Partition identities and a theorem of Zagier

Karl Mahlburg (mahlburg@math.wisc.edu)
University of Wisconsin-Madison
Department of Mathematics
480 Lincoln Dr.
Madison, WI 53706-1388
USA

Abstract. In the study of partition theory and q-series, identities that relate infinite products to infinite series are of great interest (such as the famous Rogers-Ramanujan identities). Using a recent result of Zagier, we find an infinite family of such identities that is parameterized by the integers. For example, when $m = 1$ we obtain the classical Eisenstein series identity

$$\sum_{\lambda \geq 1 \text{ odd}} \frac{(-1)^{(\lambda-1)/2} q^{\lambda}}{(1-q^{2\lambda})} = q \prod_{n=1}^{\infty} \frac{(1-q^{8n})^4}{(1-q^{4n})^2}.$$

If $m = 2$ and $\left(\frac{\cdot}{3}\right)$ denotes the usual Legendre symbol modulo 3, then we obtain

$$\sum \lambda \geq 1 \frac{\left(\frac{\lambda}{3}\right) q^{\lambda}}{(1-q^{2\lambda})} = q \prod_{n=1}^{\infty} \frac{(1-q^{n})(1-q^{6n})^6}{(1-q^{2n})^2(1-q^{3n})^3}.$$

We also describe several partition theoretic consequences of our identities. We show that the generating functions for certain types of partitions are modular forms, and are related to the generating function of the triangular numbers. In particular, we find simple formulas that solve the well-known problem of counting the number of representations of an integer as the sum of an arbitrary number of triangular numbers.