{Fil}^1(B) = JH_1^{dR}(B).$$

Define $\tilde{J}, \tilde{J}' \subset \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \tilde{R}$ in the same way. If $\tilde{B} \in M_{(m,0)}(\tilde{R})$ lifts B, then the same argument as above shows that the Hodge filtration

$$\mathrm{Fil}^1(\tilde{B}) \subset H_1^{dR}(\tilde{B}) \cong \tilde{H}_1^{dR}(B),$$

which uniquely determines $\tilde{B}$, *must* be

$$\mathrm{Fil}^1(\tilde{B}) = \tilde{J}\tilde{H}_1^{dR}(B).$$

This proves that there can be at most one such deformation $\tilde{B}$. For the existence, certainly the lift of the Hodge filtration

$$\tilde{J}\tilde{H}_1^{dR}(B) \subset \tilde{H}_1^{dR}(B)$$

determines some deformation $\tilde{B}$ of B. We leave it to the reader to verify that this lift defines an element of $M_{(m,0)}(\tilde{R})$.

For the second claim of the theorem, suppose $\tilde{B}_1$ and $\tilde{B}_2$ are the unique deformations of $B_1 \in M_{(m_1,0)}(R)$ and $B_2 \in M_{(m_2,0)}(R)$. The injectivity of the reduction map

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\tilde{B}_1, \tilde{B}_2) \to \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(B_1, B_2)$$

is the usual rigidity of morphisms between abelian schemes. The interesting part is to show that any $f \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(B_1, B_2)$ lifts to a morphism $\tilde{B}_1 \to \tilde{B}_2$. The point is that we know the lifts of the Hodge filtrations corresponding to these deformations: $\tilde{B}_1$ corresponds to

$$\tilde{J}\tilde{H}_1^{dR}(B_1) \subset \tilde{H}_1^{dR}(B_1),$$

while $\tilde{B}_2$ corresponds to

$$\tilde{J}\tilde{H}_1^{dR}(B_2) \subset \tilde{H}_1^{dR}(B_2).$$

The map

$$\tilde{f} : \tilde{H}_1^{dR}(B_1) \to \tilde{H}_1^{dR}(B_2)$$

is $\mathcal{O}_{\boldsymbol{k}}$-linear, and so commutes with action of $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \tilde{R}$. Thus $\tilde{f}$ satisfies

$$\tilde{f}\big(\tilde{J}\tilde{H}_1^{dR}(B_1)\big) = \tilde{J}\tilde{f}(\tilde{H}_1^{dR}(B_1)) \subset \tilde{J}\tilde{H}_1^{dR}(B_2).$$

This implies that f lifts, as desired. $\square$

2.3. **Integral models for signature (m,1).** Let $M_{(m,1)}$ be the moduli stack parametrizing quadruples $(A, \kappa, \psi, \mathcal{F})$ in which

- $A \to S$ is an abelian scheme of dimension $m+1$ over an $\mathcal{O}_{\boldsymbol{k}}$-scheme S,
- $\kappa : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A)$ is an action of $\mathcal{O}_{\boldsymbol{k}}$,
- $\psi : A \to A^\vee$ is an $\mathcal{O}_{\boldsymbol{k}}$-linear principal polarization,
- $\mathcal{F} \subset \mathrm{Lie}(A)$ is an $\mathcal{O}_{\boldsymbol{k}}$-stable $\mathcal{O}_S$-submodule such that $\mathrm{Lie}(A)/\mathcal{F}$ is locally free of rank 1.

We further require that $\mathcal{F}$ satisfy the *signature* $(m,1)$-*condition*: $\mathcal{O}_{\boldsymbol{k}}$ acts on $\mathcal{F}$ through the structure map $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$, and acts on $\mathrm{Lie}(A)/\mathcal{F}$ through the complex conjugate of the structure map. Note that $M_{(m,1)}$ carries over it a line bundle $\mathrm{Lie}(A)/\mathcal{F}$, where $A \to M_{(m,1)}$ is the universal object, and $\mathcal{F} \subset \mathrm{Lie}(A)$ is the universal subsheaf. This line bundle will play an essential role later.

Theorem 2.3.1 (Kramer [Kr], Pappas [Pa])**.** *The $\mathcal{O}_{\boldsymbol{k}}$-stack $M_{(m,1)}$ is regular, flat of relative dimension m, and smooth over $\mathcal{O}_{\boldsymbol{k}}[1/D]$.*

The signature condition on $\mathcal{F}$ implies that each $\alpha \in \mathcal{O}_{\boldsymbol{k}}$ acts on $\mathrm{Lie}(A)$ with characteristic polynomial

$$\det(T-\alpha) = (T - i_S(\alpha))^m \cdot (T - i_S(\overline{\alpha})), \tag{2.3.1}$$

and so the existence of $\mathcal{F}$ implies the naive signature $(m,1)$ condition.

Remark 2.3.2. Suppose $D \in \mathcal{O}_S^\times$. In this case any triple (A, κ, ψ) satisfying the naive signature condition admits a unique subsheaf $\mathcal{F} \subset \mathrm{Lie}(A)$ satisfying the signature condition above. It is characterized as the maximal subsheaf on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$.

Remark 2.3.3. At the other extreme, suppose $S = \mathrm{Spec}(\mathbb{F})$ where $\mathbb{F}$ is an algebraic closure of the field of D elements. Let $\mathbb{F}[\epsilon]$ be the ring of dual numbers of $\mathbb{F}$, and note that there is a unique $\mathbb{F}$-algebra isomorphism $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathbb{F} \cong \mathbb{F}[\epsilon]$ satisfying $\delta \otimes 1 \mapsto \epsilon$. Every finitely generated $\mathbb{F}[\epsilon]$-module is isomorphic to a direct sum of copies of $\mathbb{F} = \mathbb{F}[\epsilon]/(\epsilon)$ and $\mathbb{F}[\epsilon]$. In particular, for a triple (A, κ, ψ) as above, there is an isomorphism

$$\mathrm{Lie}(A) \cong \mathbb{F}^s \oplus \mathbb{F}[\epsilon]^t$$

for some s and t. The naive signature condition (2.3.1) is automatically satisfied regardless of the values of s and t. However,

(1) if $t > 1$ then there is no $\mathcal{F}$ satisfying the stated signature condition,
(2) if $t = 1$ then $\mathcal{F} = \epsilon\mathrm{Lie}(A)$ is the unique subsheaf of $\mathrm{Lie}(A)$ satisfying the signature condition,
(3) if $t = 0$ then *every* $\mathbb{F}$-subspace $\mathcal{F} \subset \mathrm{Lie}(A)$ of codimension 1 is $\mathcal{O}_{\boldsymbol{k}}$-stable and satisfies the signature condition.

In fact, there are only finitely many (A, κ, ψ) over $\mathbb{F}$ for which $t = 0$. If we fix one and allow $\mathcal{F}$ to vary, the resulting quadruples $(A, \kappa, \psi, \mathcal{F})$ vary over an irreducible component of the reduction of $M_{(m,1)} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathbb{F}$. Of course, there are irreducible components that do not arise in this way.

2.4. **Kudla-Rapoport divisors.** From now on we fix an integer $n \geq 2$ and define a flat, regular $\mathcal{O}_{\boldsymbol{k}}$-scheme

$$M = M_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} M_{(n-1,1)}$$

of (absolute) dimension n. A point of M is a tuple $(A_0, \kappa_0, \psi_0, A, \kappa, \psi, \mathcal{F})$, but we'll just abbreviate this to (A_0, A). Recalling that $M_{(1,0)}(\mathbb{C})$ consists of a single point corresponding to the CM elliptic curve $\mathbb{C}/\mathcal{O}_{\boldsymbol{k}}$ with automorphism group $\mathcal{O}_{\boldsymbol{k}}^\times$, the complex uniformization (2.1.2) implies that

$$M(\mathbb{C}) \cong \Gamma_L \backslash \mathcal{D}_L \tag{2.4.1}$$

where L is the unique self-dual Hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice L of signature $(n-1,1)$, $\mathcal{D}_L$ is the space of negative definite $\mathbb{C}$-lines in $L_{\mathbb{C}}$, and

$$\Gamma_L = \mathcal{O}_{\boldsymbol{k}}^\times \times \mathrm{Aut}(L).$$

The extra factor of $\mathcal{O}_{\boldsymbol{k}}^\times$ acts trivially on $\mathcal{D}_L$, and so the actual set of points of $M(\mathbb{C})$ is identical to the set of points of $M_{(n-1,1)}(\mathbb{C})$ found in (2.1.2). As *orbifolds*, (2.1.2) and (2.4.1) are not the same, as points of (2.4.1) have $|\mathcal{O}_{\boldsymbol{k}}^\times|$ as many automorphisms as points of (2.1.2).

Suppose S is a connected $\mathcal{O}_{\boldsymbol{k}}$-scheme. For any point $(A_0, A) \in M(S)$, the $\mathcal{O}_{\boldsymbol{k}}$-module $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ carries a positive definite Hermitian form

$$\langle \lambda_1, \lambda_2 \rangle = \lambda_2^\vee \circ \lambda_1 \in \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(A_0) \cong \mathcal{O}_{\boldsymbol{k}}.$$

Here we are using the principal polarizations of A_0 and A to view $\lambda_2^\vee : A^\vee \to A_0^\vee$ as a map $A \to A_0$. If S is the spectrum of an algebraically closed field then

$$\mathrm{rank}_{\mathcal{O}_{\boldsymbol{k}}} \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A) \le n,$$

with equality if and only if A_0 and A are supersingular.

Definition 2.4.1 ([KR1, KR2])**.** For each positive $m \in \mathbb{Z}$, define the *Kudla-Rapoport* (or *special*) divisor $Z(m)$ as the moduli stack of triples (A_0, A, λ) over $\mathcal{O}_{\boldsymbol{k}}$-schemes S, in which

- $(A_0, A) \in M(S)$,
- $\lambda : A_0 \to A$ is $\mathcal{O}_{\boldsymbol{k}}$-linear and satisfies $\langle \lambda, \lambda \rangle = m$.

Remark 2.4.2. The morphism $Z(m) \to M$ defined by "forget λ" is finite and unramified, and this allows us to view $Z(m)$ as a divisor, or as a line bundle, on M. More precisely, although $Z(m) \to M$ is not a closed immersion, there is an étale open cover $\bigsqcup_i U_i \to M$ of M such that over each U_i the natural map

$$Z(m) \times_M U_i \to U_i$$

is a closed immersion *when restricted to each connected component* of $Z(m) \times_M U_i$. Thus each connected component of $Z(m) \times_M U_i$ defines a divisor on U_i, and we can add these divisors together to obtain a divisor on U_i. These divisors agree on the overlaps of the U_i's, and so patch together to define a divisor on M.

We'll also need a modified version of the Kudla-Rapoport divisors, in which we allow the morphism f to have mild denominators.

Definition 2.4.3. For a positive $m \in \mathbb{Q}$ define $Z(m, \delta)$ to be the moduli stack of triples (A_0, A, λ) over $\mathcal{O}_{\boldsymbol{k}}$-schemes S, in which

- $(A_0, A) \in M(S)$,
- $\lambda \in \delta^{-1}\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ is $\mathcal{O}_{\boldsymbol{k}}$-linear and satisfies $\langle \lambda, \lambda \rangle = m$,
- the map $\delta\lambda : A_0 \to A$ induces the trivial map

$$\delta\lambda : \mathrm{Lie}(A_0) \to \mathrm{Lie}(A)/\mathcal{F}. \tag{2.4.2}$$

Remark 2.4.4. It's easy to see that $Z(m, \delta) = \emptyset$ unless $m \in D^{-1}\mathbb{Z}$.

Remark 2.4.5. The extra vanishing condition may seem unmotivated. It is necessary to add this condition to the moduli problem in order to make the main result, Theorem 3.5.1, hold. In fact the vanishing of (2.4.2) is almost automatic: Recall that $\mathcal{O}_{\boldsymbol{k}}$ acts on $\mathrm{Lie}(A_0)$ through the structure map $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$, and acts on $\mathrm{Lie}(A)/\mathcal{F}$ through the conjugate of the structure map. It follows that the image of (2.4.2) is killed by all elements of the form $\alpha - \overline{\alpha} \in \mathcal{O}_{\boldsymbol{k}}$. These elements generate the ideal (δ), and so if $D \in \mathcal{O}_S^\times$ then (2.4.2) is necessarily 0.

Proposition 2.4.6. *There are isomorphisms of complex orbifolds*

$$Z(m)(\mathbb{C}) \cong \Gamma_L \backslash \Big(\bigsqcup_{\substack{\lambda \in L \\ \langle \lambda, \lambda \rangle = m}} \mathcal{D}_L(\lambda) \Big)$$

and

$$Z(m,\delta)(\mathbb{C}) \cong \Gamma_L \backslash \Big(\bigsqcup_{\substack{\lambda \in \delta^{-1} L \\ \langle \lambda, \lambda \rangle = m}} \mathcal{D}_L(\lambda) \Big),$$

where $\mathcal{D}_L(\lambda) = \{z \in \mathcal{D}_L : z \perp \lambda\}$.

Proof. Start with a point $z \in \mathcal{D}_L$, and recall that, under the uniformization

$$\mathcal{D}_L \to M(\mathbb{C})$$

the point z is sent to the pair

$$(A_0, A_z) = (\mathbb{C}/\mathcal{O}_{\boldsymbol{k}}, L_{\mathbb{C}}/L)$$

where $L_{\mathbb{C}}$ is given the modified complex structure defined by

$$I_z(v) = \begin{cases} i \cdot v & \text{if } v \in z^{\perp} \\ -i \cdot v & \text{if } v \in z. \end{cases}$$

Certainly any $\lambda \in L \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathcal{O}_{\boldsymbol{k}}, L)$ defines an $\mathcal{O}_{\boldsymbol{k}}$-linear morphism

$$A_0(\mathbb{C}) \xrightarrow{\lambda} A_z(\mathbb{C})$$

of *real Lie groups*. In order for λ to be complex linear, the corresponding map

$$\mathbb{C} \xrightarrow{x \mapsto x\lambda} L_{\mathbb{C}}$$

on Lie algebras must be complex linear. In other words we must have

$$I_z(x\lambda) = (ix)\lambda$$

for all $x \in \mathbb{C}$. Of course this just says that $\lambda \in z^{\perp}$, or, equivalently, $z \in \mathcal{D}_L(\lambda)$.

What we have shown is that if $\lambda \in L$ satisfies $\langle \lambda, \lambda \rangle = m$, then any $z \in \mathcal{D}_L(\lambda)$ determines a triple $(A_0, A_z, \lambda) \in Z(m)(\mathbb{C})$. The proposition follows easily from this. $\square$

2.5. **Small CM points.** For the rest of this paper we fix a self-dual Hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice Λ of signature $(n-1,1)$, and let $L_0 = \mathcal{O}_{\boldsymbol{k}}$ with its negative definite Hermitian form $\langle x, y \rangle = -x\overline{y}$. A choice of isomorphism

$$L \cong L_0 \oplus \Lambda$$

(which must exist, by Proposition 2.1.5) determines a negative definite line $L_{0\mathbb{C}} \subset L_{\mathbb{C}}$, and hence a point $y_0 \in \mathcal{D}_L$. A different choice of isomorphism changes this point by an element of Γ_L, and so we obtain a well-defined point

$$Y_{\Lambda}(\mathbb{C}) = \{y_0\}$$

on $M(\mathbb{C})$. In order to make $Y_{\Lambda}(\mathbb{C})$ into an orbifold, we endow its unique point with automorphism group

$$\mathrm{Aut}(y_0) = \mathcal{O}_{\boldsymbol{k}}^{\times} \times \mathcal{O}_{\boldsymbol{k}}^{\times} \times \mathrm{Aut}(\Lambda).$$

We next construct an integral model of Y_{Λ}, which will justify the choice of automorphism group. Define an $\mathcal{O}_{\boldsymbol{k}}$-stack

$$Y = M_{(1,0)} \times M_{(0,1)} \times M_{(n-1,0)},$$

so that a point of Y is a triple (A_0, A_1, B). There is an obvious map

$$M_{(0,1)} \times M_{(n-1,0)} \to M_{(n-1,1)}$$

defined by sending (A_1, B) to $A_1 \times B$ with its product polarization and diagonal $\mathcal{O}_{\boldsymbol{k}}$-action. For the subsheaf $\mathcal{F} \subset \mathrm{Lie}(A_1 \times B)$ we simply take $\mathcal{F} = \mathrm{Lie}(B)$. The construction $(A_0, A_1, B) \mapsto (A_0, A_1 \times B)$ defines a morphism $Y \to M$.

Lemma 2.5.1. *For any geometric point*

$$(A_0, A_1, B) \in Y(\mathbb{F})$$

the Hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B)$ is positive definite of rank $n-1$. Moreover, the isomorphism class of $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B)$ is constant on connected components of Y.

Proof. First suppose $\mathbb{F} = \mathbb{C}$. Then $A_0(\mathbb{C}) \cong \mathbb{C}/\mathcal{O}_{\boldsymbol{k}}$, and $B(\mathbb{C}) \cong \Lambda_{\mathbb{C}}/\Lambda$ for some positive definite Hermitian $\mathcal{O}_{\boldsymbol{k}}$-module Λ' of rank $n-1$. But now

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B) \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathbb{C}/\mathcal{O}_{\boldsymbol{k}}, \Lambda'_{\mathbb{C}}/\Lambda') \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathcal{O}_{\boldsymbol{k}}, \Lambda') \cong \Lambda',$$

which is positive definite of rank $n-1$. This proves the first claim, assuming that $\mathbb{F}$ has characteristic 0. The characteristic p case follows from the canonical lifting theorem and the characteristic 0 case. The constancy on connected components also follows from the canonical lifting theorem. □

Let $Y_\Lambda \subset Y$ be the union of all connected components of Y on which

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B) \cong \Lambda.$$

It is clear from Corollary 2.2.2 that Y_Λ is smooth and proper of relative dimension 0 over $\mathcal{O}_{\boldsymbol{k}}$. The unique complex point of Y_Λ is

$$y_0 = (A_0, A_1, B) = (\mathbb{C}/\mathcal{O}_{\boldsymbol{k}}, \mathbb{C}/\mathcal{O}_{\boldsymbol{k}}, \Lambda_{\mathbb{C}}/\Lambda), \tag{2.5.1}$$

where the first and third entries are endowed with the obvious $\mathcal{O}_{\boldsymbol{k}}$-actions, the second entry is endowed with the complex conjugate of the obvious action, and $\Lambda_{\mathbb{C}}/\Lambda$ is given the polarization corresponding to the symplectic form $\mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}}\langle \delta^{-1}x, y\rangle$ on Λ. The canonical lifting theorem implies that every geometric point of Y_Λ lifts uniquely to characteristic 0, and hence Y_Λ has a unique geometric point in every characteristic. This unique point has automorphism group $\mathcal{O}_{\boldsymbol{k}}^\times \times \mathcal{O}_{\boldsymbol{k}}^\times \times \mathrm{Aut}(\Lambda)$.

Proposition 2.5.2. *The divisor $Z(m)(\mathbb{C})$ passes through $y_0 \in Y_\Lambda(\mathbb{C})$ if and only if Λ represents m.*

Proof. The divisor

$$Z(m)(\mathbb{C}) = \sum_{\substack{\lambda \in L \\ \langle \lambda, \lambda \rangle = m}} \mathcal{D}_L(\lambda)$$

on $\mathcal{D}_L$ passes through y_0 if and only if there is some $\lambda \in L$ of Hermitian norm m such that $y_0 \perp \lambda$. Of course the decomposition $L_{\mathbb{C}} = y_0 \oplus \Lambda_{\mathbb{C}}$ shows that $y_0 \perp \lambda$ if and only if $\lambda \in \Lambda$.

Here is a second proof, based on the moduli interpretations. The image of y_0 in $M(\mathbb{C})$ is

$$y_0 = (A_0, A_1 \times B),$$

with (A_0, A_1, B) as in (2.5.1), and

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A_1 \times B) \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A_1) \times \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B).$$

Any nonzero $\mathcal{O}_{\boldsymbol{k}}$-linear map $A_0 \to A_1$ would induce an *isomorphism* of Lie algebras, and the signature conditions imply that these Lie algebras are not $\mathcal{O}_{\boldsymbol{k}}$-linearly isomorphism. Therefore $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A_1) = 0$. On the other hand,

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B) \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathcal{O}_{\boldsymbol{k}}, \Lambda) \cong \Lambda,$$

and so in fact

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A_1 \times B) \cong \Lambda.$$

By definition, $y_0 \in Z(m)(\mathbb{C})$ if and only if there is some

$$\lambda \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A_1 \times B) \cong \Lambda$$

of Hermitian norm m, and so the claim is clear. □

2.6. **Arithmetic intersection.** We need at least some rudiments of the Gillet-Soulé theory [SABK] of arithmetic Chow groups.

Definition 2.6.1. If Z is a divisor on M, a *Green function* for Z is a smooth function Φ on $M(\mathbb{Z}) \smallsetminus Z(\mathbb{C})$ with a logarithmic singularity along $Z(\mathbb{C})$ in the following sense: around every point $M(\mathbb{C})$ there is a neighborhood U such that $\Phi(z) + \log|\psi(z)|^2$ extends to a smooth function on U, where $\psi = 0$ is any defining equation of $Z(\mathbb{C})$ on U.

Definition 2.6.2. An *arithmetic divisor* on M is a pair (Z, Φ) where Z is a divisor on M and Φ is a Green function for Z. Each nonzero rational function f on M determines an arithmetic divisor $(\mathrm{div}(f), -\log|f|^2)$, and arithmetic divisors of this form are *rationally equivalent to* 0.

Definition 2.6.3. The group of arithmetic divisors modulo rational equivalence is the *Gillet-Soulé arithmetic Chow group* $\widehat{\mathrm{Ch}}^1(M)$.

Given an arithmetic divisor (Z, Φ), we can form the usual line bundle $\mathcal{O}(Z)$. The constant function 1 on M defines a section $\sigma(Z)$ of $\mathcal{O}(Z)$, whose zero locus is precisely Z. Define a smoothly varying family of metrics on the complex fibers of $\mathcal{O}(Z)$ by

$$-\log||\sigma(Z)||_z^2 = \Phi(z).$$

This establishes an isomorphism

$$\widehat{\mathrm{Ch}}^1(M) \cong \widehat{\mathrm{Pic}}(M),$$

where $\widehat{\mathrm{Pic}}(M)$ is the group of metrized line bundles on M. The inverse sends

$$\widehat{\mathcal{E}} \mapsto (\mathrm{div}(\sigma), -\log||\sigma||^2)$$

for any nonzero rational section σ of $\mathcal{E}$. Similar remarks hold with M replaced by Y_Λ.

If X is any irreducible divisor on Y_Λ, define

$$\widehat{\deg}_{\mathrm{fin}}(X) = \sum_{\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}} \sum_{x \in X(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}})} \frac{\log(\mathrm{N}(\mathfrak{p}))}{|\mathrm{Aut}(x)|}$$

where $\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$ is an algebraic closure of $\mathcal{O}_{\boldsymbol{k}}/\mathfrak{p}$. Extend the definition linearly to all divisors X. Given a metrized line bundle $\widehat{\mathcal{E}}$ on Y_Λ, pick a nonzero rational section σ of $\mathcal{E}$, and set

$$\widehat{\deg}\ \widehat{\mathcal{E}} = \widehat{\deg}_{\mathrm{fin}}\,\mathrm{div}(\sigma) - \frac{\log||\sigma||_{y_0}^2}{\mathrm{Aut}(y_0)}$$

where y_0 is the unique complex point of Y_Λ. This does not depend on the choice of section, and defines a homomorphism

$$\widehat{\deg} : \widehat{\mathrm{Pic}}(Y_\Lambda) \to \mathbb{R}$$

called the *arithmetic degree.*

There is an obvious notion of pullback

$$\widehat{\mathrm{Pic}}(M) \to \widehat{\mathrm{Pic}}(Y_\Lambda)$$

induced by the morphism $Y_\Lambda \to M$, and the composition

$$\widehat{\mathrm{Pic}}(M) \to \widehat{\mathrm{Pic}}(Y_\Lambda) \to \mathbb{R},$$

called *arithmetic intersection with* Y_Λ, is denoted

$$\widehat{Z} \mapsto [\widehat{Z} : Y_\Lambda].$$

In terms of arithmetic divisors, the arithmetic intersection can be computed as follows. The usual *moving lemma* in characteristic 0 implies that every class of $\widehat{\mathrm{Ch}}^1(M)$ is represented by an arithmetic divisor (Z, Φ) for which $y_0 \notin Z(\mathbb{C})$. In other words, such that Y_Λ and Z intersect in dimension 0. The arithmetic intersection can then be computed as

$$[\widehat{Z} : Y_\Lambda] = \widehat{\deg}_{\mathrm{fin}}(Z \cap Y_\Lambda) + \frac{\Phi(y_0)}{|\mathrm{Aut}(y_0)|}.$$

2.7. **The cotautological bundle.** Let A_0 and A be the universal objects over $M_{(1,0)}$ and $M_{(n-1,1)}$, respectively. Over $M_{(1,0)}$ we have the line bundle $\mathrm{Lie}(A_0)$, and over $M_{(n-1,1)}$ we have the line bundle $\mathrm{Lie}(A)/\mathcal{F}$. Over the product

$$M = M_{(1,0)} \times M_{(n-1,1)}$$

we can form the *cotautological bundle*

$$T = \mathrm{Lie}(A_0) \otimes_{\mathcal{O}_M} (\mathrm{Lie}(A)/\mathcal{F}).$$

Note that the principal polarization of A_0 induces a perfect alternating form on $H_1^{dR}(A_0)$, which identifies

$$\mathrm{Fil}^1(A_0)^\vee \cong \mathrm{Lie}(A_0),$$

and hence identifies

$$T \cong \mathrm{Hom}_{\mathcal{O}_M}(\mathrm{Fil}^1(A_0), \mathrm{Lie}(A)/\mathcal{F}).$$

It is this second realization of T that will be more useful to us in practice.

The cotautological bundle T is easy to describe under the uniformization (2.4.1). The proof of the following proposition is tedious but not hard. It involves only tracing through the details of the isomorphism of Proposition 2.1.2.

Proposition 2.7.1. *For any negative line $z \in \mathcal{D}_L$ there is a canonical $\mathbb{C}$-linear isomorphism*

$$T_z \cong \mathrm{Hom}_{\mathbb{C}}(z, \mathbb{C}).$$

Because each line $z \in \mathcal{D}_L$ comes endowed with a negative definite Hermitian form $\langle \cdot, \cdot \rangle$, obtained by restricting the Hermitian form on $L_{\mathbb{C}}$, the line $\mathrm{Hom}_{\mathbb{C}}(z, \mathbb{C})$ carries a *natural metric* defined by the relation

$$||\sigma||^{\mathrm{nat}} \cdot |\langle s, s \rangle|^{1/2} = |\sigma(s)|$$

for all $\sigma \in \mathrm{Hom}_{\mathbb{C}}(z, \mathbb{C})$ and $s \in z$. Using Proposition 2.7.1 we obtain a metric on the cotautological bundle T, still denoted $|| \cdot ||^{\mathrm{nat}}$. Define the *normalized metric* on T by

$$||\sigma|| = \sqrt{4\pi e^\gamma} \cdot ||\sigma||^{\mathrm{nat}}$$

where $\gamma = -\Gamma'(1) = 0.57721\ldots$ is Euler's constant. More concretely, every $\lambda \in L$ defines a linear functional $\langle \cdot, \lambda \rangle \in \mathrm{Hom}_{\mathbb{C}}(z, \mathbb{C})$, and this linear functional has norm

$$||\langle \cdot, \lambda \rangle||_z^2 = -4\pi e^{\gamma} \langle \lambda_z, \lambda_z \rangle,$$

where λ_z is the orthogonal projection of λ to the negative line z. The cotautological bundle with its normalized metric is denoted

$$\widehat{T} \in \widehat{\mathrm{Pic}}(M).$$

Theorem 2.7.2 (Chowla-Selberg)**.** *The arithmetic intersection of $\widehat{T}$ with Y_Λ is*

$$[\widehat{T} : Y_\Lambda] = \frac{1}{|\mathrm{Aut}(y_0)|} \left(2\frac{L'(\chi, 0)}{L(\chi, 0)} + \log \left| \frac{D}{4\pi} \right| - \gamma \right).$$

Proof. First let's understand what happens when T is restricted to Y_Λ. The morphism $Y_\Lambda \to M$ was defined by

$$(A_0, A_1, B) \mapsto (A_0, A_1 \times B)$$

where the subsheaf $\mathcal{F} \subset \mathrm{Lie}(A_1 \times B)$ is simply $\mathcal{F} = \mathrm{Lie}(B)$. The line bundle T on

$$M = M_{(1,0)} \times M_{(n-1,1)}$$

was defined as

$$T = \mathrm{Lie}(A_0) \otimes \mathrm{Lie}(A)/\mathcal{F}$$

where A_0 and A are the universal objects over $M_{(1,0)}$ and $M_{(n-1,1)}$, and when we restrict this to Y_Λ we obtain

$$\begin{aligned} T|_{Y_\Lambda} &\cong \mathrm{Lie}(A_0) \otimes \mathrm{Lie}(A_1 \times B)/\mathrm{Lie}(B) \\ &\cong \mathrm{Lie}(A_0) \otimes \mathrm{Lie}(A_1), \end{aligned} \tag{2.7.1}$$

where (A_0, A_1, B) is the universal object over Y_Λ.

Next we recall the Chowla-Selberg formula, as in [Co]. Suppose E is an elliptic curve over $\mathbb{Q}^{\mathrm{alg}}$. Fix a model of E over a finite extension $K/\mathbb{Q}$ contained in $\mathbb{Q}^{\mathrm{alg}}$, large enough that E has semistable reduction, and let $\pi : \mathcal{E} \to \mathrm{Spec}(\mathcal{O}_K)$ be the Néron model of E over $\mathcal{O}_K$. Let ω be a nonzero rational section of the line bundle $\pi_* \Omega^1_{\mathcal{E}/\mathcal{O}_K}$ on $\mathrm{Spec}(\mathcal{O}_K)$ with divisor

$$\mathrm{div}(\omega) = \sum_{\mathfrak{q}} m(\mathfrak{q}) \cdot \mathfrak{q},$$

where the sum is over the closed points $\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_K)$. The *Faltings height* of E is defined as

$$h_{\mathrm{Falt}}(E) = \frac{1}{[K : \mathbb{Q}]} \left(\sum_{\mathfrak{q}} \log(\mathrm{N}(\mathfrak{q})) \cdot m(\mathfrak{q}) - \frac{1}{2} \sum_{\tau : K \to \mathbb{C}} \log \left| \int_{\mathcal{E}^\tau(\mathbb{C})} \omega^\tau \wedge \overline{\omega^\tau} \right| \right).$$

It is independent of the choice of K, the model of E over K, and the section ω. The Chowla-Selberg formula implies that if E has complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$, then

$$-2h_{\mathrm{Falt}}(E) = \log(2\pi) + \frac{1}{2} \log D + \frac{L'(\chi, 0)}{L(\chi, 0)}. \tag{2.7.2}$$

Define a line bundle

$$\mathrm{coLie}(A_0) = \pi_* \Omega^1_{A_0/Y_\Lambda}$$

on Y_Λ, where $\pi : A_0 \to Y_\Lambda$ is the universal object. A vector $\omega \in \mathrm{coLie}(A_{0,y_0})$ in the fiber at the unique complex point $y_0 \in Y_\Lambda(\mathbb{C})$ is a holomorphic 1-form on $A_{0,y_0}(\mathbb{C})$, and we denote by

$$\widehat{\mathrm{coLie}}(A_0) \in \widehat{\mathrm{Pic}}(Y_\Lambda)$$

the line bundle $\mathrm{coLie}(A_0)$ endowed with the metric

$$||\omega||^2_{y_0} = \left| \int_{A_{0,y_0}(\mathbb{C})} \omega \wedge \overline{\omega} \right|. \tag{2.7.3}$$

Unwinding the definitions, one can show that

$$\widehat{\deg}\ \widehat{\mathrm{coLie}}(A_0) = \frac{2h_{\mathrm{Falt}}(A_{0,y_0})}{|\mathrm{Aut}(y_0)|}. \tag{2.7.4}$$

Denote by

$$\widehat{\mathrm{Lie}}(A_0) \in \widehat{\mathrm{Pic}}(Y_\Lambda)$$

the line bundle $\mathrm{Lie}(A_0)$ with the metric dual to (2.7.3). Combining (2.7.2) and (2.7.4) shows that

$$\widehat{\deg}\ \widehat{\mathrm{Lie}}(A_0) = \frac{1}{|\mathrm{Aut}(y_0)|} \cdot \left(\log(2\pi) + \frac{1}{2}\log D + \frac{L'(\chi,0)}{L(\chi,0)} \right). \tag{2.7.5}$$

The same formula holds if we replace A_0 by A_1 everywhere.

If we go back to the isomorphism (2.7.1) and tediously keep track of the metrics, we find an isomorphism

$$\widehat{T}|_{Y_\Lambda} \cong \widehat{\mathrm{Lie}}(A_0) \otimes \widehat{\mathrm{Lie}}(A_1) \otimes \widehat{\mathcal{O}}_{Y_\Lambda}(16\pi^3 e^\gamma)$$

of metrized line bundles on Y_Λ, where $\widehat{\mathcal{O}}_{Y_\Lambda}(16\pi^3 e^\gamma)$ denotes the trivial bundle on Y_Λ with the metric defined by

$$||f||^2_{y_0} = 16\pi^3 e^\gamma |f(y_0)|^2$$

for any rational function f on Y_Λ. It's easy to see that

$$\widehat{\deg}\, \widehat{\mathcal{O}}_{Y_\Lambda}(16\pi^3 e^\gamma) = \frac{-\log(16\pi^3 e^\gamma)}{\mathrm{Aut}(y_0)}.$$

The theorem follows by combining this, the equality

$$[\widehat{T} : Y_\Lambda] = \widehat{\deg}\, \widehat{\mathrm{Lie}}(A_0) + \widehat{\deg}\, \widehat{\mathrm{Lie}}(A_1) + \widehat{\deg}\, \widehat{\mathcal{O}}_{Y_\Lambda}(16\pi^3 e^\gamma),$$

and (2.7.5). □

3. Green functions and regularized theta lifts

The Hermitian form $\langle \cdot, \cdot \rangle$ on L induces a $D^{-1}\mathbb{Z}/\mathbb{Z}$-valued quadratic form

$$Q_L(\lambda) = \langle \lambda, \lambda \rangle$$

on the finite $\mathbb{Z}$-module $\delta^{-1}L/L$. Let Δ be the automorphism group of $\delta^{-1}L/L$ with its quadratic form Q_L, let S_L denote the space of all complex valued functions on $\delta^{-1}L/L$, and let S_L^Δ be the Δ-invariant functions in S_L.

3.1. **Vector-valued modular forms.** There is a *Weil representation*

$$\omega_L : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_L)$$

attached to the quadratic form $\langle \lambda, \lambda \rangle$ on L. As S_L comes with a complex conjugation, there is a *conjugate representation* $\overline{\omega}_L$ defined by

$$\overline{\omega}_L(g)(f) = \overline{\omega_L(g)(\overline{f})}.$$

Denote by $\omega_L^\vee : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_L^\vee)$ the dual representation, and by

$$\{\cdot, \cdot\} : S_L \times S_L^\vee \to \mathbb{C}$$

the tautological $\mathbb{C}$-bilinear pairing.

Fix $k \in \mathbb{Z}$, and let $\tau = u + iv$ be the variable on the upper half-plane. Suppose $f(\tau)$ is a smooth S_L-valued function $f(\tau)$ on $\mathcal{H}$ satisfying

$$(cz + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = \omega_L(\gamma) f \tag{3.1.1}$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. The transformation law implies that $f(\tau)$ admits a Fourier expansion

$$f(\tau) = \sum_{m \in \mathbb{Q}} a_f(m, v) q^m$$

with coefficients in S_L, and that the function $a_f(m, v) \in S_L$ is supported on the subset

$$\{\lambda \in \delta^{-1}L/L : Q_L(\lambda) = m \text{ in } \mathbb{Q}/\mathbb{Z}\}.$$

In particular, only $m \in D^{-1}\mathbb{Z}$ can contribute to the sum.

We define spaces of modular forms as follows.

(1) Let $M_k^!(\omega_L)$ be the space of *weakly holomorphic modular forms*: holomorphic functions $f(\tau)$ satisfying (3.1.1) with a Fourier expansion of the form

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} a_f(m) q^m.$$

(2) Let $M_k(\omega_L)$ be the space of *holomorphic modular forms*: holomorphic functions $f(\tau)$ satisfying (3.1.1) with a Fourier expansion of the form

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \geq 0}} a_f(m) q^m,$$

(3) Let $S_k(\omega_L)$ be the space of *cuspidal modular forms*: holomorphic modular forms with $a_f(0) = 0$.

(4) Let $H_k(\omega_L)$ be the space of *(weakly) harmonic Maass forms*: smooth functions on $\mathcal{H}$ which satisfy the transformation law (3.1.1), are annihilated by the weight k Laplacian

$$\Delta_k = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right),$$

and satisfy the growth condition $f(\tau) = P_f(\tau) + O(e^{-cv})$ as $v \to \infty$ for some constant $c > 0$ depending on f and some finite sum

$$P_f(\tau) = \sum_{m \leq 0} a_f^+(m) q^m.$$

There are obvious inclusions

$$S_k(\omega_L) \subset M_k(\omega_L) \subset M_k^!(\omega_L) \subset H_k(\omega_L).$$

All of the above spaces of modular forms make sense if ω_L is replaced by $\overline{\omega}_L$ or $\omega_L^\vee$. The Weil representation ω_L commutes with the action of Δ, and so acts on all spaces of modular forms defined above.

We will be working mostly with harmonic forms of weight $2-n$. Any $f \in H_{2-n}(\omega_L)$ has a unique decomposition $f = f^+ + f^-$ into a *holomorphic part* and an *antiholomorphic part* (which are not modular forms) with Fourier expansions of the form

$$f^+(\tau) = \sum_{\substack{m\in\mathbb{Q} \\ m\gg-\infty}} a_f^+(m) q^m$$

and

$$f^-(\tau) = \sum_{\substack{m\in\mathbb{Q} \\ m<0}} a_f^-(m)\Gamma(n-1, 4\pi|m|v) q^m.$$

Here $\Gamma(s,x) = \int_x^\infty e^{-t}t^{s-1}\,dt$ is the incomplete gamma function. Note that

$$\Gamma(n-1, v) \sim v^{n-2}e^{-v}$$

as $v \to \infty$, and so

$$a_f^+(m) = \lim_{v\to\infty} a_f(m,v).$$

Definition 3.1.1. Given vector-valued modular forms $F(\tau) \in S_k(\overline{\omega}_L)$ and $G(\tau) \in M_k(\omega_L^\vee)$, define the *Petersson inner product*

$$\langle F, G\rangle^{\mathrm{Pet}} = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \{\overline{F}, G\} v^{k-2}\,du\,dv$$

Remark 3.1.2. A scalar-valued modular form $f \in S_k(\Gamma_0(D), \chi^k)$ can be promoted to a Δ-invariant $\vec{f} \in S_k(\overline{\omega}_L)$ via the construction

$$\vec{f} = \sum_{\gamma\in\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})} (f|_k\gamma)\cdot\overline{\omega}_L(\gamma^{-1})\varphi_0,$$

where $\varphi_0 \in S_L$ is the characteristic function of $0 \in \delta^{-1}L/L$. This defines a *surjection*

$$S_k(\Gamma_0(D), \chi^k) \to S_k(\overline{\omega}_L)^\Delta.$$

3.2. **Divisors associated with harmonic forms.** To each harmonic modular form $f \in H_{2-n}(\omega_L)^\Delta$ we will attach a certain linear combination $Z(f)$ of the Kudla-Rapoport divisors on M. The easiest way to do this is to specify $Z(f)$ for a particularly simple set of generators of $H_{2-n}(\omega_L)^\Delta$. The following proposition follows from the results of [BF, Section 3].

Proposition 3.2.1.

(1) *Suppose $m \in \mathbb{Z}$ is positive. There is an $f_m \in H_{2-n}(\omega_L)^\Delta$ with holomorphic part of the form*

$$f_m^+ = \varphi_0 \cdot q^{-m} + O(1),$$

where $\varphi_0 \in S_L$ is the characteristic function of $\{0\} \subset \delta^{-1}L/L$.

(2) *Suppose $m \in D^{-1}\mathbb{Z}$ is positive. There is some $f_{m,\delta} \in H_{2-n}(\omega_L)^\Delta$ with holomorphic part of the form*
$$f^+_{m,\delta} = \varphi_{m,\delta} \cdot q^{-m} + O(1),$$
where $\varphi_{m,\delta} \in S_L$ is the characteristic function of
$$\{\lambda \in \delta^{-1}L/L : Q_L(\lambda) = m\}.$$
(3) *If $n > 2$ then f_m and $f_{m,\delta}$ are unique, and as m varies these harmonic forms span $H_{2-n}(\omega_L)^\Delta$.*
(4) *If $n = 2$ then any two f_m as above differ by a constant form, and similarly for $f_{m,\delta}$. As m varies these harmonic forms, together with the constant forms, span $H_{2-n}(\omega_L)^\Delta$.*

There is a unique linear map
$$Z : H_{2-n}(\omega_L)^\Delta \to \mathrm{Div}(M)_\mathbb{C}$$
satisfying
$$Z(f_m) = Z(m) \quad \text{and} \quad Z(f_{m,\delta}) = Z(m,\delta),$$
and such that all constant forms in $H_{2-n}(\omega_L)^\Delta$ are sent to the divisor 0. Such constant forms only exist when $n = 2$. Thus for every $f \in H_{2-n}(\omega_L)^\Delta$ we obtain a divisor $Z(f)$ on M, and a line bundle
$$\Theta(f) = Z(f) + a^+_f(0,0) \cdot T \in \mathrm{Pic}(M) \otimes_\mathbb{Z} \mathbb{C},$$
where $a^+_f(0,0)$ is the value of the constant term $a^+_f(0) \in S_L$ at the trivial coset $0 \in \delta^{-1}L/L$.

3.3. **Construction of Green functions.** Define the Siegel theta kernel
$$\vartheta_L : \mathcal{H} \times \mathcal{D}_L \to S_L^\vee$$
by
$$\vartheta_L(\tau, z, \varphi) = v \sum_{\lambda \in \delta^{-1}L} \varphi(\lambda) \cdot e^{2\pi i \langle \lambda_{z^\perp}, \lambda_{z^\perp}\rangle \tau} \cdot e^{2\pi i \langle \lambda_z, \lambda_z \rangle \overline{\tau}},$$
where λ_z and $\lambda_{z^\perp}$ are the orthogonal projections of λ to the negative lines z and $z^\perp$. As a function of τ, ϑ_L transforms like a weight $n-2$ modular form of representation $\omega_L^\vee$. As a function of z the theta kernel is Γ_L-invariant. For any
$$f \in H_{2-n}(\omega_L)^\Delta$$
the scalar-valued function $\{f, \vartheta_L\}$ is therefore $\mathrm{SL}_2(\mathbb{Z})$-invariant, and we may attempt to form a function on $\mathcal{D}_L$ by
$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \{f, \vartheta_L\} \, \frac{du\,dv}{v^2}.$$
This integral diverges, due to the growth of $f(\tau)$ at ∞. However, the integral can be regularized by first defining
$$\Phi(z, f, s) = \lim_{T \to \infty} \int_{\mathcal{F}_T} \{f, \vartheta_L\} v^{-s} \, \frac{du\,dv}{v^2},$$
where $\mathcal{F}_T$ is the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ truncated at height T. This function has meromorphic continuation to all s, and we define the *regularized theta lift* $\Phi(z, f)$ as the constant term in the Laurent expansion at $s = 0$. The result is a function on $M(\mathbb{C})$, which by the theorem below, is smooth away from

the support of $Z(f)$, and has a logarithmic singularity along $Z(f)(\mathbb{C})$. However, and this will be essential later, the function $\Phi(z,f)$ is defined at *every* point of $M(\mathbb{C})$. Expressed differently, the function $\Phi(z,f)$, smooth on the complement of $Z(f)(\mathbb{C})$, has a natural extension to a discontinuous function on all of $M(\mathbb{C})$.

The following theorem is due to Borcherds [Bo1] and Bruinier [Br]. Borcherds studied the functions $\Phi(f)$ only for weakly holomorphic forms $f \in M^!_{2-n}(\omega_L)$, and Bruinier subsequently extended the ideas to harmonic forms. Both Borcherds and Bruinier worked in the context of orthogonal Shimura varieties, but the unity case is almost identical.

Theorem 3.3.1. *The function $\Phi(f)$ is a Green function for the divisor $Z(f)$ on M. Furthermore, at any point $z_0 \in M(\mathbb{C})$, including points of $Z(f)(\mathbb{C})$, the value of $\Phi(z_0,f)$ can be computed as*

$$\Phi(z_0,f) = \lim_{z\to z_0} \left(\Phi(z,f) + \sum_{\substack{m\in\mathbb{Q} \\ m>0}} \sum_{\substack{\lambda\in L \\ \langle\lambda,\lambda\rangle = m \\ \lambda\perp z_0}} a_f^+(-m,\lambda)\cdot \log|4\pi e^{\gamma}\langle\lambda_z,\lambda_z\rangle| \right)$$

for any lift $z_0 \in \mathcal{D}_L$, where the limit is over $z \notin Z(f)(\mathbb{C})$.

In the special case of $f = f_m$, the theorem says that $\Phi(z_0,f_m)$ is a Green function for the divisor $Z(m)$, and that

$$\Phi(z_0,f_m) = \lim_{z\to z_0} \left(\Phi(z,f_m) + \sum_{\substack{\lambda\in L \\ \langle\lambda,\lambda\rangle = m \\ \lambda\perp z_0}} \log|4\pi e^{\gamma}\langle\lambda_z,\lambda_z\rangle| \right), \tag{3.3.1}$$

where the limit is over $z \notin Z(m)(\mathbb{C})$. This is actually quite natural. Recall that

$$Z(m)(\mathbb{C}) = \sum_{\substack{\lambda\in L \\ \langle\lambda,\lambda\rangle = m}} \mathcal{D}_L(\lambda)$$

as divisors on $\mathcal{D}_L$. The only $\mathcal{D}_L(\lambda)$ that pass through z_0 are those with $\lambda \perp z_0$, and so in a neighborhood of z_0 we have

$$Z(m)(\mathbb{C}) = \sum_{\substack{\lambda\in L \\ \langle\lambda,\lambda\rangle = m \\ \lambda\perp z_0}} \mathcal{D}_L(\lambda).$$

It's not hard to see that the function $z \mapsto -\log|4\pi e^{\gamma}\langle\lambda_z,\lambda_z\rangle|$ has a logarithmic singularity along $\mathcal{D}_L(\lambda)$, and the above formula just says that we can compute the value $\Phi(z_0,f_m)$ by subtracting of from $\Phi(z,f_m)$ a function with the same type of singularity near z_0, and then taking the limit as $z \mapsto z_0$.

For every $f \in H_{2-n}(\omega_L)^{\Delta}$, we now have an arithmetic divisor with complex coefficients

$$\widehat{Z}(f) = (Z(f),\Phi(f))$$

which we also view as a metrized line bundle on M. We also define

$$\widehat{\Theta}(f) = \widehat{Z}(f) + a_f^+(0,0)\cdot \widehat{T} \in \widehat{\mathrm{Pic}}(M)_{\mathbb{C}},$$

where $a_f^+(0,0)$ is the value of the function $a_f^+(0) \in S_L^{\Delta}$ at the coset $0 \in \delta^{-1}L/L$.

3.4. **Rankin-Selberg L-functions.** Recall that we have fixed an isomorphism

$$L \cong L_0 \oplus \Lambda.$$

Exactly as with L, we may form the spaces S_{L_0} and S_Λ of complex-valued functions on $\delta^{-1}L_0/L_0$ and $\delta^{-1}\Lambda/\Lambda$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on these spaces by Weil representations ω_{L_0} and ω_Λ.

Let S_Λ be the space of all $\mathbb{C}$-valued functions on $\delta^{-1}\Lambda/\Lambda$. For each $\varphi \in S_\Lambda$ define the representation number

$$R_\Lambda(m,\varphi) = \sum_{\substack{\lambda\in\delta^{-1}\Lambda \\ \langle\lambda,\lambda\rangle = m}} \varphi(\lambda).$$

We view each $R_\Lambda(m)$ as an element of the dual space $S_\Lambda^\vee$, and assemble them into the weight $n-1$ theta series

$$\vartheta_\Lambda(\tau) = \sum_{m\in\mathbb{Q}} R_\Lambda(m)\cdot q^m \in M_{n-1}(\omega_\Lambda^\vee).$$

A function on $\delta^{-1}\Lambda/\Lambda$ extends to a function on

$$\delta^{-1}L/L = \delta^{-1}L_0/L_0 \oplus \delta^{-1}\Lambda/\Lambda,$$

trivial off of $\delta^{-1}L_0/L_0$, thereby inducing maps $S_\Lambda \to S_L$ and $S_\Lambda^\vee \to S_L^\vee$. Thus we may view ϑ_Λ as an element

$$\vartheta_\Lambda \in M_{n-1}(\omega_L^\vee).$$

For any cusp form

$$F = \sum_m b_F(m)\cdot q^m \in S_n(\overline{\omega}_L)^\Delta$$

it therefore makes sense to form the *convolution L-function*

$$L(F,\vartheta_\Lambda,s) = \Gamma\left(\frac{s}{2}+n-1\right)\sum_{m\geq 0}\frac{\{\overline{b_F(m)},R_\Lambda(m)\}}{m^{\frac{s}{2}+n-1}}.$$

Define a nonholomorphic $S_{L_0}^\vee$-valued Eisenstein series

$$E(\tau,s,\varphi) = \sum_{\gamma\in B\backslash\mathrm{SL}_2(\mathbb{Z})}\big(\omega_{L_0}(\gamma)\varphi\big)(0)\cdot\frac{\mathrm{Im}(\gamma\tau)^{s/2}}{(c\tau+d)}$$

of weight 1 and representation $\omega_{L_0}^\vee$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ and B is the subgroup of upper-triangular matrices. Using the isomorphism

$$S_L^\vee \cong S_{L_0}^\vee \otimes S_\Lambda^\vee$$

induced by $L \cong L_0\oplus\Lambda$, we can form a weight n real analytic $S_L^\vee$-valued modular form

$$E(\tau,s)\otimes\vartheta_\Lambda(\tau)$$

of representation $\omega_L^\vee$. The usual Rankin-Selberg unfolding method shows that

$$L(F,\vartheta_\Lambda,s) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}}\left\{\overline{F(\tau)},E(\tau,s)\otimes\vartheta_\Lambda(\tau)\right\}\,v^{n-2}du\,dv.$$

As the Eisenstein series satisfies the functional equation $E(\tau,-s) = -E(\tau,s)$, the convolution L-function satisfies

$$L(F,\vartheta_\Lambda,-s) = -L(F,\vartheta_\Lambda,s)$$

and

$$L'(F,\vartheta_\Lambda,0)=\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}}\left\{\overline{F(\tau)},E'(\tau,0)\otimes\vartheta_\Lambda(\tau)\right\}\,v^{n-2}du\,dv.$$

3.5. **The main result.** We need one more ingredient before we state our main result: Bruinier and Funke [BF] show that there is a short exact sequence

$$0\to M^!_{2-n}(\omega_L)\to H_{2-n}(\omega_L)\xrightarrow{\xi}S_n(\overline{\omega}_L)\to 0$$

where ξ is the $\Delta-$ *invariant* conjugate-linear differential operator

$$\xi(f)=2iv^{2-n}\cdot\overline{\frac{\partial f}{\partial\overline{\tau}}}.$$

Theorem 3.5.1. *Fix any Δ-invariant $f\in H_{2-n}(\omega_L)$ and set $F=\xi(f)$. The central derivative of $L(F,\vartheta_\Lambda,s)$ satisfies*

$$[\widehat{\Theta}(f):Y_\Lambda]=\frac{-1}{|\mathrm{Aut}(y_0)|}\cdot L'(F,\vartheta_\Lambda,0). \tag{3.5.1}$$

Conjecture 3.5.2. *If $\xi(f)=0$ then $\widehat{\Theta}(f)=0$.*

Assuming the conjecture, the formation of $f\mapsto\widehat{\Theta}(f)$ factors through the differential operator ξ to yield a conjugate-linear map

$$\widehat{\Theta}:S_n(\overline{\omega}_L)^\Delta\to\widehat{\mathrm{Pic}}(M)_{\mathbb{C}},$$

and the theorem can be restated as

$$[\widehat{\Theta}(F):Y_\Lambda]=\frac{-1}{|\mathrm{Aut}(y_0)|}\cdot L'(F,\vartheta_\Lambda,0).$$

Moreover, the theorem gives some evidence for the conjecture, as the right hand side of (3.5.1) is obviously 0 if $\xi(f)=0$.

As both sides of (3.5.1) are linear in f, it suffices to treat the cases $f=f_m$ and $f=f_{m,\delta}$ (and, when $n=2$, the case of f a constant form). The goal of the rest of these notes is to treat the case $f=f_m$. The case $f=f_{m,\delta}$ can be treated in very much the same way. The case of a constant form follows from the CM value formula, which we now discuss.

3.6. **The CM value formula.** We will derive a formula for the value of the Green function $\Phi(f_m)$ at the CM point y_0, even when y_0 lies on the singularity of the Green function! This formula was first proved by Bruinier and Yang for Green functions for special divisors on orthogonal Shimura varieties (see also related work of Schofer [Scho]). The proof in the unitary case is virtually identical.

Let's go back to the real analytic $S_{L_0}^\vee$-valued weight 1 Eisenstein series

$$E(\tau,s,\varphi)=\sum_{\gamma\in B\backslash\mathrm{SL}_2(\mathbb{Z})}\big(\omega_{L_0}(\gamma)\varphi\big)(0)\cdot\frac{\mathrm{Im}(\gamma\tau)^{s/2}}{(c\tau+d)}$$

used in the Rankin-Selberg integral. Its derivative

$$\mathcal{E}(\tau)=E'(\tau,0)$$

is harmonic and transforms like a weight 1 modular form with representation $\omega_{L_0}^\vee$. It does not satisfy the correct growth condition required of forms in $H_1(\omega_{L_0}^\vee)$, but it still has decomposition $\mathcal{E}=\mathcal{E}^++\mathcal{E}^-$ into a holomorphic part and a nonholomorphic part. The coefficients of $\mathcal{E}(\tau)$ were computed by Shofer [Scho], and the holomorphic and nonholomorphic parts have the form

$$\mathcal{E}^+(\tau) = \sum_{m\ge 0} a^+_{\mathcal{E}}(m) q^m$$

and

$$\mathcal{E}^-(\tau) = \mathrm{ev}_0 \cdot \log(v) + \sum_{m<0} a^-_{\mathcal{E}}(m)\Gamma(0, 4\pi|m|v) q^m,$$

where $\mathrm{ev}_0 \in S^\vee_{L_0}$ is evaluation at $0 \in \delta^{-1}L_0/L_0$. In particular

$$a^+_{\mathcal{E}}(m) = \lim_{v\to\infty} a_{\mathcal{E}}(m, v)$$

for $m > 0$, and that

$$a^+_{\mathcal{E}}(0) = \lim_{v\to\infty} \big(a_{\mathcal{E}}(0, v) - \mathrm{ev}_0 \cdot \log(v)\big).$$

We now give Shofer's formulas for the coefficients $a^+_{\mathcal{E}}(m) \in S^\vee_{L_0}$. For the sake of simplicity we give only the value of $a^+_{\mathcal{E}}(m)$ at the characteristic function $\varphi_0 \in S_{L_0}$ of $0 \in \delta^{-1}L_0/L_0$. For a positive $m \in \mathbb{Q}$ define a finite set of rational primes

$$\mathrm{Diff}_{L_0}(m) = \{p < \infty : L_0 \otimes_{\mathcal{O}_{\boldsymbol{k}}} \boldsymbol{k} \text{ does not represent } m\}.$$

The set of places of $\mathbb{Q}$ at which $L_0 \otimes_{\mathcal{O}_{\boldsymbol{k}}} \boldsymbol{k}$ does not represent m has even cardinality, and includes the prime at ∞. Therefore $\mathrm{Diff}_{L_0}(m)$ has odd cardinality. Note that every prime in $\mathrm{Diff}_{L_0}(m)$ is nonsplit in $\boldsymbol{k}$. Define

$$\rho(m) = |\{\mathfrak{a} \subset \mathcal{O}_{\boldsymbol{k}} : \mathrm{N}(\mathfrak{a}) = m\}|.$$

In particular $\rho(m) = 0$ unless $m \in \mathbb{Z}$.

Proposition 3.6.1 (Schofer [Scho])**.** *For a nonnegative rational number m, the coefficient $a^+_{\mathcal{E}}(m, \varphi_0)$ is given by the following formulas.*

1. *If $m \notin \mathbb{Z}$, then $a^+_{\mathcal{E}}(m, \varphi_0) = 0$.*
2. *If $m > 0$ and $|\mathrm{Diff}_{L_0}(m)| > 1$, then $a^+_{\mathcal{E}}(m, \varphi_0) = 0$.*
3. *If $m > 0$ and $\mathrm{Diff}_{L_0}(m) = \{p\}$ for a single prime p, then*

$$a^+_{\mathcal{E}}(m, \varphi_0) = -|\mathcal{O}^\times_{\boldsymbol{k}}| \cdot \rho\left(\frac{mD}{p^\epsilon}\right) \cdot \mathrm{ord}_p(pm) \cdot \log(p),$$

where

$$\epsilon = \begin{cases} 1 & \text{if } p \text{ is inert in } \boldsymbol{k}, \\ 0 & \text{if } p \text{ is ramified in } \boldsymbol{k}. \end{cases}$$

4. *The constant term is*

$$a^+_{\mathcal{E}}(0, \varphi_0) = \gamma + \log\left|\frac{4\pi}{d_{\boldsymbol{k}}}\right| - 2\frac{L'(\chi_{\boldsymbol{k}}, 0)}{L(\chi_{\boldsymbol{k}}, 0)}.$$

Comparing the formula for $a^+_{\mathcal{E}}(0, \varphi_0)$ with the Chowla-Selberg formula immediately proves the following.

Corollary 3.6.2. *The metrized cotautological bundle satisfies*

$$[\widehat{T} : Y_\Lambda] = -\frac{1}{|\mathrm{Aut}(y_0)|} \cdot a^+_{\mathcal{E}}(0, \varphi_0).$$

Theorem 3.6.3 (CM value formula)**.** *The harmonic form f_m satisfies*

$$-\frac{\Phi(y_0,f_m)}{|\mathrm{Aut}(y_0)|}+\sum_{\substack{m_1,m_2\in\mathbb{Z}_{\geq 0}\\ m_1+m_2=m}}\frac{a^+_{\mathcal{E}}(m_1,\varphi_0)\cdot R_\Lambda(m_2,\varphi_0)}{|\mathrm{Aut}(y_0)|}$$
$$=\frac{L'(\xi(f_m),\vartheta_\Lambda,0)}{|\mathrm{Aut}(y_0)|}+a^+_{f_m}(0,0)\cdot[\widehat{T}:Y_\Lambda].$$

Here we are using the same notation for both the characteristic function $\varphi_0\in S_{L_0}$ of $0\in\delta^{-1}L_0/L_0$, and the characteristic function $\varphi_0\in S_\Lambda$ of $0\in\delta^{-1}\Lambda/\Lambda$.

Remark 3.6.4. In light of the CM value formula, to prove Theorem 3.5.1 in the case $f=f_m$ it suffices to prove

$$[\widehat{Z}(m):Y_\Lambda]=\frac{\Phi(y_0,f_m)}{|\mathrm{Aut}(y_0)|}-\sum_{\substack{m_1,m_2\in\mathbb{Z}_{\geq 0}\\ m_1+m_2=m}}\frac{a^+_{\mathcal{E}}(m_1,\varphi_0)\cdot R_\Lambda(m_2,\varphi_0)}{|\mathrm{Aut}(y_0)|}.$$

In order to prove the CM value formula, let's go back to the theta kernel

$$\vartheta_L:\mathcal{H}\times\mathcal{D}_L\to S_L^\vee$$

defined by

$$\vartheta_L(\tau,z,\varphi)=v\sum_{\lambda\in\delta^{-1}L}\varphi(\lambda)\cdot e^{2\pi i\langle\lambda_{z^\perp},\lambda_{z^\perp}\rangle\tau}\cdot e^{2\pi i\langle\lambda_z,\lambda_z\rangle\overline{\tau}},$$

used in the construction of the Green functions $\Phi(z,f)$, and think about what happens at the point $z=y_0$ corresponding to the splitting $L=L_0\oplus\Lambda$. Using the isomorphism

$$S_L\cong S_{L_0}\otimes S_\Lambda,$$

we may assume that our $\varphi\in S_L$ factors as $\varphi_1\otimes\varphi_2$, and then

$$\vartheta_L(\tau,y_0,\varphi)=v\sum_{\substack{\lambda_1\in\delta^{-1}L_0\\ \lambda_2\in\delta^{-1}\Lambda}}\varphi_1(\lambda_1)\varphi_2(\lambda_2)\cdot e^{2\pi i\langle\lambda_2,\lambda_2\rangle\tau}\cdot e^{2\pi i\langle\lambda_1,\lambda_1\rangle\overline{\tau}}$$
$$=\left(v\sum_{\lambda_1\in\delta^{-1}L_0}\varphi_1(\lambda_1)e^{2\pi i\langle\lambda_1,\lambda_1\rangle\overline{\tau}}\right)\cdot\left(\sum_{\lambda_2\in\delta^{-1}\Lambda}\varphi_2(\lambda_2)e^{2\pi i\langle\lambda_2,\lambda_2\rangle\tau}\right).$$

This factorization says precisely that, as functions

$$\mathcal{H}\to S_L^\vee\cong S_{L_0}^\vee\otimes S_\Lambda^\vee,$$

the value of ϑ_L at y_0 factors as

$$\vartheta_L(\tau,y_0)=\vartheta_{L_0}(\tau)\otimes\vartheta_\Lambda(\tau),$$

where $\vartheta_{L_0}:\mathcal{H}\to S_{L_0}^\vee$ is the weight -1 nonholomorphic modular form

$$\vartheta_{L_0}(\tau,\varphi)=v\sum_{\lambda\in\delta^{-1}L_0}\varphi(\lambda)e^{2\pi i\overline{\tau}\langle\lambda,\lambda\rangle}.$$

The proof of the CM value formula hinges on two properties of the real analytic $S_{L_0}^\vee$-valued Eisenstein series $G(\tau,s)$ of weight -1 defined by

$$G(\tau,s,\varphi)=\sum_{\gamma\in B\backslash\mathrm{SL}_2(\mathbb{Z})}(\omega_{L_0}(\gamma)\varphi)(0)\cdot(c\tau+d)\cdot\mathrm{Im}(\gamma\tau)^{\frac{s}{2}+1}.$$

The first property,

$$G(\tau,0) = 2\cdot\vartheta_{L_0}(\tau), \tag{3.6.1}$$

is a special case of the Siegel-Weil formula. The second,

$$-2\cdot\bar\partial\big(\mathcal{E}(\tau)d\tau\big) = G(\tau,0)\cdot\frac{du\wedge dv}{v^2}, \tag{3.6.2}$$

was first observed by Kudla [Ku1].

Lemma 3.6.5. *For any $f\in H_{2-n}(\omega_L)^{\Delta}$, we have*

$$\Phi(y_0,f) = \lim_{T\to\infty}\left(\int_{\mathcal{F}_T}\{f(\tau),\vartheta_L(\tau,y_0)\}\,\frac{du\wedge dv}{v^2} - \mathrm{CT}\{f^+,\vartheta_\Lambda\}\cdot\log(T)\right)$$

where

$$\mathrm{CT}\{f^+,\vartheta_\Lambda\} = \sum_{m\in\mathbb{Q}}\{a_f^+(-m),R_\Lambda(m)\}$$

is the constant term in the q-expansion of $\{f^+,\vartheta_\Lambda\}$.

Proof. Let $\mathcal{S}=\{\tau\in\mathcal{F}_T : u<1\}$, so that $\Phi(y_0,f)$ is the constant term at $s=0$ of

$$\int_{\mathcal{S}}\{f(\tau),\vartheta_L(\tau,y_0)\}\,\frac{du\wedge dv}{v^{s+2}} + \lim_{T\to\infty}\left(\int_{u=0}^{1}\int_{v=1}^{T}\{f(\tau),\vartheta_L(\tau,y_0)\}\frac{du\,dv}{v^{s+2}}\right).$$

Consider the integral

$$\int_{v=1}^{T}\int_{u=0}^{1}\{f(\tau),\vartheta_L(\tau,y_0)\}\frac{du\,dv}{v^{s+2}} = \int_{v=1}^{T}\int_{u=0}^{1}\{f,\vartheta_{L_0}\otimes\vartheta_\Lambda\}\frac{du\,dv}{v^{s+2}}. \tag{3.6.3}$$

The inner integral over u just picks out the constant term in the q-expansion of the integrand, and so (3.6.3) becomes

$$\sum_{m_1+m_2+m_3=0}\int_{v=1}^{T}\{a_f(m_1,v),a_{\vartheta_{L_0}}(m_2,v)\otimes R_\Lambda(m_3)\}\frac{dv}{v^{s+2}}.$$

The most interesting contribution comes from the sum

$$\begin{aligned}
&\sum_m\int_{v=1}^{T}\{a_f^+(-m),a_{\vartheta_{L_0}}(0,v)\otimes R_\Lambda(m)\}\frac{dv}{v^{s+2}}\\
&\quad= \sum_m\{a_f^+(-m),\varphi_0\otimes R_\Lambda(m)\}\int_{v=1}^{T}\frac{dv}{v^{s+1}}\\
&\quad= \mathrm{CT}\{f^+,\vartheta_\Lambda\}\cdot\int_{v=1}^{T}\frac{dv}{v^{s+1}}\\
&\quad= \mathrm{CT}\{f^+,\vartheta_\Lambda\}\cdot\frac{1-T^{-s}}{s},
\end{aligned}$$

whose constant term at $s=0$ is $\mathrm{CT}\{f^+,\vartheta_\Lambda\}\log(T)$. The sum of the remaining terms converges (as $T\to\infty$) uniformly on compact subsets to a *holomorphic* function of s. What this shows is that for a fixed T the function

$$\int_{\mathcal{F}_T}\{f(\tau),\vartheta_L(\tau,y_0)\}\,\frac{du\wedge dv}{v^{s+2}} - \mathrm{CT}\{f^+,\vartheta_\Lambda\}\cdot\frac{1-T^{-s}}{s} \tag{3.6.4}$$

extends holomorphically to a neighborhood of $s=0$, and as $T\to\infty$ these functions converge uniformly to a holomorphic function in that neighborhood. If we first let

$T \to \infty$ in (3.6.4), and then take the constant term at $s = 0$, the result is $\Phi(y_0, f)$. If we instead compute the constant term at $s = 0$ and then let $T \to \infty$, we get

$$\lim_{T\to\infty} \left(\int_{\mathcal{F}_T} \{f(\tau), \vartheta_L(\tau, y_0)\} \frac{du \wedge dv}{v^2} - \mathrm{CT}\{f^+, \vartheta_\Lambda\} \cdot \log(T) \right).$$

The lemma follows. □

Proof of the CM value formula. Fix any $f \in H_{2-n}(\omega_L)^\Delta$. The relations (3.6.1) and (3.6.2) imply

$$\begin{aligned} d\big[\mathcal{E}(\tau) \otimes \vartheta_\Lambda(\tau)\, d\tau\big] &= \overline{\partial}\big(\mathcal{E}(\tau)\, d\tau\big) \otimes \vartheta_\Lambda(\tau) \\ &= -\frac{1}{2} G(\tau, 0) \otimes \vartheta_\Lambda(\tau) \cdot \frac{du \wedge dv}{v^2} \\ &= -\vartheta_{L_0}(\tau) \otimes \vartheta_\Lambda(\tau) \cdot \frac{du \wedge dv}{v^2} \\ &= -\vartheta_L(\tau, y_0) \cdot \frac{du \wedge dv}{v^2}, \end{aligned}$$

combining this with Lemma 3.6.5 shows that

$$\begin{aligned} \Phi(y_0, f) &= \lim_{T\to\infty} \left(\int_{\mathcal{F}_T} \{f(\tau), \vartheta_L(\tau, y_0)\} \frac{du \wedge dv}{v^2} - \mathrm{CT}\{f^+, \vartheta_\Lambda\} \cdot \log(T) \right) \\ &= \lim_{T\to\infty} \left(- \int_{\mathcal{F}_T} \big\{f, d\big[\mathcal{E} \otimes \vartheta_\Lambda\, d\tau\big]\big\} - \mathrm{CT}\{f^+, \vartheta_\Lambda\} \cdot \log(T) \right). \end{aligned}$$

Directly from the definition of the differential operator ξ, a simple calculation shows

$$df \wedge d\tau = -\overline{\xi(f)} v^n \cdot \frac{du \wedge dv}{v^2},$$

and so

$$\begin{aligned} \Phi(y_0, f) = &- \lim_{T\to\infty} \int_{\mathcal{F}_T} \Big\{\overline{\xi(f)}, \mathcal{E} \otimes \vartheta_\Lambda\Big\} v^n \cdot \frac{du \wedge dv}{v^2} \\ &- \lim_{T\to\infty} \Big(\int_{\mathcal{F}_T} d\big\{f(\tau), \mathcal{E} \otimes \vartheta_\Lambda\, d\tau\big\} + \mathrm{CT}\{f^+, \vartheta_\Lambda\} \cdot \log(T) \Big). \end{aligned}$$

The first term is just

$$\lim_{T\to\infty} \int_{\mathcal{F}_T} \{\overline{\xi(f)}, \mathcal{E} \otimes \vartheta_\Lambda\} \cdot \frac{du \wedge dv}{v^{2-n}} = L'(\xi(f), \vartheta_\Lambda, 0).$$

For the second term we note that

$$\int_{\mathcal{F}_T} d\big\{f, \mathcal{E} \otimes \vartheta_\Lambda\, d\tau\big\} = -\int_0^1 \big\{f(u + iT), E'(u + iT, 0) \otimes \vartheta_\Lambda(u + iT)\big\}\, du,$$

and the integral on the right just picks out the constant term

$$\sum_m \sum_{m_1 + m_2 = m} \{a_f(-m, T), a_{\mathcal{E}}(m_1, T) \otimes R_\Lambda(m_2)\}.$$

in the q-expansion of the integrand. Therefore

$$\begin{aligned}
&\lim_{T\to\infty}\Big(\mathrm{CT}\{f^+,\vartheta_\Lambda\}\log(T)+\int_{\mathcal{F}_T} d\{f,\mathcal{E}\otimes\vartheta_\Lambda\,d\tau\}\Big)\\
&= \lim_{T\to\infty}\Big(\mathrm{CT}\{f^+,\vartheta_\Lambda\}\log(T)-\sum_m\sum_{m_1+m_2=m}\big\{a_f(-m,T),a_{\mathcal{E}}(m_1,T)\otimes R_\Lambda(m_2)\big\}\Big)\\
&= -\lim_{T\to\infty}\sum_m\big\{a_f(-m,T),\big(a_{\mathcal{E}}(0,T)-\mathrm{ev}_0\log(T)\big)\otimes R_\Lambda(m)\big\}\\
&\qquad -\lim_{T\to\infty}\sum_{m\neq 0}\sum_{m_1+m_2=m}\big\{a_f(-m,T),a_{\mathcal{E}}(m_1,T)\otimes R_\Lambda(m_2)\big\}\Big)\\
&= -\sum_m\sum_{m_1+m_2=m}\big\{a_f^+(-m),a_{\mathcal{E}}^+(m_1)\otimes R_\Lambda(m_2)\big\}.
\end{aligned}$$

Putting everything together, we find

$$\Phi(y_0,f)=-L'(\xi(f),\vartheta_\Lambda,0)+\sum_m\sum_{m_1+m_2=m}\big\{a_f^+(-m),a_{\mathcal{E}}^+(m_1)\otimes R_\Lambda(m_2)\big\}.$$

Now take $f=f_m$ and recall that

$$f_m^+=\varphi_0 q^{-m}+a_{f_m}^+(0)+o(1).$$

Our formula reduces to

$$\begin{aligned}
\Phi(y_0,f_m) &= -L'(\xi(f_m),\vartheta_\Lambda,0)\\
&\qquad+\big\{a_{f_m}^+(0),a_{\mathcal{E}}^+(0)\otimes R_\Lambda(0)\big\}+\sum_{m_1+m_2=m}\big\{\varphi_0,a_{\mathcal{E}}^+(m_1)\otimes R_\Lambda(m_2)\big\}\\
&= -L'(\xi(f_m),\vartheta_\Lambda,0)\\
&\qquad - a_{f_m}^+(0,0)[\widehat{T}:Y_\Lambda]\cdot|\mathrm{Aut}(y_0)|+\sum_{m_1+m_2=m} a_{\mathcal{E}}^+(m_1,\varphi_0)\otimes R_\Lambda(m_2,\varphi_0),
\end{aligned}$$

and the CM value formula is proved. □

4. Calculation of the arithmetic intersection

In this section we prove the formula

$$[\widehat{Z}(m):Y_\Lambda]=\frac{\Phi(y_0,f_m)}{|\mathrm{Aut}(y_0)|}-\sum_{\substack{m_1,m_2\in\mathbb{Z}_{\geq 0}\\ m_1+m_2=m}}\frac{a_{\mathcal{E}}^+(m_1,\varphi_0)\cdot R_\Lambda(m_2,\varphi_0)}{|\mathrm{Aut}(y_0)|}$$

of Remark 3.6.4, and so complete the proof of Theorem 3.5.1. Recall that

$$R_\Lambda(m,\varphi_0)=\sum_{\substack{\lambda\in\delta^{-1}\Lambda\\ \langle\lambda,\lambda\rangle=m}}\varphi_0(\lambda)=\sum_{\substack{\lambda\in\Lambda\\ \langle\lambda,\lambda\rangle=m}}1$$

is just the number of $\lambda\in\Lambda$ such that $\langle\lambda,\lambda\rangle=m$.

4.1. **Gross's calculation.** Fix a prime p nonsplit in $\boldsymbol{k}$, and let $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ be the unique prime above it. Denote by R the completion of the ring of integers of the maximal unramified extension of $\boldsymbol{k}_{\mathfrak{p}}$, and let $\mathbb{F} = R/\mathfrak{p}$ be its residue field.

Let $E \in M_{(1,0)}(\mathbb{F})$ be an elliptic curve with complex multiplication $\mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(E)$. In particular, E is supersingular. The canonical lifting theorem tells us that E admits a unique deformation

$$E_r \in M_{(1,0)}(R/\mathfrak{p}^r)$$

for every r.

Theorem 4.1.1 (Gross [Gr])**.** *Suppose $f : E \to E$ is a nonzero $\mathcal{O}_{\boldsymbol{k}}$-conjugate-linear endomorphism, and set*

$$r = \mathrm{ord}_p(p \deg(f)) \cdot \begin{cases} 1/2 & \text{if } p \text{ is inert in } \boldsymbol{k} \\ 1 & \text{if } p \text{ is ramified in } \boldsymbol{k}. \end{cases}$$

Then r is an integer, and f lifts to an endomorphism of E_r, but does not lift to an endomorphism of E_{r+1}. Equivalently, the formal deformation functor of the pair (E, f) is represented by the Artinian ring $R/\mathfrak{p}^r$.

4.2. **Decomposition of the intersection.** Consider the Cartesian diagram

$$\begin{tikzcd} Z(m) \cap Y_\Lambda \arrow[r] \arrow[d] & Y_\Lambda \arrow[d] \\ Z(m) \arrow[r] & M. \end{tikzcd}$$

Our goal is to decompose the intersection $Z(m) \cap Y_\Lambda$ into smaller, more manageable substacks. Given $m_1, m_2 \in \mathbb{Z}_{\geq 0}$, denote by $X_\Lambda(m_1, m_2)$ the algebraic stack over $\mathcal{O}_{\boldsymbol{k}}$ whose functor of points assigns to a connected $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid of tuples $(A_0, A_1, B, \lambda_1, \lambda_2)$ in which

- $(A_0, A_1, B) \in Y_\Lambda(S)$,
- $\lambda_1 \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A_1)$ satisfies $\langle \lambda_1, \lambda_1 \rangle = m_1$,
- $\lambda_2 \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B)$ satisfies $\langle \lambda_2, \lambda_2 \rangle = m_2$.

Proposition 4.2.1. *For every positive integer m there is an isomorphism of $\mathcal{O}_{\boldsymbol{k}}$-stacks*

$$Z(m) \cap Y_\Lambda \cong \bigsqcup_{\substack{m_1, m_2 \in \mathbb{Z}_{\geq 0} \\ m_1 + m_2 = m}} \mathtt{X}_\Lambda(m_1, m_2).$$

Proof. Let S be an $\mathcal{O}_{\boldsymbol{k}}$-scheme, and suppose we have an S-valued point of $Z(m) \cap Y_\Lambda$. This point consists of a triple $(A_0, A, \lambda) \in Z(m)(S)$ and a triple $(A_0', A_1, B) \in Y_\Lambda(S)$. These triples have the same image in $M(S)$, which means that we are given an isomorphism

$$(A_0, A) \cong (A_0', A_1 \times B)$$

in the category $M(S)$. If we use this isomorphism to identify $A_0 = A_0'$ and $A = A_1 \times B$, then the orthogonal decomposition

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A) \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A_1) \times \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B)$$

induces a decomposition $\lambda = \lambda_1 + \lambda_2$, with

$$\langle \lambda_1, \lambda_1 \rangle + \langle \lambda_2, \lambda_2 \rangle = \langle \lambda, \lambda \rangle = m.$$

If we set $m_1 = \langle \lambda_1, \lambda_1 \rangle$ and $m_2 = \langle \lambda_2, \lambda_2 \rangle$, we obtain a point

$$(A_0, A_1, B, \lambda_1, \lambda_2) \in X_\Lambda(m_1, m_2)(S).$$

The inverse construction is obvious. $\square$

Proposition 4.2.2. *Fix $m > 0$.*

(1) *If Λ does not represent m, then $X_\Lambda(0, m) = \emptyset$.*
(2) *If Λ does represent m, then $X_\Lambda(0, m)$ is nonempty, and is smooth of relative dimension 0 over $\mathcal{O}_{\boldsymbol{k}}$. In particular it is of dimension 1.*

Proof. A geometric point of $X_\Lambda(0, m)$ is a tuple $(A_0, A_1, B, \lambda_1, \lambda_2)$ where (A_0, A_1, B) is a point of Y_Λ, and

$$\lambda_1 \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A_1)$$

and

$$\lambda_2 \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B) \cong \Lambda$$

have Hermitian norms 0 and m, respectively (so $\lambda_1 = 0$). In particular, if Λ doesn't represent m then $X_\Lambda(0, m)$ has no geometric points.

The fact that $X_\Lambda(0, m)$ is smooth of relative dimension 0 follows from the canonical lifting theorem: every tuple $(A_0, A_1, B, 0, \lambda_2)$ has a unique deformation through a nilpotent thickening. To prove $X_\Lambda(0, m) \neq \emptyset$, assuming that Λ represents m, we construct a complex point. Just let $y_0 = (A_0, A_1, B) \in Y_\Lambda(\mathbb{C})$ be the unique complex point, and let $\lambda_2 \in \Lambda \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B)$ be an element of Hermitian norm m. Then

$$(A_0, A_1, B, 0, \lambda_2) \in X_\Lambda(m_1, m_2)(\mathbb{C}).$$

$\square$

4.3. **Degrees of zero cycles.** When $m_1 > 0$, each stack $X_\Lambda(m_1, m_2)$ defines a divisor on Y_Λ. In this subsection we compute its degree $\widehat{\deg}_{\mathrm{fin}} X_\Lambda(m_1, m_2)$, and compare with the earlier calculation of Fourier coefficients of $\mathcal{E}^+(\tau)$. Recall that for any positive $m \in \mathbb{Q}$ we defined a finite set

$$\mathrm{Diff}_{L_0}(m) = \{p < \infty : L_0 \otimes_{\mathcal{O}_{\boldsymbol{k}}} \boldsymbol{k} \text{ does not represent } m\}$$

of odd cardinality, all of whose elements are nonsplit in $\boldsymbol{k}$.

Theorem 4.3.1. *Fix $m_1, m_2 \in \mathbb{Z}$ with $m_1 > 0$ and $m_2 \geq 0$.*

(1) *If $|\mathrm{Diff}_{L_0}(m_1)| > 1$, then $X_\Lambda(m_1, m_2) = \emptyset$.*
(2) *If $\mathrm{Diff}_{L_0}(m_1) = \{p\}$, then $X_\Lambda(m_1, m_2)$ has dimension 0 and is supported in characteristic p. Furthermore, the étale local ring of every geometric point $z \in X_\Lambda(m_1, m_2)(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}})$ has length*

$$\mathrm{length}(\mathcal{O}^{et}_{X_\Lambda(m_1,m_2),z}) = \mathrm{ord}_p(pm_1) \cdot \begin{cases} 1/2 & \text{if } p \text{ is inert in } \boldsymbol{k}, \\ 1 & \text{if } p \text{ is ramified in } \boldsymbol{k}, \end{cases}$$

and the number of geometric points of $X(m_1, m_2)$, counted with multiplicities, is

$$\sum_{z \in X_\Lambda(m_1,m_2)(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}})} \frac{1}{|\mathrm{Aut}(z)|} = \frac{R_\Lambda(m_2, \varphi_0)}{|\mathcal{O}_{\boldsymbol{k}}^\times \times \mathrm{Aut}(\Lambda)|} \cdot \rho(m_1/p^\epsilon)$$

where $\mathfrak{p}$ is the unique prime of $\boldsymbol{k}$ above p, $\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$ is an algebraic closure of its residue field,

$$\rho(m) = |\{\mathfrak{a} \subset \mathcal{O}_{\boldsymbol{k}} : \mathrm{N}(\mathfrak{a}) = m\}|,$$

and

$$\epsilon = \begin{cases} 1 & \text{if } p \text{ is inert in } \boldsymbol{k}, \\ 0 & \text{if } p \text{ is ramified in } \boldsymbol{k}. \end{cases}$$

Proof. Let $\mathbb{F}$ be any algebraically closed field, and suppose

$$z = (A_0, A_1, B, \lambda_1, \lambda_2) \in X_\Lambda(m_1, m_2)(\mathbb{F})$$

is a geometric point. In particular $A_0 \in M_{(1,0)}(\mathbb{F})$ and $A_1 \in M_{(0,1)}(\mathbb{F})$. Recall that each of $M_{(1,0)}$ and $M_{(0,1)}$ has a unique geometric point in every characteristic. It follows from this that the underlying elliptic curves A_0 and A_1 are $\mathcal{O}_{\boldsymbol{k}}$-conjugate-linearly isomorphic. If we fix one such conjugate linear isomorphism $A_0 \cong A_1$, then $\lambda_1 \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A_1)$ can be viewed as a $\mathcal{O}_{\boldsymbol{k}}$-conjugate-linear endomorphism $\lambda_1 \in \mathrm{End}(A_0)$ of degree m_1. This already that $\mathbb{F}$ has characteristic $p > 0$, and that A_0 is supersingular. Using the action $\mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A_0)$ we find a decomposition

$$\mathrm{End}(A_0)_{\mathbb{Q}} = \boldsymbol{k} \oplus \boldsymbol{k}\lambda_1.$$

Using the fact that $\mathrm{End}(A_0)_{\mathbb{Q}}$ is the quaternion algebra ramified at ∞ and p, a routine calculation shows that the $\boldsymbol{k}$-vector space $\boldsymbol{k}\lambda_1$, with its Hermitian form

$$\langle x\lambda_1, y\lambda_1 \rangle = \deg(x\lambda_1 y\lambda_1) = x\overline{y}\deg(\lambda_1),$$

is isomorphic to $L_{0\mathbb{Q}}$ everywhere locally *except* at ∞ and p. As $\boldsymbol{k}\lambda_1$ represesnts $m_1 = \deg(\lambda_1)$ everywhere locally, it follows that $L_{0\mathbb{Q}}$ represents m_1 everywhere locally *except* at ∞ and p. Therefore $\mathrm{Diff}_{L_0}(m_1) = \{p\}$.

Now take $\mathbb{F} = \mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$. The (completed) étale local ring at the point z pro-represents the formal deformation functor of the tuple $(A_0, A_1, B, \lambda_1, \lambda_2)$. The canonical lifting theorem tells us that everything in the tuple, except λ_1, deforms *uniquely* through any Artinian thickening of $\mathbb{F}$. If we fix an $\mathcal{O}_{\boldsymbol{k}}$-conjugate linear isomorphism $A_0 \cong A_1$, the completed étale local ring at z therefore pro-represents the deformation functor of (A_0, λ_1). This length is precisely what Gross's theorem computes, and we obtain the desired formula for $\mathrm{length}(\mathcal{O}^{et}_{X_\Lambda(m_1,m_2),z})$.

Finally, we must count the number of points $(A_0, A_1, B, \lambda_1, \lambda_2) \in X_\Lambda(m_1, m_2)(\mathbb{F})$. As noted before, Y_Λ has a unique point in characteristic 0, and so the canonical lifting theorem implies that it has a unique $\mathbb{F}$-point, (A_0, A_1, B). How many choices are there for λ_1 and λ_2? As $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B) \cong \Lambda$, there are

$$R_\Lambda(m_2, \varphi_0) = |\{\lambda_2 \in \Lambda : \langle \lambda_2, \lambda_2 \rangle = m_2\}|$$

choices for λ_2. The number of choices for λ_1 is

$$|\{\lambda_1 \in \mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(A_0) : \deg(\lambda_1) = m_1\}|,$$

where $\mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(A_0)$ is the $\mathcal{O}_{\boldsymbol{k}}$-submodule of conjugate-linear endomorphisms. As a quadratic space, $\mathrm{End}_{\overline{\mathcal{O}}_{\boldsymbol{k}}}(A_0)$ with its degree form is isomorphic to $\mathcal{O}_{\boldsymbol{k}}$ with the quadratic form $Q(x) = p^\epsilon x\overline{x}$, and hence the number of choices for λ_1 is

$$|\{x \in \mathcal{O}_{\boldsymbol{k}} : p^\epsilon x\overline{x} = m_1\}| = \rho(m_1/p^\epsilon) \cdot |\mathcal{O}_{\boldsymbol{k}}^\times|.$$

This gives

$$R_\Lambda(m_2, \varphi_0) \cdot \rho(m_1/p^\epsilon) \cdot |\mathcal{O}_{\boldsymbol{k}}^\times|$$

ways to extend (A_0, A_1, B) to a point of $X_\Lambda(m_1, m_2)(\mathbb{F})$, but some of these extensions are isomorphic: if $\xi_1 \in \mathcal{O}_{\boldsymbol{k}}^\times$ and $\xi_2 \in \mathrm{Aut}(\Lambda)$ then

$$(A_0, A_1, B, \lambda_1, \lambda_2) \cong (A_0, A_1, B, \xi_1 \circ \lambda_1, \xi_2 \circ \lambda_2).$$

Counting tuples up to isomorphism,

$$|X_\Lambda(m_1, m_2)(\mathbb{F})| = \frac{R_\Lambda(m_2, \varphi_0) \cdot \rho(m_1/p^\epsilon)}{\mathrm{Aut}(\Lambda)},$$

and each point of $X_\Lambda(m_1, m_2)(\mathbb{F})$ has automorphism group $\mathcal{O}_{\boldsymbol{k}}^\times$. Thus

$$\sum_{x \in X_\Lambda(m_1, m_2)(\mathbb{F}_\mathfrak{p}^{\mathrm{alg}})} \frac{1}{|\mathrm{Aut}(x)|} = \frac{R_\Lambda(m_2, \varphi_0) \cdot \rho(m_1/p^\epsilon)}{|\mathcal{O}_{\boldsymbol{k}}^\times \times \mathrm{Aut}(\Lambda)|}.$$

□

Corollary 4.3.2. *For any $m_1, m_2 \in \mathbb{Z}$ with $m_1 > 0$ and $m_2 \geq 0$, we have*

$$\widehat{\deg}_{\mathrm{fin}} X_\Lambda(m_1, m_2) = -\frac{a_{\mathcal{E}}^+(m_1, \varphi_0) R_\Lambda(m_2, \varphi_0)}{|\mathrm{Aut}(y_0)|}.$$

Proof. Compare Theorem 4.3.1 with Proposition 3.6.1. If $|\mathrm{Diff}_{L_0}(m_1)| > 1$ then both sides are 0. If $\mathrm{Diff}_{L_0}(m_1) = \{p\}$ then

$$-\frac{a_{\mathcal{E}}^+(m_1, \varphi_0) R_\Lambda(m_2, \varphi_0)}{|\mathrm{Aut}(y_0)|} = \frac{\mathrm{ord}_p(pm_1) R_\Lambda(m_2, \varphi_0)}{|\mathcal{O}_{\boldsymbol{k}}^\times \times \mathrm{Aut}(\Lambda)|} \cdot \rho\left(\frac{m_1 D}{p^\epsilon}\right) \cdot \log(p),$$

while

$$\widehat{\deg}_{\mathrm{fin}} X_\Lambda(m_1, m_2) = \frac{\mathrm{ord}_p(pm_1) R_\Lambda(m_2, \varphi_0)}{|\mathcal{O}_{\boldsymbol{k}}^\times \times \mathrm{Aut}(\Lambda)|} \cdot \rho\left(\frac{m_1}{p^\epsilon}\right) \cdot \log(p).$$

We leave it to the reader to show that the right hand sides are equal. □

4.4. **The case of proper intersection.** Although not logically necessary, we now have enough information to prove our main result under the simplifying hypothesis that Λ does not represent m. Recall (Proposition 2.5.2) that this is precisely the case where the unique complex point of Y_Λ does not lie on $Z(m)$. Moreover, Proposition 4.2.2 implies $X_\Lambda(0, m) = \emptyset$, and so the decomposition of Proposition 4.2.1 simplifies to

$$Z(m) \cap Y_\Lambda \cong \bigsqcup_{\substack{m_1, m_2 \in \mathbb{Z}_{\geq 0} \\ m_1 + m_2 = m \\ m_1 > 0}} \mathtt{X}_\Lambda(m_1, m_2),$$

which has dimension 0 by Theorem 4.3.1. Furthermore, Corollary 4.3.2 implies

$$\widehat{\deg}_{\mathrm{fin}} \big(Z(m) \cap Y_\Lambda\big) = -\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{\geq 0} \\ m_1 + m_2 = m \\ m_1 > 0}} \frac{a_{\mathcal{E}}^+(m_1, \varphi_0) R_\Lambda(m_2, \varphi_0)}{|\mathrm{Aut}(y_0)|}.$$

Adding $\Phi(y_0, f_m)/|\mathrm{Aut}(y_0)|$ to both sides proves

$$[\widehat{Z}(m) : Y_\Lambda] = \frac{\Phi(y_0, f_m)}{|\mathrm{Aut}(y_0)|} - \sum_{\substack{m_1, m_2 \in \mathbb{Z}_{\geq 0} \\ m_1 + m_2 = m \\ m_1 > 0}} \frac{a_{\mathcal{E}}^+(m_1, \varphi_0) \cdot R_\Lambda(m_2, \varphi_0)}{|\mathrm{Aut}(y_0)|}.$$

As we assume $R_\Lambda(m,\varphi_0)=0$, we can add the term $(m_1,m_2)=(0,m)$ into the sum without changing its value, leaving us with

$$\begin{aligned}[\widehat{Z}(m):Y_\Lambda] &= \frac{\Phi(y_0,f_m)}{|\mathrm{Aut}(y_0)|} - \frac{1}{|\mathrm{Aut}(y_0)|}\sum_{\substack{m_1,m_2\in\mathbb{Z}_{\ge 0}\\ m_1+m_2=m}} a^+_{\mathcal{E}}(m_1,\varphi_0)\cdot R_\Lambda(m_2,\varphi_0)\\ &= -\frac{L'(\xi(f_m),\vartheta_\Lambda,0)}{|\mathrm{Aut}(y_0)|} - a^+_{f_m}(0,0)\cdot[\widehat{T}:Y_\Lambda],\end{aligned}$$

where the second equality is precisely the CM value formula. Thus

$$[\widehat{\Theta}(f_m):Y_\Lambda] = -\frac{1}{|\mathrm{Aut}(y_0)|}\cdot L'(\xi(f_m),\vartheta_\Lambda,0),$$

as desired.

4.5. **The adjunction formula.** As pointed out before, the morphisms $Y_\Lambda\to M$ and $Z(m)\to M$ are not closed immersions, but over an étale cover of M they become the next best thing. A *sufficiently small étale open subscheme* of M is a scheme $\mathcal{U}$ and an étale morphism $\mathcal{U}\to M$ such that

- the base change $Z(m)_{/\mathcal{U}}\to\mathcal{U}$ restricts to a closed immersion on each connected component of $Z(m)_{/\mathcal{U}}$,
- the base change $Y_{\Lambda/\mathcal{U}}\to\mathcal{U}$ restricts to a closed immersion on each connected component of $Y_{\Lambda/\mathcal{U}}$,
- the universal triple (A_0,A_1,B) over $Y_{\Lambda/\mathcal{U}}\to\mathcal{U}$ satisfies
$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0,B)\cong\Lambda.$$

The stack M admits a cover by sufficiently small étale open subschemes.

Fix a sufficiently small étale open subscheme Y, and one connected component

$$\mathcal{Y}\subset Y_{\Lambda/\mathcal{U}}.$$

We view $\mathcal{Y}$ and all connected components of $Z(m)_{/\mathcal{U}}$ as closed subschemes of $\mathcal{U}$. Note that as $\mathcal{Y}$ is smooth over $\mathcal{O}_{\boldsymbol{k}}$ and connected, it is reduced and irreducible. Thus every connected component of $Z(m)_{/\mathcal{U}}$ either contains $\mathcal{Y}$, or intersects $\mathcal{Y}$ in dimension 0.

Proposition 4.5.1. *There are exactly $R_\Lambda(m,\varphi_0)$ connected components of $Z(m)_{/\mathcal{U}}$ that contain $\mathcal{Y}$.*

Proof. Because $\mathcal{Y}$ is flat over $\mathcal{O}_{\boldsymbol{k}}$, these connected components can be counted in characteristic 0. Recall that the pullback of $Z(m)$ to $\mathcal{D}_L$ is

$$Z(m)(\mathbb{C})\cong\bigsqcup_{\substack{\lambda\in L\\ \langle\lambda,\lambda\rangle=m}}\mathcal{D}_L(\lambda)$$

where $\mathcal{D}_L(\lambda)\subset\mathcal{D}_L$ is the divisor of negative lines othogonal to λ. Under the isomorphism $L\cong L_0\oplus\Lambda$ the point y_0 is just the negative line $L_{0\mathbb{C}}\subset L_{\mathbb{C}}$, *i.e.* the orthononal complement of $\Lambda_{\mathbb{C}}$. Thus $y_0\in\mathcal{D}_L(\lambda)$ if and only if $\lambda\in\Lambda$, and so in a small neighboorhood of y_0 we have

$$Z(m)(\mathbb{C})=\bigsqcup_{\substack{\lambda\in\Lambda\\ \langle\lambda,\lambda\rangle=m}}\mathcal{D}_L(\lambda).$$

In this neighborhood, the $\mathcal{D}_L(\lambda)$'s appearing in the disjoint union are exactly the connected components of $Z(m)(\mathbb{C})$ passing through y_0, and there are visibly $R_\Lambda(m, \varphi_0)$ such components. $\square$

Theorem 4.5.2 (The adjunction isomorphism)**.** *Suppose $\mathcal{Z} \subset Z(m)_{/\mathcal{U}}$ is a connected component, and denote by $\mathcal{O}(\mathcal{Z})$ the line bundle on $\mathcal{U}$ defined by the divisor $\mathcal{Z}$. If $\mathcal{Y} \subset \mathcal{Z}$, then there is a canonical isomorphism*

$$\mathcal{O}(\mathcal{Z})_{/\mathcal{Y}} \cong T_{/\mathcal{Y}}$$

of line bundles on $\mathcal{Y}$.

Let's first describe the adjunction isomorphism using the complex uniformization (2.4.1). As noted above, a typical connected component of $\mathbb{Z}(m)(\mathbb{C})$ passing through y_0 has the form

$$\mathcal{Z} = \mathcal{D}_L(\lambda)$$

for some $\lambda \in L$. On the one hand, the associated line bundle $\mathcal{O}(\mathcal{Z})$ on $\mathcal{D}_L$ has a canonical section $\sigma(\mathcal{Z})$ with divisor $\mathcal{Z}$, corresponding to the constant function 1 in $\mathcal{O}(\mathcal{D}_L)$. On the other hand, recall from Proposition 2.7.1 the isomorphism

$$T_z \cong \mathrm{Hom}_{\mathbb{C}}(z, \mathbb{C})$$

at each point $z \in \mathcal{D}_L$. The vector $\lambda \in L$ determines a linear functional

$$\mathrm{obst}(\lambda)_z = \langle \cdot, \lambda \rangle \in \mathrm{Hom}_{\mathbb{C}}(z, \mathbb{C}),$$

and varying z defines a holomorphic section $\mathrm{obst}(\lambda)$ of T with divisor $\mathcal{D}_L(\lambda)$. There is therefore a unique isomorphism of line bundles

$$\mathcal{O}(\mathcal{Z}) \cong T$$

on $\mathcal{D}_L$ identifying $\sigma(\mathcal{Z}) = \mathrm{obst}(\lambda)$. The adjunction isomorphism is defined by restricting this isomophism to fibers at y_0. For future reference, we note that the section $\mathrm{obst}(\lambda)$ has norm

$$||\mathrm{obst}(\lambda)||_z^2 = -4\pi e^{\gamma} \cdot \langle \lambda_z, \lambda_z \rangle.$$

To extend this isomorphism to integral models, we need to translate the construction of $\mathrm{obst}(\lambda)$ into the language of moduli problems. Let (A_0, A) be the pullback to $\mathcal{D}_L$ of the universal pair over $M(\mathbb{C})$. As $z \in \mathcal{D}_L$ varies, the fiber $A_{0,z}$ is constant and is just the elliptic curve $\mathbb{C}/\mathcal{O}_{\boldsymbol{k}}$. In particular $H_1(A_{0,z}(\mathbb{C}), \mathbb{Z}) \cong \mathcal{O}_{\boldsymbol{k}}$. The abelian variety A_z is described in the proof of Proposition 2.1.2, and has first homology $H_1(A_z(\mathbb{C}), \mathbb{Z}) \cong L$. Identifying $L \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathcal{O}_{\boldsymbol{k}}, L)$ in the obvious way, we find an isomorphism

$$L \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(H_1(A_{0,z}(\mathbb{C}), \mathbb{Z}), H_1(A_z(\mathbb{C}), \mathbb{Z})\big).$$

The deRham comparison theorem therefore identifies

$$L \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(H_1^{dR}(A_{0,z}), H_1^{dR}(A_z)\big)$$

at each z, and as z varies these isomorphisms arise from an isomorphism of $\mathcal{O}_{\mathcal{D}_L}$-modules

$$L \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{D}_L} \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(H_1^{dR}(A_0), H_1^{dR}(A)\big).$$

Moreover, the Gauss-Manin connection ∇ on the right is, virtually by definition, the unique connection under which each $\lambda \in L$ defines a *parallel* section $\lambda \otimes 1$. At each $z \in \mathcal{D}_L$ there is a commutative diagram

$$\begin{tikzcd}
L \arrow[rr, "\lambda \mapsto \langle \cdot, \lambda \rangle"] \arrow[d] & & \mathrm{Hom}_{\mathbb{C}}(z, \mathbb{C}) \arrow[d] \\
\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(H_1^{dR}(A_0), H_1^{dR}(A)\big)_z \arrow[r] & \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(\mathrm{Fil}^1(A_0), \mathrm{Lie}(A)/\mathcal{F}\big)_z \arrow[r] & T_z
\end{tikzcd}$$

Now suppose we have a connected component $\mathcal{Z}$ of $Z(m)(\mathbb{C})$ passing through y_0. Resticting the universal object over $Z(m)$ to the point y_0 yields the triple $(A_{0,y_0}, A_{y_0}, \lambda_{y_0})$ for some $\mathcal{O}_{\boldsymbol{k}}$-linear map

$$\lambda_{y_0} \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0,y_0}, A_{y_0}).$$

This λ_{y_0} then determines a vector in the fiber

$$\lambda_{y_0} \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(H_1^{dR}(A_0), H_1^{dR}(A)\big)_{y_0}$$

which we can parallel transport to a section λ of $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(H_1^{dR}(A_0), H_1^{dR}(A)\big)$ defined in a neighborhood of y_0. The commutative diagram above shows that the image of λ under the natural map

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(H_1^{dR}(A_0), H_1^{dR}(A)\big) \to \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(\mathrm{Fil}^1(A_0), \mathrm{Lie}(A)/\mathcal{F}\big) \cong T$$

is precisely the section $\mathrm{obst}(\lambda)$ defined above.

To complete the construction of the adjunction isomorphism on integral models, all we have to do is replace the parallel transport by its deformation theoretic analogue.

Proof of Theorem 4.5.2. Let $\mathcal{Z}$ be a connected component of $Z(m)_{/\mathcal{U}}$ containing $\mathcal{Y}$. If $I_{\mathcal{Y}} \subset \mathcal{O}_{\mathcal{U}}$ denotes the ideal sheaf defining $\mathcal{Y}$, then $I_{\mathcal{Y}}^2$ is the ideal sheaf of a closed subscheme $\tilde{\mathcal{Y}} \subset \mathcal{U}$, called the *first order infinitesimal neighborhood of* $\mathcal{Y}$. The picture is

$$\begin{tikzcd}
 & \tilde{\mathcal{Y}} \arrow[dr] & \\
\mathcal{Y} \arrow[ur] \arrow[dr] & & \mathcal{U} \\
 & \mathcal{Z}. \arrow[ur] &
\end{tikzcd}$$

The universal pair over M pulls back to a pair $(\tilde{A}_0, \tilde{A})$ over $\tilde{\mathcal{Y}}$, and the universal triple over $Z(m)$ pulls back to a triple over $\mathcal{Y}$, denoted (A_0, A, λ). Of course (A_0, A) is nothing more than the restriction of $(\tilde{A}_0, \tilde{A})$ to $\mathcal{Y}$. The map $\lambda \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ induces a map

$$\lambda \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(H_1^{dR}(A_0), H_1^{dR}(A)\big)$$

on deRham cohomology, which by deformation theory has a canonical extension (parallel transport!) to

$$\tilde{\lambda} \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(H_1^{dR}(\tilde{A}_0), H_1^{dR}(\tilde{A})\big).$$

Define the *obstruction to deforming* λ,

$$\mathrm{obst}(\lambda) \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathrm{Fil}^1(\tilde{A}_0), \mathrm{Lie}(\tilde{A})/\mathcal{F})$$

to be the composition

$$\mathrm{Fil}^1(\tilde{A}_0) \to H_1^{dR}(\tilde{A}_0) \xrightarrow{\tilde{\lambda}} H_1^{dR}(\tilde{A}) \to \mathrm{Lie}(\tilde{A})/\mathcal{F},$$

and view $\mathrm{obst}(\lambda)$ as a section of $T_{/\tilde{\mathcal{Y}}}$.

Lemma 4.5.3. *The maximal closed subscheme of* $\tilde{\mathcal{Y}}$ *on which* $\mathrm{obst}(\lambda) = 0$ *is* $\tilde{\mathcal{Y}} \cap \mathcal{Z}$.

Proof. Define $\mathrm{obst}^*(\lambda)$ to be the composition

$$\mathrm{Fil}^1(\tilde{A}_0) \to H_1^{dR}(\tilde{A}_0) \xrightarrow{\tilde{\lambda}} H_1^{dR}(\tilde{A}) \to \mathrm{Lie}(\tilde{A}),$$

so that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Fil}^1(\tilde{A}_0) \arrow[r, "\mathrm{obst}^*(\lambda)"] \arrow[dr, "\mathrm{obst}(\lambda)"'] & \mathrm{Lie}(\tilde{A}) \arrow[d] \\ & \mathrm{Lie}(\tilde{A})/\mathcal{F}. \end{tikzcd}$$

of coherent sheaves on $\tilde{Y}$. The largest closed subscheme of $\tilde{\mathcal{Y}}$ on which $\mathrm{obst}^*(\lambda)$ vanishes is the largest closed subscheme between $\mathcal{Y} \subset \tilde{\mathcal{Y}}$ to which λ deforms. But this is precisely $\tilde{\mathcal{Y}} \cap \mathcal{Z}$, and the deformation is simply the pullback of the universal object via

$$\tilde{\mathcal{Y}} \cap \mathcal{Z} \to \mathcal{Z} \to Z(m).$$

It only remains to show that $\mathrm{obst}(\lambda)$ and $\mathrm{obst}^*(\lambda)$ have the same zero locus. We only give the proof under the simplifying hypothesis that $D \in \mathcal{O}_{\tilde{\mathcal{Y}}}^\times$, as the general case is quite a bit more technical. Under this hypothesis

$$\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\tilde{\mathcal{Y}}} \cong \mathcal{O}_{\tilde{\mathcal{Y}}} \times \mathcal{O}_{\tilde{\mathcal{Y}}},$$

and the idempotents e and e' on the right hand side induce a splitting

$$H_1^{dR}(\tilde{A}_0) = eH_1^{dR}(\tilde{A}_0) \oplus e'H_1^{dR}(\tilde{A}_0)$$

in which $eH_1^{dR}(\tilde{A}_0)$ is maximal submodule on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_{\tilde{\mathcal{Y}}}$, and $e'H_1^{dR}(\tilde{A}_0)$ is the maximal submodule on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the complex conjugate. The signature condition on $\tilde{A}_0$ implies that

$$e'H_1^{dR}(\tilde{A}_0) = \mathrm{Fil}^1(\tilde{A}_0).$$

Similarly, the signature condition on the subsheaf $\mathcal{F}$ implies that $\mathcal{F} = e\mathrm{Lie}(\tilde{Y})$, and in particular the map

$$e'\mathrm{Lie}(\tilde{A}) \to \mathrm{Lie}(\tilde{A})/\mathcal{F}$$

is an isomorphism. Thus we can replace the diagram above by

$$\begin{tikzcd} e'H_1^{dR}(\tilde{A}_0) \arrow[r, "\mathrm{obst}^*(\lambda)"] \arrow[dr, "\mathrm{obst}(\lambda)"'] & e'\mathrm{Lie}(\tilde{A}) \arrow[d, "\cong"] \\ & \mathrm{Lie}(\tilde{A})/\mathcal{F}, \end{tikzcd}$$

from which we see that $\mathrm{obst}(\lambda)$ and $\mathrm{obst}^*(\lambda)$ have the same zero locus. $\square$

Returning to the main proof, let $\mathcal{O}(\mathcal{Z})$ be the line bundle on $\mathcal{U}$ determined by the divisor $\mathcal{Z}$, and let $\sigma(\mathcal{Z})$ be the section of $\mathcal{O}(\mathcal{Z})$ defined by the constant function 1 on $\mathcal{U}$. Note that $\sigma(\mathcal{Z})$ has zero locus $\mathcal{Z}$, and so $\sigma(\mathcal{Z})|_{\tilde{\mathcal{Y}}}$ has zero locus $\tilde{\mathcal{Y}} \cap \mathcal{Z}$. Using the fact that $\sigma(\mathcal{Z})|_{\tilde{\mathcal{Y}}}$ and $\mathrm{obst}(\lambda)$ have the same zero locus, an elementary argument shows that, Zariski locally on $\tilde{\mathcal{Y}}$, one can find an isomorphism

$$\mathcal{O}(\mathcal{Z})_{/\tilde{\mathcal{Y}}} \cong T_{\tilde{\mathcal{Y}}}$$

taking $\sigma(\mathcal{Z}) \mapsto \mathrm{obst}(\lambda)$. Furthermore, any two such isomorphisms agree upon restriction to $\mathcal{Y}$, and so we can glue together the resulting isomorphisms on an open cover of $\mathcal{Y}$ to obtain the adjunction isomorphism. $\square$

4.6. **The case of improper intersection.** Now we prove our main theorem in the general case. Define a metrized line bundle

$$\widehat{Z}^{\heartsuit}(m) = \widehat{Z}(m) \otimes \widehat{T}^{\otimes - R_\Lambda(m,\varphi_0)} \in \widehat{\mathrm{Pic}}(M).$$

It will turn out to be easier to compute $[\widehat{Z}^{\heartsuit}(m) : Y_\Lambda]$ than to directly compute $[\widehat{Z}(m) : Y_\Lambda]$, and we will carry out this computation by constructing a nonzero section

$$\sigma_m^{\heartsuit} \in H^0(Y_\Lambda, Z^{\heartsuit}(m)_{/Y_\Lambda})$$

of $Z^{\heartsuit}(m)$ restricted to Y_Λ.

The desired section will be constructed by passing to a cover $\{\mathcal{U} \to M\}$ of M by sufficiently small étale open subschemes, and working on each connected component $\mathcal{Y} \subset Y_{\Lambda/\mathcal{U}}$. In terms of line bundles on $\mathcal{U}$ we have

$$Z(m)_{/\mathcal{U}} \cong \bigotimes_{\mathcal{Z} \subset Z(m)_{/\mathcal{U}}} \mathcal{O}(\mathcal{Z}),$$

and so Proposition 4.5.1 and the adjunction isomorphism provide isomorphisms

$$\begin{aligned} Z^{\heartsuit}(m)_{/\mathcal{Y}} &\cong Z(m)_{/\mathcal{Y}} \otimes T_{/\mathcal{Y}}^{-\otimes R_\Lambda(m,\varphi_0)} \\ &\cong \bigotimes_{\substack{\mathcal{Z} \subset Z(m)_{/\mathcal{U}} \\ \mathcal{Z} \not\supset \mathcal{Y}}} \mathcal{O}(\mathcal{Z})_{/\mathcal{Y}} \otimes \bigotimes_{\substack{\mathcal{Z} \subset Z(m)_{/\mathcal{U}} \\ \mathcal{Z} \supset \mathcal{Y}}} \big(\mathcal{O}(\mathcal{Z}) \otimes T^{-1}\big)_{/\mathcal{Y}} \\ &\cong \bigotimes_{\substack{\mathcal{Z} \subset Z(m)_{/\mathcal{U}} \\ \mathcal{Z} \not\supset \mathcal{Y}}} \mathcal{O}(\mathcal{Z})_{/\mathcal{Y}} \end{aligned}$$

of line bundles on $\mathcal{Y}$. Each line bundle $\mathcal{O}(\mathcal{Z}) \supset \mathcal{O}_{\mathcal{U}}$ has a canonical section $\sigma(\mathcal{Z})$, corresponding to the constant function $1 \in \mathcal{O}_{\mathcal{U}}$, and their tensor product $\otimes_{\mathcal{Z}} \sigma(\mathcal{Z})$ defines a canonical nonzero section of

$$\bigotimes_{\substack{\mathcal{Z} \subset Z(m)_{/\mathcal{U}} \\ \mathcal{Z} \not\supset \mathcal{Y}}} \mathcal{O}(\mathcal{Z}).$$

Restricting this section to $\mathcal{Y}$ and applying the isomorphism above yields the desired section $\sigma_m^{\heartsuit}$ of $Z^{\heartsuit}(m)_{/\mathcal{Y}}$.

Each section $\sigma(\mathcal{Z})$ has divisor $\mathcal{Z}$, and hence its restriction to $\mathcal{Y}$ has divisor $\mathcal{Y} \cap \mathcal{Z}$. Therefore

$$\mathrm{div}(\sigma_m^{\heartsuit}) = \sum_{\substack{\mathcal{Z} \subset Z(m)_{/\mathcal{U}} \\ \mathcal{Z} \not\supset \mathcal{Y}}} (\mathcal{Z} \cap \mathcal{Y}).$$

We can think of the right hand side as obtained by taking the 0-dimensional part (that is, the disjoint union of all 0-dimensional connected components) of

$$Z(m) \times_M \mathcal{Y} \cong Z(m)_{/\mathcal{U}} \times_{\mathcal{U}} \mathcal{Y} \cong \bigsqcup_{\mathcal{Z} \subset Z(m)_{/\mathcal{U}}} (\mathcal{Z} \cap \mathcal{Y}),$$

and viewing it as a divisor on $\mathcal{Y}$. But

$$Z(m) \times_M \mathcal{Y} \cong \bigsqcup_{\substack{m_1, m_2 \in \mathbb{Z}_{\geq 0} \\ m_1 + m_2 = m}} \mathtt{X}_\Lambda(m_1, m_2)_{/\mathcal{Y}}$$

by Proposition 4.2.1, and the 0-dimensional part of the right hand side is

$$\bigsqcup_{\substack{m_1, m_2 \in \mathbb{Z}_{\geq 0} \\ m_1 + m_2 = m \\ m_1 > 0}} \mathtt{X}_\Lambda(m_1, m_2)_{/\mathcal{Y}}.$$

Combining this with Corollary 4.3.2, we have just proved the following proposition.

Proposition 4.6.1. *The divisor of $\sigma_m^\heartsuit$ is the 0-cycle*

$$\mathrm{div}(\sigma_m^\heartsuit) = \sum_{\substack{m_1, m_2 \in \mathbb{Z}_{\geq 0} \\ m_1 + m_2 = m \\ m_1 > 0}} \mathtt{X}_\Lambda(m_1, m_2)$$

on Y_Λ. In particular

$$\widehat{\deg}_{\mathrm{fin}} \mathrm{div}(\sigma_m^\heartsuit) = -\sum_{\substack{m_1, m_2 \in \mathbb{Z}_{\geq 0} \\ m_1 + m_2 = m \\ m_1 > 0}} \frac{a_{\mathcal{E}}^+(m_1, \varphi_0) R_\Lambda(m_2, \varphi_0)}{|\mathrm{Aut}(y_0)|}.$$

Proposition 4.6.2. *The norm of $\sigma_m^\heartsuit$ at the unique point $y_0 \in Y_\Lambda(\mathbb{C})$ satisfies*

$$-\log ||\sigma_m^\heartsuit||_{y_0}^2 = \Phi(y_0, f_m).$$

Proof. Recall that in a small neighborhood of y_0 we have

$$Z(m)(\mathbb{C}) = \bigsqcup_{\substack{\lambda \in \Lambda \\ \langle \lambda, \lambda \rangle = m}} \mathcal{D}_L(\lambda).$$

Let σ_m be the section of the line bundle $Z(m)$ corresponding to the constant function 1 on $\mathcal{D}_L$. By definition of the metric on $Z(m)$,

$$-\log ||\sigma_m||_z^2 = \Phi(z, f_m).$$

The connected components $\mathcal{Z} \subset Z(m)(\mathbb{C})$ passing through y_0 are precisely the $\mathcal{D}_L(\lambda)$'s appearing in the disjoint union. For each such component the associated line bundle $\mathcal{O}(\mathcal{Z})$ has a canonical section $\sigma(\mathcal{Z})$ corresponding to the constant function 1 on $\mathcal{D}_L$. Moreover, for each λ appearing in the disjoint union we have a canonical section $\mathrm{obst}(\lambda)$ of the cotautological bundle T, having norm

$$||\mathrm{obst}(\lambda)||_z^2 = -4\pi e^\gamma \langle \lambda_z, \lambda_z \rangle.$$

If we trace through the construction of $\sigma_m^\heartsuit$, the isomorphism of line bundles

$$Z^\heartsuit(m) \cong Z(m) \otimes T^{\otimes - R_\Lambda(m, \varphi_0)} \cong Z(m) \otimes \bigotimes_{\substack{\lambda \in \Lambda \\ \langle \lambda, \lambda \rangle = m}} T^{-1}$$

identifies

$$\sigma_m^{\heartsuit} = \sigma_m \otimes \bigotimes_{\substack{\lambda\in\Lambda \\ \langle\lambda,\lambda\rangle = m}} \mathrm{obst}(\lambda)^{-1}.$$

Therefore

$$\begin{aligned} -\log ||\sigma_m^{\heartsuit}||_z^2 &= -\log ||\sigma_m||_z^2 + \sum_{\substack{\lambda\in\Lambda \\ \langle\lambda,\lambda\rangle = m}} \log ||\mathrm{obst}(\lambda)||^2 \\ &= \Phi(z, f_m) + \sum_{\substack{\lambda\in\Lambda \\ \langle\lambda,\lambda\rangle = m}} \log |4\pi e^{\gamma}\langle \lambda_z, \lambda_z\rangle|^2. \end{aligned}$$

Now take the limit as $z \to y_0$ and apply (3.3.1). □

Proof of Theorem 3.5.1. Combining the two propositions gives

$$\begin{aligned} [\widehat{Z}^{\heartsuit}(m) : Y_\Lambda] &= \widehat{\deg}_f \, \mathrm{div}(\sigma_m^{\heartsuit}) - \frac{\log ||\sigma_m^{\heartsuit}||_{y_0}^2}{\mathrm{Aut}(y_0)} \\ &= \frac{\Phi(y_0, f_m)}{|\mathrm{Aut}(y_0)|} - \sum_{\substack{m_1, m_2 \in \mathbb{Z}_{\geq 0} \\ m_1 + m_2 = m \\ m_1 > 0}} \frac{a_{\mathcal{E}}^{+}(m_1, \varphi_0) R_\Lambda(m_2, \varphi_0)}{|\mathrm{Aut}(y_0)|}. \end{aligned}$$

Using the equality

$$[\widehat{T}^{\otimes R_\Lambda(m,\varphi_0)} : Y_\Lambda] = -\frac{a_{\mathcal{E}}^{+}(0, \varphi_0) R_\Lambda(m, \varphi_0)}{|\mathrm{Aut}(y_0)|}$$

of Corollary 3.6.2, this can be rewritten as

$$[\widehat{Z}(m) : Y_\Lambda] = \frac{\Phi(y_0, f_m)}{|\mathrm{Aut}(y_0)|} - \sum_{\substack{m_1, m_2 \in \mathbb{Z}_{\geq 0} \\ m_1 + m_2 = m}} \frac{a_{\mathcal{E}}^{+}(m_1, \varphi_0) R_\Lambda(m_2, \varphi_0)}{|\mathrm{Aut}(y_0)|}$$

and the CM value formula then gives

$$[\widehat{Z}(m) : Y_\Lambda] = -\frac{L'(\xi(f_m), \vartheta_\Lambda, 0)}{|\mathrm{Aut}(y_0)|} - a^{+} +_{f_m} (0,0) \cdot [\widehat{T} : Y_\Lambda].$$

Thus

$$[\widehat{\Theta}(f_m) : Y_\Lambda] = -\frac{1}{|\mathrm{Aut}(y_0)|} \cdot L'(\xi(f_m), \vartheta_\Lambda, 0),$$

as desired. This proves Theorem 3.5.1 for $f = f_m$. The proof for $f = f_{m,\delta}$ is very similar. □

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