Exercises given in the second semester.

(1) Let $f : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves on a topological space. Prove that $\mathrm{Ker}(f)$ is a sheaf.

(2) Let X be a scheme and k a field. To give a morphism $\mathrm{Spec}(k) \longrightarrow X$is equivalent to giving a point $x \in X$ and an embedding of $k(x) \hookrightarrow k$, where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$.

(3) Let K be a field of characteristic not equal to 2 and V a finite dimensional vector space over K. Show by brute force that $w \in \bigwedge^2 V$ is totally decomposable if and only if $w \wedge w = 0$. Conclude that the Grassmannian $\mathrm{Grass}(2, V) \subseteq \mathbb{P}(\bigwedge^2 V)$ is a variety cut by $\binom{n}{2}$ quadratic relations.

(4) Show that the flag variety $\mathrm{Drap}_m(V)$ is a projective variety.

(5) Let X be a scheme over a field k. Show that $\mathrm{Hom}(\mathrm{Spec}(k[\epsilon]), X) = \{(x, v) : x \in X, k(x) = k, v \in T_{X,x}\}$.

(6) Show: finite $\Rightarrow$ finite type $\Rightarrow$ locally of finite type; finite $\Rightarrow$ quasi finite; all converse implications do not hold in general.

(7) Show that a finite morphism is closed.

(8) Prove that open and closed immersions are separated.

(9) Let $f : X \longrightarrow Y$ be a morphism of proper schemes over S. Let $Z \subset X$ be a closed subscheme. Then Z and $f(Z)$ are proper over S.

(10) Let X be a variety oven an algebraically closed field. Let $x_1 \neq x_2$ be two distinct closed points of X. Let $U_i = X - \{x_i\}$. Calculate $H^1(X, \{U_1, U_2\}, \mathrm{GL}_1)$.

(11) Let $X = \mathrm{Spec}(k[x, y, z]/(xy - z^n))$ where k is an algebraically closed field. Prove that $Cl(X) \cong \mathbb{Z}/n\mathbb{Z}$.

(12) Do exercise II 6.5 (a), (b) in Hartshorne.

(13) Do exercise II 4.7 in Hartshorne.

(14) Let k be a field of characteristic p. Let $X = \mathrm{Spec}(k[x_0, \ldots, x_n]/(f_1, \ldots, f_m)$ and define $X^{(p)}$ as $\mathrm{Spec}(k[x_0, \ldots, x_n]/(f_1^\sigma, \ldots, f_m^\sigma)$ where if $f_i = \sum a_I x^I$ then $f_i^\sigma = \sum a_I^p x^I$. The map

$$k[x_0, \ldots, x_n]/(f_1^\sigma, \ldots, f_m^\sigma) \longrightarrow k[x_0, \ldots, x_n]/(f_1, \ldots, f_m),$$

determined by $x_i \mapsto x_i^p$, induces morphism

$$\mathrm{Fr}_X : X \longrightarrow X^{(p)}$$

called the Frobenius morphism. Describe $\Omega_{X/X^{(p)}}$ and give examples where it is not trivial. Note that if k is an algebraically closed field then the induced map on points $X(k) \longrightarrow X^{(p)}(k)$ is a bijection.

(15) Let C/k be a curve of genus g over an algebraically closed field. Then there exists a surjective morphism $f : C \longrightarrow \mathbb{P}^1$ such that $\deg(F) \leq g+1$. Show that for $g = 1$ this is sharp, while for $g > 1$ there are always curves with a morphism of smaller degree.

(16) A hyperelliptic curve is a non-singular curve with affine model

$$C : y^2 = (x - \alpha_1) \dots (x - \alpha_{2g+2}).$$

(N.B. The projective model of C is not just the closure in $\mathbb{P}^2$.) Prove that C has genus g.

Let $(x, y) \mapsto (x, -y)$ be the involution of C. Prove that it has $2g+2$ fixed points. It is called the hyperelliptic involution.

Conversely, Let C be a curve of genus g with an involution $h : C \longrightarrow C$ with $2g+2$ fixed points. Show that C is hyperelliptic and that h is the hyperelliptic involution.