Criteria for irrationality of Euler's constant

Jonathan Sondow (jsondow@alumni.princeton.edu)
209 West 97th Street Apt 6F
New York, NY 10025
USA

Abstract. By modifying the integrals in Beukers' version of Apery's proof that Zeta(2) and Zeta(3) are irrational, we find necessary and sufficient conditions for irrationality of Euler's constant.

For integers $n > 0$, we define a double integral $I(n)$ and a linear combination $L(n)$ of $\log(n+1), \ldots, \log(2n)$ over the positive rationals, and prove:

THEOREM . *If* $d(n) = \mathrm{LCM}(1, \ldots, n)$*, then the following are equivalent*:

1. For some $n > 0$, the fractional part of the product $d(2n)\,L(n)$ satisfies
$$\{d(2n)L(n)\} = d(2n)I(n).$$
2. This equation holds for all sufficiently large n.
3. Euler's constant is a rational number.

Corollaries give sufficient conditions for irrationality, involving the logarithm $L(n)$ but not the integral $I(n)$. For example:

COROLLARY . *If the inequality*
$$\{d(2n)L(n)\} > 1/2^n$$
holds for infinitely many n*, then Euler's constant is irrational.*