A diophantine approximation property characterizing both algebraic and Mahler's numbers

Florian Luca (fluca@matmor.unam.mx)
UNAM
Mathematical Institute
Ap. Postal 61-3 (Xangari)
CP 58 089
Morelia, Michoacan
Mexico

Abstract. A real number α is called a *Mahler U-number* if there exists a positive integer m such that for every $t \in \mathbf{R}$ there exist integers $c_0, c_1, \ldots, c_m$ such that

$$0 < \left| \sum_{i=0}^{m} c_i \alpha^i \right| < \left(1 + \sum_{i=0}^{m} |c_i| \right)^t. \tag{1}$$

If m is the least such integer for α, then α is a *Mahler U_m-number.* The U_1-numbers are precisely the Liouville numbers. Like the Liouville numbers, the U-numbers are transcendental. In this talk, we shall point out a certain simultaneous diophantine approximation property on all the powers of a real number α which is fulfilled precisely when α is either algebraic or a U-number. For any real number x we write $\|x\|$ for the distance from x to the nearest integer. Our result is the following:

THEOREM 1. *Let $e \geq 1$ be any fixed real number and let α be a real number. Then α is either algebraic of degree $d \leq e+1$, or a Mahler U_m-number for some $m \leq e$, if and only if, for each positive integer s, there exists $M_s \in \mathbf{R}$ and infinitely many positive integers $q \in \mathbf{Z}$ such that*

$$\|q\alpha^k\| < M_s q^{-1/e}, \qquad \textit{holds for all} \;\; k = 1,\ 2,\ \ldots,\ s.$$

Take, for example, $e = 1$. Then the above Theorem says that if α is a real number such that for all positive integers $s \geq 1$ the s-uple of real numbers $(\alpha,\ \alpha^2, \ldots, \alpha^s)$ can be simultaneously approximated with the fixed simultaneous approximation exponent 2 (up to a constant M_s depending on s), then α is either rational, or quadratic, or Liouville, and conversely, the rational, quadratic and Liouville numbers do fulfill the above simultaneous approximation property.

This is joint work with A. Dress (Bielefeld) and N. D. Elkies (Harvard).