On the index of a number field

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Abstract. We deal with the problem of the computation of the index, $\mathrm{ind}(K)$, of a number field K. Factorizing $\mathrm{ind}(K)$ as $\mathrm{ind}(K) = \prod_p p_p^{\alpha}$, let $\mathrm{ind}_p(K) = \alpha_p$. It is well known that, from the factorization type of $p\mathcal{O}_K$, one can decide if the p-component of the index, $\mathrm{ind}_p(K)$, is equal to 0 or different from 0, but the mere form of the factorization of $p\mathcal{O}_K$ is not sufficient, in general, for deciding what is the actual value of $\mathrm{ind}_p(K)$.

However, the value of $\mathrm{ind}_p(K)$ is determined by the isomorphism classes of the completions of K at the primes over p: more precisely, $\mathrm{ind}_p(K)$ coincides with the index, $I_p(K \otimes \mathbb{Q}_p)$, of $K \otimes \mathbb{Q}_p$. In the particular case when $K/\mathbb{Q}$ is Galois, $K \otimes \mathbb{Q}_p \cong L \oplus \cdots \oplus L = nL$ for some integer n and some Galois extension L of $\mathbb{Q}_p$, and $\mathrm{ind}_p(K) = I_p(nL)$.

We describe a method for explicitly computing $I_p(nL)$ for all n and all tamely ramified extensions of $\mathbb{Q}_p$, hence for explicitly computing the p-component of index of a Galois number field, tame at p. Using this method we exhibit two Galois number fields, of degree 52 over $\mathbb{Q}$, with the same ramification numbers at the primes over 3, but with different 3-component of the index: this gives a counterexample to Nart's conjecture on the value of the index of normal extension of $\mathbb{Q}$.

In view of this, it remains the problem to study under which conditions on two local fields L, L', tamely ramified over $\mathbb{Q}_p$, one can say that

$$I_p(nL) = I_p(nL') \quad \text{forall } n \in \mathbb{N}. \tag{1}$$

and, more generally, when it is true that

$$I_p(n_1 L_1 + \dots n_k L_k) = I_p(n_1 L_1' + \dots + n_k L_k') \quad \text{forall } n_1, \dots, n_k \in \mathbb{N}. \tag{2}$$

We shall show, among others, that for any two tamely ramified extensions L and L' of $\mathbb{Q}_p$, (1) holds if and only if the defining equations of L and L' are related by an arithmetical condition. We shall give interpretations of the arithmetical conditions involved both for the particular case (1) and for the general case (2) in terms of Galois theory.