On polynomials taking small values at integral arguments

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Abstract. In a recent paper, S. P. Tung considers the problem of estimating from below the quantity

$$S_F(T) = \max_{\substack{x\in\mathbb{N} \\ x\le T}} \min_{y\in\mathbb{Z}} |F(x,y)|,$$

where $F \in \mathbb{Q}[X,Y]$ is a given polynomial and $T \in \mathbb{N}$ is a variable growing to infinity. If there exists a polynomial $f(X) \in \mathbb{Q}[X]$, taking integral values on $\mathbb{Z}$, such that $F\big(X, f(X)\big)$ is a constant, then $S_F(T)$ is bounded. However this is essentially the only case when $S_F(T)$ is bounded. More precisely, define, for all $\mathcal{A} \subseteq \mathbb{N}$,

$$S_{\mathcal{A},F}(T) = \max_{x\in\mathcal{A}(T)} \min_{y\in\mathbb{Z}} |F(x,y)|.$$

Then Tung proves the following: *There exists a number $c > 0$, depending only on $\deg F$, with the following property. Either there exists a polynomial $f(X) \in \mathbb{Q}[X]$ such that $F\big(X, f(X)\big)$ is constant, or, for all sequences $\mathcal{A} \subseteq \mathbb{N}$ of positive density, we have $S_{\mathcal{A},F}(T) \gg T^c$.*

We are concerned with a question in a different direction: *how large can one choose the exponent c in the above statement?*

Tung points out that c cannot exceed 1/2 and, under the Generalized Riemann Hypothesis, he obtains the inequality $S_{\mathcal{A},F}(T) \gg \frac{\sqrt{T}}{\log^2 T}$, proving in particular that one can choose $c = (1/2) - \epsilon$, for any $\epsilon > 0$.

The purpose of the present talk is to show, unconditionally, that in fact one can take $c = 1/2$. We state this as the following

THEOREM 1. *Let $\mathcal{A} \in \mathbb{N}$ be a set of positive lower asymptotic density and let $F \in \mathbb{Q}[X,Y]$. Then, either there exists $f \in \mathbb{Q}[X]$ such that $F\big(X, f(X)\big)$ is constant, or $S_{\mathcal{A},F}(T) \gg \sqrt{T}$ for $T \to \infty$.*

We shall deduce Theorem 1 from a similar statement, namely

THEOREM 2. *If $\mathcal{A}$ is a set of positive upper asymptotic density and $y(a)$, $a \in \mathcal{A}$ are integers such that $\big|F\big(a, y(a)\big)\big| = o(\sqrt{a})$, then there exists a polynomial $f \in \mathbf{Q}[X]$ such that $F\big(X, f(X)\big)$ is constant.*