Dirichlet series of squares of sums of squares

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Abstract. Hardy and Wright recorded elegant closed forms for the generating functions of the divisor functions $\sigma_k(n)$ and $\sigma_k^2(n)$:

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k)$$

and

$$\sum_{n=1}^{\infty} \frac{\sigma_k^2(n)}{n^s} = \frac{\zeta(s)\zeta(s-k)^2\zeta(s-2k)}{\zeta(2s-2k)}.$$

In this work, we explore other arithmetical functions enjoying this remarkable property.

1. In our basic Theorem, we are able to generalize the above result and prove that if f_i and g_i are completely multiplicative, then we have

   $$\sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n) \cdot (f_2 * g_2)(n)}{n^s} = \frac{L_{f_1 f_2}(s) L_{g_1 g_2}(s) L_{f_1 g_2}(s) L_{g_1 f_2}(s)}{L_{f_1 f_2 g_1 g_2}(2s)}$$

   where $L_f(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$ is the Dirichlet series corresponding to f.
2. Let $r_N(n)$ be the number of solutions of $x_1^2 + \cdots + x_N^2 = n$ and let $r_{2,P}(n)$ be the number of solutions of $x^2 + Py^2 = n$. A central application of our Theorem is to obtain concise closed forms, in terms of $\zeta(s)$ and Dirichlet L-functions, for the generating functions of $r_N(n), r_N^2(n), r_{2,P}(n)$ and $r_{2,P}(n)^2$ for certain P and (even) $N = 2, 4, 6, 8$. We also use these generating functions to obtain asymptotics for the average values of each function for which we obtain a Dirichlet series, and more generally.

We finish by discussing the more vexing cases $N = 3, 12, 24$.

This is joint work with Stephen Choi (SFU). These transparencies are at www.cecm.sfu.ca/personal/jborwein/talks.html

The corresponding CECM preprint 01:167, to appear in the Ramanujan Journal, is at www.cecm.sfu.ca/preprints/2001pp.html