Points of small height outside of a collection of subspaces

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Abstract. We consider an "anti Sigel's Lemma." Suppose $L_1(\boldsymbol{X}), \dots, L_M(\boldsymbol{X})$ are M linear forms in N variables with integer coefficients. Does there exist a vector $\boldsymbol{x}$ in $\mathbb{Z}^N$ with $|\boldsymbol{x}| = \max\{|x_1|, \dots, |x_N|\}$ relatively small such that

$$L_i(\boldsymbol{x}) \neq 0$$

for every $i = 1, \dots, M$? We prove that there does exist such a vector with

$$|\boldsymbol{x}| \leq \frac{M+1}{2},$$

that is the upper bound is linear in the number of linear forms. This readily generalizes to the number field case, that is if K is a number field of degree d over $\mathbb{Q}$, and the linear forms have coefficients in K, then there exists a vector $\boldsymbol{x} \in K^N$ at which none of the linear forms vanish, and

$$H(\boldsymbol{x}) \ll_{K,N} M^{1/d},$$

where H stands for the height function on K^N.

This question is also closely related to a classical question in the geometry of numbers. Suppose Λ is a sublattice of $\mathbb{Z}^N$ of maximal rank and determinant D, and suppose that C_R^N is a closed cube in $\mathbb{R}^N$ centered at the origin with sidelength $2R$. What would be an upper bound on the number of points of Λ in C_R^N? We provide an upper bound of the form

$$\min\left\{ \frac{2^N}{D}([R] + \alpha_N D^{1/N})^N, (2[R]+1)^N \right\},$$

where [ ] is the integer part function, and α_N is a constant that depends only on N.