Heights on finite étale K-algebras

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Abstract. Let K be a number field and $\mathcal{M}_K$ the set of places of K. An adelic norm on a finite dimensional K-vector space V is a family $\mathcal{F} = \{N_v : V \otimes_K K_v \to \mathbf{R}, v \in \mathcal{M}_K\}$ of norms, on the completions of V, satisfying certain compatibility conditions. To any adelic norm $\mathcal{F}$ it can be attached a height function $H_{\mathcal{F}}$ on V, essentially by taking the product over all v's of the norms of the family. In this talk we will focus on the case in which $V = A$ is a finite, separable, commutative (hence étale) K-algebra with unit. In this situation is possible to define a canonical adelic norm $\mathcal{F}_A$, and hence a canonical height function H_A, on A which depend only the structure of A as a K-algebra (see also [1]). We will present some new results on H_A including the followings:

THEOREM 1. *Let A be a finite étale K-algebra as above. A Banach adelic norm is an adelic norm $\mathcal{F} = \{N_v, v\in\mathcal{M}_K\}$ such that $(A \otimes_K K_v, N_v)$ is a K_v-Banach algebra. If $\mathcal{B}(A)$ denote the set of Banach adelic norm on A, then*

$$H_A(a) = \min_{\mathcal{F}\in\mathcal{B}(A)} H_{\mathcal{F}}(a),$$

for all $a\in A$. Moreover:

$$H_A(a) = \lim_{k\to\infty} H_{\mathcal{F}}(a^k)^{1/k}$$

for all $a\in A$ and for all $\mathcal{F}\in\mathcal{B}(A)$.

THEOREM 2. *Let A be a finite étale K-algebra as above. If $T : A \to A$ is a K-linear transformation such that $H_A\big(T(a)\big) = H_A(a)$ for all $a\in A$, then $b = T(1_A)$ is invertible and $L_{b^{-1}}T$ is a K-algebra automorphism of A.*

The second theorem generalize a result contained in [1]. Time permitting we will also give a functorial interpretation of this last result.

Reference

1. V. Talamanca, *Height preserving linear transformations on semisimple K-algebras*, Proceeding of the $XIX^{\text{èmes}}$ Journées Arithmétiques, Collectanea Mathematica, vol. XLVII fasc. 1-2, 1997, pp. 217–234.